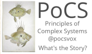
Optimal Supply Networks I: Branching

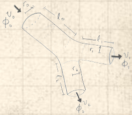
Last updated: 2023/01/26, 11:38:42 EST

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394, 2022–2023| @pocsvox

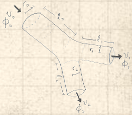
Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont





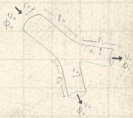
These slides are brought to you by:



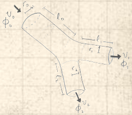
These slides are also brought to you by:

Special Guest Executive Producer

On Instagram at pratchett_the_cat



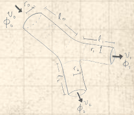
Outline



Optimal supply networks

What's the best way to distribute stuff?

- Stuff = medical services, energy, people, ...
- Some fundamental network problems:
  1. Distribute stuff from a single source to many sinks
  2. Distribute stuff from many sources to many sinks
  3. Redistribute stuff between nodes that are both sources and sinks
- Supply and Collection are equivalent problems




THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


THE BRAID OR THE BRANCH

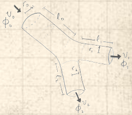
Single source optimal supply

Basic question for distribution/supply networks:

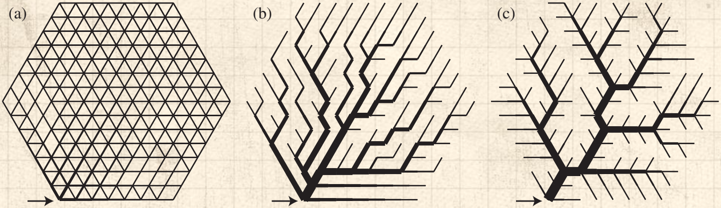
- How does flow behave given cost:

$$C = \sum_j I_j^{\gamma} Z_j$$

  where
  I_j = current on link j
  and
  Z_j = link j's impedance.
- Example: $\gamma = 2$ for electrical networks.



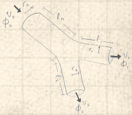
Single source optimal supply



(a) $\gamma > 1$: Braided (bulk) flow
(b) $\gamma < 1$: Local minimum: Branching flow
(c) $\gamma < 1$: Global minimum: Branching flow

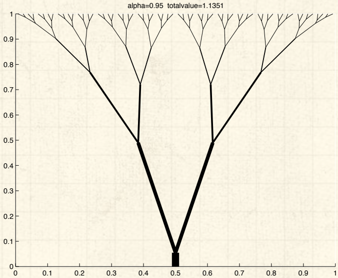
- Note: This is a single source supplying a region.

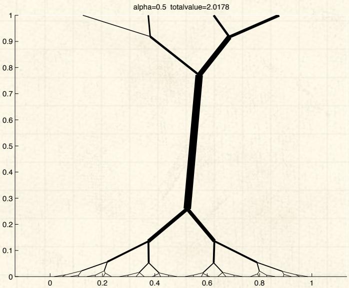
From Bohn and Magnasco [3]
See also Banavar *et al.* [1]: "Topology of the Fittest Transportation Network"; focus is on presence or absence of loops—same story



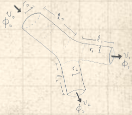
Single source optimal supply

Optimal paths related to transport (Monge) problems:



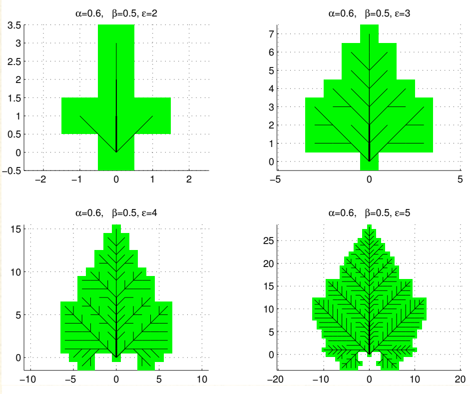


"Optimal paths related to transport problems"
Qinglan Xia,
Communications in Contemporary Mathematics, **5**, 251–279, 2003. [19]

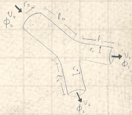


Growing networks—two parameter model: [20]

FIGURE 1. $\alpha = 0.6, \beta = 0.5$

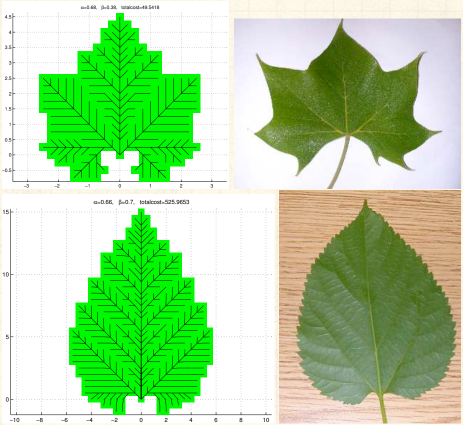


- Parameters control impedance ($0 \leq \alpha < 1$) and angles of junctions ($0 < \beta$)
- For this example: $\alpha = 0.6$ and $\beta = 0.5$



Growing networks: [20]

Figure 3. A maple leaf



Top: $\alpha = 0.66$, $\beta = 0.38$; Bottom: $\alpha = 0.66$, $\beta = 0.70$

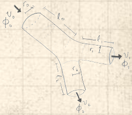
Single source optimal supply

An immensely controversial issue ...

- The form of natural branching networks: Random, optimal, or some combination? [6, 18, 2, 5, 4]
- River networks, blood networks, trees, ...

Two observations:

- Self-similar networks appear everywhere in nature for single source supply/single sink collection.
- Real networks differ in details of scaling but reasonably agree in scaling relations.



River network models

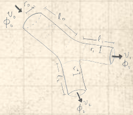
Optimality:

- Optimal channel networks [13]
- Thermodynamic analogy [14]

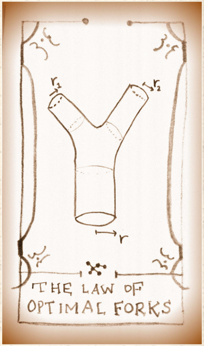
versus ...

Randomness:

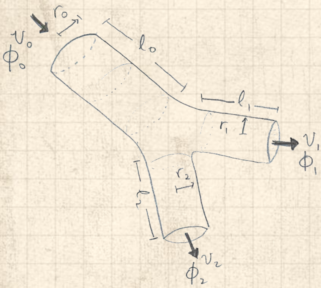
- Scheidegger's directed random networks
- Undirected random networks




THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


r_1
r_2
r
THE LAW OF OPTIMAL FORKS

Optimization—Murray's law

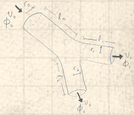


- Murray's law (1926) connects branch radii at forks: [11, 10, 12, 7, 16]

$$r_0^3 = r_1^3 + r_2^3$$

where r_0 = radius of main branch, and r_1 and r_2 are radii of sub-branches.

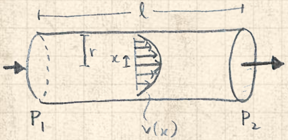
- Holds up well for outer branchings of blood networks.
- Also found to hold for trees [12, 8] when xylem is not a supporting structure [9].
- See D'Arcy Thompson's "On Growth and Form" for background and general inspiration [15, 16].



- Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

where Δp = pressure difference, Φ = flux.



- Fluid mechanics: Poiseuille impedance for smooth Poiseuille flow in a tube of radius r and length ℓ:

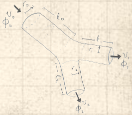
$$Z = \frac{8\eta\ell}{\pi r^4}$$

- η = dynamic viscosity (units: $ML^{-1}T^{-1}$).
- Power required to overcome impedance:

$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z.$$

- Also have rate of energy expenditure in maintaining blood given metabolic constant c:

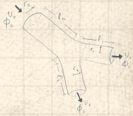
$$P_{\text{metabolic}} = cr^2\ell$$



Optimization—Murray's law

Aside on P_{drag}

- Work done = $F \cdot d$ = energy transferred by force F
- Power = P = rate work is done = $F \cdot v$
- Δp = Pressure differential = Force per unit area
- Φ = Volume flow per unit time (current)
  = cross-sectional area · velocity
- So $\Phi \Delta p$ = Force · velocity



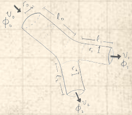
Optimization—Murray's law

Murray's law:

- Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell$$

- Observe power increases linearly with ℓ
- But r's effect is nonlinear:
  - increasing r makes flow easier but increases metabolic cost (as r^2)
  - decreasing r decrease metabolic cost but impedance goes up (as r^{-4})



Optimization—Murray's law

Murray's law:

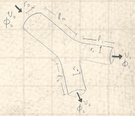
- Minimize P with respect to r:

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell\right)$$

- Flow rates at each branching have to add up (else our organism is in serious trouble ...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches



Optimization—Murray's law

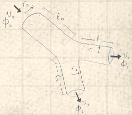
Murray's law:

- Find:

$$\Phi = kr^3$$

- Insert question from assignment 16
- All of this means we have a groovy cube-law:

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$



Optimization

Murray meets Tokunaga:

- Φ_ω = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

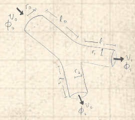
$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

- Using $\phi_\omega = k r_\omega^3$

$$(r_\omega)^3 = 2\left(r_{\omega-1}\right)^3 + \sum_{k=1}^{\omega-1} T_k \left(r_{\omega-k}\right)^3$$

- Same form as:

$$n_\omega = \underbrace{2 n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega} n_{\omega'}}_{\text{absorption}}$$



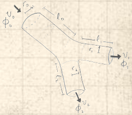
Optimization

Murray meets Tokunaga:

- Find Horton ratio for vessel radius $R_r = r_\omega / r_{\omega - 1}$.
- Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

$$\boxed{R_r^3 = R_n = R_v}$$

- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?



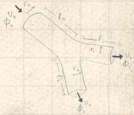
Optimization

Murray meets Tokunaga:

- Isometry: $V_\omega \propto \ell_\omega^3$
- Gives

$$\boxed{R_\ell^3 = R_r^3 = R_n = R_v}$$

- We need one more constraint ...
- West *et al.* (1997) [18] achieve similar results following Horton's laws (but this work is a disaster).
- So does Turcotte *et al.* (1998) [17] using Tokunaga (sort of).

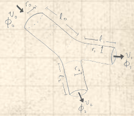


References I

[1] J. R. Banavar, F. Colaiori, A. Flammini, A. Maritan, and A. Rinaldo.
Topology of the fittest transportation network.
Phys. Rev. Lett., 84:4745–4748, 2000. pdf

[2] J. R. Banavar, A. Maritan, and A. Rinaldo.
Size and form in efficient transportation networks.
Nature, 399:130–132, 1999. pdf

[3] S. Bohn and M. O. Magnasco.
Structure, scaling, and phase transition in the optimal transport network.
Phys. Rev. Lett., 98:088702, 2007. pdf

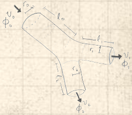


References II

[4] P. S. Dodds.
Optimal form of branching supply and collection networks.
Phys. Rev. Lett., 104(4):048702, 2010. pdf

[5] P. S. Dodds and D. H. Rothman.
Geometry of river networks. I. Scaling, fluctuations, and deviations.
Physical Review E, 63(1):016115, 2001. pdf

[6] J. W. Kirchner.
Statistical inevitability of Horton's laws and the apparent randomness of stream channel networks.
Geology, 21:591–594, 1993. pdf



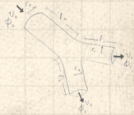
References III

[7] P. La Barbera and R. Rosso.
Reply.
Water Resources Research, 26(9):2245–2248, 1990. pdf

[8] K. A. McCulloh, J. S. Sperry, and F. R. Adler.
Water transport in plants obeys Murray's law.
Nature, 421:939–942, 2003. pdf

[9] K. A. McCulloh, J. S. Sperry, and F. R. Adler.
Murray's law and the hydraulic vs mechanical functioning of wood.
Functional Ecology, 18:931–938, 2004. pdf

[10] C. D. Murray.
The physiological principle of minimum work applied to the angle of branching of arteries.
J. Gen. Physiol., 9(9):835–841, 1926. pdf

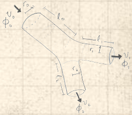


References IV

[11] C. D. Murray.
The physiological principle of minimum work. I. The vascular system and the cost of blood volume.
Proc. Natl. Acad. Sci., 12:207–214, 1926. pdf

[12] C. D. Murray.
A relationship between circumference and weight in trees and its bearing on branching angles.
J. Gen. Physiol., 10:725–729, 1927. pdf

[13] I. Rodríguez-Iturbe and A. Rinaldo.
Fractal River Basins: Chance and Self-Organization.
Cambridge University Press, Cambrigde, UK, 1997.



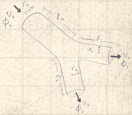
References V

[14] A. E. Scheidegger.
Theoretical Geomorphology.
Springer-Verlag, New York, third edition, 1991.

[15] D. W. Thompson.
On Growth and Form.
Cambridge University Pres, Great Britain, 2nd edition, 1952.

[16] D. W. Thompson.
On Growth and Form — Abridged Edition.
Cambridge University Press, Great Britain, 1961.

[17] D. L. Turcotte, J. D. Pelletier, and W. I. Newman.
Networks with side branching in biology.
Journal of Theoretical Biology, 193:577–592, 1998.
pdf



References VI

[18] G. B. West, J. H. Brown, and B. J. Enquist.
A general model for the origin of allometric scaling laws in biology.
Science, 276:122–126, 1997. pdf

[19] Q. Xia.
Optimal paths related to transport problems.
Communications in Contemporary Mathematics, 5:251–279, 2003. pdf

[20] Q. Xia.
The formation of a tree leaf.
ESAIM: Control, Optimisation and Calculus of Variations, 13:359–377, 2007. pdf