Optimal Supply Networks III: Redistribution

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Outline

Distributed Sources

Size-density law Cartograms A reasonable derivation Global redistribution Public versus Private

References

Many sources, many sinks

How do we distribute sources?

- 🗞 Focus on 2-d (results generalize to higher dimensions).
- Sources = hospitals, post offices, pubs, ...
- Key problem: How do we cope with uneven population densities?
- local Sector Sec best distributed uniformly.
- Representation of the second s
- 🚯 Q2: Given population density is uneven, what do we do?
- & We'll follow work by Stephan (1977, 1984)^[4, 5], Gastner and Newman (2006)^[2], Um et al. (2009)^[6], and work cited by them.

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Solidifying the basic problem

- 🗞 Given a region with some population distribution $\rho_{\rm r}$ most likely uneven.
- & Given resources to build and maintain N facilities.
- \mathbb{Q} : How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?

"Optimal design of spatial distribution networks" Gastner and Newman, Phys. Rev. E, 74, 016117, 2006.^[2]



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References

🚳 Why?



Optimal source allocation

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 ρ_{pop} .

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 \clubsuit Fit is $\rho_{\text{fac}} \propto \rho_{\text{pop}}^{0.66}$ with $r^2 = 0.94$.

Looking good for a 2/3 power ...

10 100

& Optimal facility density ρ_{fac} vs. population density

population density ρ (in km⁻²)



1000 10000

- Again: Different story to branching networks where there was either one source or one sink.
- Now sources & sinks are distributed throughout region.

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Optimal source allocation



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References

Science, 196, 523-524, 1977.^[4] 🗞 We first examine Stephan's treatment (1977)^[4, 5]

Subnational Boundaries"

G. Edward Stephan,

'Territorial Division: The Least-Time

Constraint Behind the Formation of

- Zipf-like approach: invokes principle of minimal effort.
- lso known as the Homer Simpson principle.





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Approximately optimal location of 5000 facilities. 🙈 Based on 2000 Census data.

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Optimal source allocation

- \bigotimes Consider a region of area A and population P with a single functional center that everyone needs to access every day.
- Build up a general cost function based on time expended to access and maintain center.
- \Im Write average travel distance to center as \overline{d} and assume average speed of travel is \bar{v} .
- Assume isometry: average travel distance \bar{d} will be on the length scale of the region which is $\sim A^{1/2}$
- Average time expended per person in accessing facility is therefore

 $\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$

where c is an unimportant shape factor.

Optimal source allocation

- & Next assume facility requires regular maintenance (person-hours per day).
- \mathfrak{L} Call this quantity τ .
- lf burden of mainenance is shared then average cost per person is τ/P where P = population.

 \bigotimes Replace *P* by $\rho_{pop}A$ where ρ_{pop} is density.

- lmportant assumption: uniform density.
- Total average time cost per person:

$$T = \bar{d}/\bar{v} + \tau/(\rho_{\mathsf{pop}}A) = cA^{1/2}/\bar{v} + \tau/(\rho_{\mathsf{pop}}A).$$

Now Minimize with respect to A ...

Optimal source allocation

🚳 Differentiating ...

$$\begin{split} \frac{\partial T}{\partial A} &= \frac{\partial}{\partial A} \left(c A^{1/2} / \bar{v} + \tau / (\rho_{\mathsf{pop}} A) \right) \\ &= \frac{c}{2 \bar{v} A^{1/2}} - \frac{\tau}{\rho_{\mathsf{pop}} A^2} = 0 \end{split}$$

🚷 Rearrange:

$$A = \left(\frac{2\bar{v}\tau}{c\rho_{\rm pop}}\right)^{2/3} \propto \rho_{\rm pop}^{-2/3}$$

 \clubsuit # facilities per unit area ρ_{fac} :

$$ho_{
m fac} \propto A^{-1} \propto
ho_{
m pop}^{2/3}$$

\delta Groovy ...

Optimal source allocation

An issue:

- \Re Maintenance (τ) is assumed to be independent of population and area (P and A)
- 🚳 Stephan's online book "The Division of Territory in Society" is here 🗹.
- & (It used to be here \square .)
- ♣ The Readme I is well worth reading (1995).

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Cartogram of countries 'rescaled' by population:



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Diffusion-based cartograms:

- ldea of cartograms is to distort areas to more accurately represent some local density $\rho_{\rm pop}$ (e.g. population).
- Many methods put forward—typically involve some kind of physical analogy to spreading or repulsion.
- Algorithm due to Gastner and Newman (2004)^[1] is based on standard diffusion:

$$\nabla^2 \rho_{\rm pop} - \frac{\partial \rho_{\rm pop}}{\partial t} = 0. \label{eq:pop}$$

- Allow density to diffuse and trace the movement of individual elements and boundaries.
- Diffusion is constrained by boundary condition of surrounding area having density $\bar{\rho}_{pop}$.

Cartograms

Child mortality:



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Standard world map:

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Gross domestic product:

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People living with HIV:

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Greenhouse gas emissions:



The preceding sampling of Gastner & Newman's cartograms lives here Z.

A larger collection can be found at worldmapper.org

WSRLDMAPPER The world as you've never seen is bef

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Size-density law



- left: population density-equalized cartogram. \Re Right: (population density)^{2/3}-equalized cartogram.
- & Facility density is uniform for $\rho_{\text{pop}}^{2/3}$ cartogram.

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From Gastner and Newman (2006)^[2]

la Cartogram's Voronoi cells are somewhat hexagonal.

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Approximations:

one per source.

Deriving the optimal source distribution:

- Basic idea: Minimize the average distance from a random individual to the nearest facility.^[2]
- & Assume given a fixed population density ρ_{pop} defined on a spatial region Ω .
- \mathfrak{F} Formally, we want to find the locations of nsources $\{\vec{x}_1, \dots, \vec{x}_n\}$ that minimizes the cost function

$$F(\{\vec{x}_1,\ldots,\vec{x}_n\}) = \int_{\Omega} \rho_{\mathsf{pop}}(\vec{x}) \min_i ||\vec{x} - \vec{x}_i|| \mathrm{d}\vec{x} \, .$$

- Also known as the p-median problem.
- 🗞 Not easy ... in fact this one is an NP-hard problem.^[2]
- Approximate solution originally due to Gusein-Zade^[3].

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- Befine $A(\vec{x})$ as the area of the Voronoi cell containing \vec{x} .
- As per Stephan's calculation, estimate typical distance from \vec{x} to the nearest source (say *i*) as

 \mathfrak{F} For a given set of source placements $\{\vec{x}_1, \dots, \vec{x}_n\}$,

the region Ω is divided up into Voronoi cells \mathbb{Z} ,

$c_i A(\vec{x})^{1/2}$

where c_i is a shape factor for the *i*th Voronoi cell. Approximate c_i as a constant c_i .

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Spending on healthcare:



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Carrying on:

The cost function is now

 $F = c \int_{\Omega} \rho_{\mathsf{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathsf{d}\vec{x} \,.$

- & We also have that the constraint that Voronoi cells divide up the overall area of Ω : $\sum_{i=1}^{n} A(\vec{x}_i) = A_{\Omega}$.
- Sneakily turn this into an integral constraint:

$$\int_\Omega \frac{{\rm d}\vec x}{A(\vec x)}=n.$$

- \bigotimes Within each cell, $A(\vec{x})$ is constant.
- So ... integral over each of the n cells equals 1.

Now a Lagrange multiplier story:

 \bigotimes By varying $\{\vec{x}_1, \dots, \vec{x}_n\}$, minimize

$$G(A) = c \int_{\Omega} \rho_{\mathrm{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} - \lambda \left(n - \int_{\Omega} \left[A(\vec{x}) \right]^{-1} \mathrm{d}\vec{x} \right)_{\mathrm{Suppose}}^{\mathrm{Distributed}}_{\mathrm{Suppose}} \int_{\mathrm{Comparison of the formula of the set of the s$$

- 🚯 I Can Haz Calculus of Variations 🖉?
- Sompute $\delta G/\delta A$, the functional derivative \mathbb{Z} of the References functional G(A).
- 🚳 This gives

$$\int_{\Omega} \left[\frac{c}{2} \rho_{\mathrm{pop}}(\vec{x}) A(\vec{x})^{-1/2} - \lambda \left[A(\vec{x}) \right]^{-2} \right] \mathrm{d}\vec{x} \, = 0.$$

Setting the integrand to be zilch, we have:

 $\rho_{\rm non}(\vec{x}) = 2\lambda c^{-1} A(\vec{x})^{-3/2}.$

Size-density law

Now a Lagrange multiplier story:

Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho_{\rm pop}^{-2/3}.$$

Similar Finally, we indentify $1/A(\vec{x})$ as $\rho_{fac}(\vec{x})$, an approximation of the local source density.

Substituting $\rho_{fac} = 1/A$, we have

$$\rho_{\rm fac}(\vec{x}) = \left(\frac{c}{2\lambda}\rho_{\rm pop}\right)^{2/3}$$

 \aleph Normalizing (or solving for λ):

$$\rho_{\rm fac}(\vec{x}) = n \frac{[\rho_{\rm pop}(\vec{x})]^{2/3}}{\int_{\Omega} [\rho_{\rm pop}(\vec{x})]^{2/3} {\rm d}\vec{x}} \propto [\rho_{\rm pop}(\vec{x})]^{2/3}.$$

Global redistribution networks

One more thing:

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Optimal Supply

A reasonable derivation

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- How do we supply these facilities?
- How do we best redistribute mail? People?
- How do we get beer to the pubs?
- langle Gastner and Newman model: cost is a function of basic maintenance and travel time:

 $C_{\mathsf{maint}} + \gamma C_{\mathsf{travel}}.$

Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance ℓ_{ii} and number of legs to journey:

 $(1-\delta)\ell_{ii}+\delta(\#\mathsf{hops}).$

& When $\delta = 1$, only number of hops matters.

Global redistribution networks



From Gastner and Newman (2006)^[2]

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Beyond minimizing distances:

- Scaling laws between population and facility densities" by Um et al., Proc. Natl. Acad. Sci., 2009. [6]
- local states and a second state of the second states and a second state of the second states and st that the connection between facility and population density

$ho_{\rm fac} \propto ho_{\rm DOD}^{lpha}$

does not universally hold with $\alpha = 2/3$.

Two idealized limiting classes:

- 1. For-profit, commercial facilities: $\alpha = 1$;
- 2. Pro-social, public facilities: $\alpha = 2/3$.
- locations in the United With the United with the With the United and the United with the Unite States and South Korea.

Public versus private facilities: evidence



Left plot: ambulatory hospitals in the U.S.

Right plot: public schools in the U.S.

Note: break in scaling for public schools. Transition from $\alpha \simeq 2/3$ to $\alpha = 1$ around $\rho_{\rm pop} \simeq 100.$

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Public versus private facilities: evidence

US facility	α (SE)	R ²	
Ambulatory hospital	1.13(1)	0.93	
Beauty care	1.08(1)	0.86	
Laundry	1.05(1)	0.90	
Automotive repair	0.99(1)	0.92	
Private school	0.95(1)	0.82	
Restaurant	0.93(1)	0.89	
Accommodation	0.89(1)	0.70	Rough transition
Bank	0.88(1)	0.89	hatwaan nublic
Gas station	0.86(1)	0.94	between public
Death care	0.79(1)	0.80	and private at
* Fire station	0.78(3)	0.93	~~ 0 8
* Police station	0.71(6)	0.75	$\alpha = 0.8$.
Public school	0.69(1)	0.87	
SK facility	α (SE)	R ²	Note: * indicates
Bank	1 18(2)	0.96	analysis is at
Parking place	1.13(2)	0.91	stato/provinco
* Primary clinic	1.09(2)	1.00	state/province
* Hospital	0.96(5)	0.97	level: otherwise
* University/college	0.93(9)	0.89	level, otherwise
Market place	0.87(2)	0.90	county level.
* Secondary school	0.77(3)	0.98	
* Primary school	0.77(3)	0.97	
Social welfare org.	0.75(2)	0.84	
* Police station	0.71(5)	0.94	
Government office	0.70(1)	0.93	
* Fire station	0.60(4)	0.93	
* Public health center	0.09(5)	0.19	

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Public versus Private



A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.

Public versus private facilities: the story So what's going on?

- Social institutions seek to minimize distance of travel.
- Commercial institutions seek to maximize the number of visitors.
- \bigotimes Defns: For the *i*th facility and its Voronoi cell V_i , define

 n_i = population of the *i*th cell;

 \hat{r}_i $\langle r_i \rangle$ = the average travel distance to the *i*th facility.

 \bigcirc A_i = area of *i*th cell (s_i in Um *et al.*^[6])

line with the second se constructed):

 $v_i = n_i \langle r_i \rangle^{\beta}$ with $0 \le \beta \le 1$.

```
🚳 Limits:
    f
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Ø	$\beta = 0$: purely commercial.	
Ø	$\beta = 1$: purely social.	

Public versus private facilities: the story

🗞 Either proceeding as per the Gastner-Newman-Gusein-Zade calculation or, as Um et al. do, observing that the cost for each cell should be the same, we have:

$$\rho_{\rm fac}(\vec{x}) = n \frac{[\rho_{\rm pop}(\vec{x})]^{2/(\beta+2)}}{\int_{\Omega} [\rho_{\rm pop}(\vec{x})]^{2/(\beta+2)} {\rm d}\vec{x}} \propto [\rho_{\rm pop}(\vec{x})]^{2/(\beta+2)}.$$

So For $\beta = 0$, $\alpha = 1$: commercial scaling is linear. So For $\beta = 1$, $\alpha = 2/3$: social scaling is sublinear.

Proc. Natl. Acad. Sci., 101:7499–7504, 2004. pdf

Optimal design of spatial distribution networks.

Bunge's problem in central place theory and its

Territorial division: The least-time constraint behind the formation of subnational boundaries.

[1] M. T. Gastner and M. E. J. Newman. Diffusion-based method for producing

[2] M. T. Gastner and M. E. J. Newman.

Phys. Rev. E, 74:016117, 2006. pdf

Geogr. Anal., 14:246–252, 1982. pdf

Science, 196:523–524, 1977. pdf 🗹

density-equalizing maps.

[3] S. M. Gusein-Zade.

generalizations.

[4] G. E. Stephan.

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References II Optimal Supply

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[5] G. E. Stephan. Territorial subdivision. Social Forces, 63:145–159, 1984. pdf 🕑

[6] J. Um, S.-W. Son, S.-I. Lee, H. Jeong, and B. J. Kim. Scaling laws between population and facility densities. Proc. Natl. Acad. Sci., 106:14236-14240, 2009. pdf 🖸

Distributed Sources References

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