Scale-free networks

Last updated: 2022/08/28, 08:34:20 EDT

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394, 2022–2023| @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont

Scale-free networks

These slides are brought to you by:

PoCS
Principles of Complex Systems
@pocsvox
What's the Story?

These slides are also brought to you by:

Special Guest Executive Producer

On Instagram at pratchett_the_cat

PoCS
Principles of Complex Systems
@pocsvox
What's the Story?

Outline


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


THE NETWORKS FREE OF SCALE

Outline

Scale-free networks

- Main story
- Model details
- Analysis
- A more plausible mechanism
- Robustness
- Krapivsky & Redner's model
- Generalized model
- Analysis
- Universality?
- Sublinear attachment kernels
- Superlinear attachment kernels
- Nutshell

References

Scale-free networks

- Networks with power-law degree distributions have become known as scale-free networks.

Scale-free networks

- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

Scale-free networks

- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

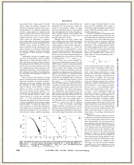
$$P_k \sim k^{-\gamma} \text{ for 'large' } k$$

Scale-free networks

- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma} \text{ for 'large' } k$$

- One of the seminal works in complex networks:



"Emergence of scaling in random networks"
Barabási and Albert,
Science, **286**, 509–511, 1999. [2]

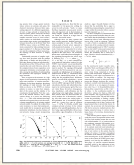
Times cited: $\sim 23,532$ (as of October 8, 2015)

Scale-free networks

- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma} \text{ for 'large' } k$$

- One of the seminal works in complex networks:



"Emergence of scaling in random networks"
Barabási and Albert,
Science, **286**, 509–511, 1999. [2]

Times cited: $\sim 23,532$ (as of October 8, 2015)

- Somewhat misleading nomenclature...

Scale-free networks

- Scale-free networks are not fractal in any sense.

Scale-free networks

- Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, ...(non-physical)

Scale-free networks

- Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, ...(non-physical)
- Primary example: hyperlink network of the Web

Scale-free networks

- Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, ...(non-physical)
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not...

Some real data (we are feeling brave):

From Barabási and Albert's original paper [2]:

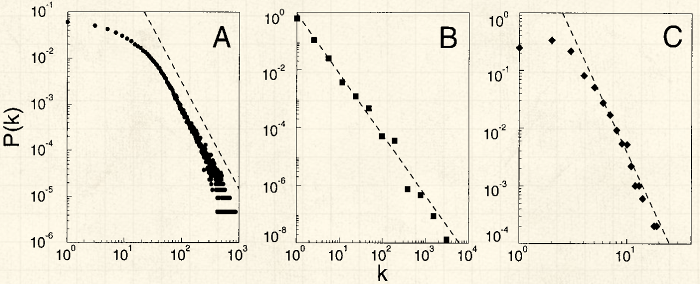
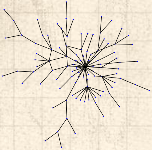
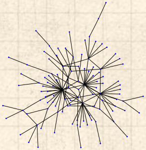


Fig. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\text{actor}} = 2.3$, (B) $\gamma_{\text{www}} = 2.1$ and (C) $\gamma_{\text{power}} = 4$.

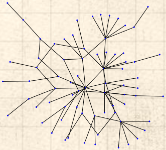
Random networks: largest components



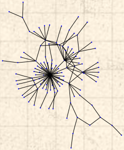
γ = 2.5
$\langle k \rangle$ = 1.8



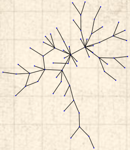
γ = 2.5
$\langle k \rangle$ = 2.05333



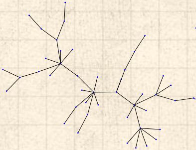
γ = 2.5
$\langle k \rangle$ = 1.66667



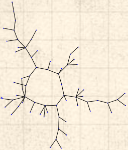
γ = 2.5
$\langle k \rangle$ = 1.92



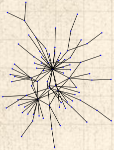
γ = 2.5
$\langle k \rangle$ = 1.6



γ = 2.5
$\langle k \rangle$ = 1.50667



γ = 2.5
$\langle k \rangle$ = 1.62667



γ = 2.5
$\langle k \rangle$ = 1.8

Scale-free networks

The big deal:

- We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

Scale-free networks

The big deal:

- We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

A big deal for scale-free networks:

- How does the exponent γ depend on the mechanism?

Scale-free networks

The big deal:

- We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

A big deal for scale-free networks:

- How does the exponent γ depend on the mechanism?
- Do the mechanism details matter?

Outline

BA model

- Barabási-Albert model = BA model.

BA model

- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).

BA model

- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).
- Step 1: start with m_0 disconnected nodes.

BA model

- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).
- Step 1: start with m_0 disconnected nodes.
- Step 2:

BA model

- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).
- Step 1: start with m_0 disconnected nodes.
- Step 2:
  1. Growth—a new node appears at each time step $t = 0, 1, 2, \ldots$.

BA model

- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).
- Step 1: start with m_0 disconnected nodes.
- Step 2:
  1. Growth—a new node appears at each time step $t = 0, 1, 2, \ldots$.
  2. Each new node makes m links to nodes already present.

BA model

- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).
- Step 1: start with m_0 disconnected nodes.
- Step 2:
  1. Growth—a new node appears at each time step $t = 0, 1, 2, \ldots$.
  2. Each new node makes m links to nodes already present.
  3. Preferential attachment—Probability of connecting to ith node is $\propto k_i$.

BA model

- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).
- Step 1: start with m_0 disconnected nodes.
- Step 2:
  1. Growth—a new node appears at each time step $t = 0, 1, 2, \ldots$.
  2. Each new node makes m links to nodes already present.
  3. Preferential attachment—Probability of connecting to ith node is $\propto k_i$.
- In essence, we have a rich-gets-richer scheme.

BA model

- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).
- Step 1: start with m_0 disconnected nodes.
- Step 2:
  1. Growth—a new node appears at each time step $t = 0, 1, 2, \ldots$.
  2. Each new node makes m links to nodes already present.
  3. Preferential attachment—Probability of connecting to ith node is $\propto k_i$.
- In essence, we have a rich-gets-richer scheme.
- Yes, we've seen this all before in Simon's model.

Outline

BA model

Definition: A_k is the attachment kernel for a node with degree k.

BA model

- Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

BA model

- Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- Definition: $P_{\text{attach}}(k, t)$ is the attachment probability.

BA model

- Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- Definition: $P_{\text{attach}}(k, t)$ is the attachment probability.
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

BA model

- **Definition:** A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- **Definition:** $P_{\text{attach}}(k, t)$ is the attachment probability.
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t

BA model

- Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- Definition: $P_{\text{attach}}(k,t)$ is the attachment probability.
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\text{max}}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t

BA model

- Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- Definition: $P_{\text{attach}}(k, t)$ is the attachment probability.
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\text{max}}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

Approximate analysis

- When $(N+1)$th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

Approximate analysis

- When $(N+1)$th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.

Approximate analysis

- When $(N+1)$th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.

Approximate analysis

- When $(N+1)$th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- Approximate $k_{i,N+1} - k_{i,N}$ with $\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t}$:

Approximate analysis

- When $(N+1)$th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- Approximate $k_{i,N+1} - k_{i,N}$ with $\frac{\mathsf{d}}{\mathsf{d}t} k_{i,t}$:

$$\frac{\mathsf{d}}{\mathsf{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where $t = N(t) - m_0$.

- Deal with denominator: each added node brings m new edges.

- Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

- Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt}$$

- Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt} = \frac{1}{2t} k_i(t)$$

- Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

- Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t}$$

- Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt} = \frac{1}{2t} k_i(t)$$

- Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i\, t^{1/2}.}$$

- Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt} = \frac{1}{2t} k_i(t)$$

- Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i\, t^{1/2}.}$$

- Next find c_i ...

Know ith node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

- Know ith node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

- So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \ge t_{i,\text{start}}.$$

- Know ith node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

- So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$

- All node degrees grow as $t^{1/2}$

- Know ith node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

- So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$

- All node degrees grow as $t^{1/2}$ but later nodes have larger $t_{i,\text{start}}$ which flattens out growth curve.

- Know ith node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

- So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \ge t_{i,\text{start}}.$$

- All node degrees grow as $t^{1/2}$ but later nodes have larger $t_{i,\text{start}}$ which flattens out growth curve.
- First-mover advantage: Early nodes do best.

- Know ith node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

- So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$

- All node degrees grow as $t^{1/2}$ but later nodes have larger $t_{i,\text{start}}$ which flattens out growth curve.
- First-mover advantage: Early nodes do best.
- Clearly, a Ponzi scheme.

We are already at the Zipf distribution:

- Degree of node i is the size of the ith ranked node:

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$

We are already at the Zipf distribution:

- Degree of node i is the size of the ith ranked node:

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$

- From before:

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

so $t_{i,\text{start}} \sim i$ which is the rank.

We are already at the Zipf distribution:

- Degree of node i is the size of the ith ranked node:

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$

- From before:

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

so $t_{i,\text{start}} \sim i$ which is the rank.

- We then have:

$$k_i \propto i^{-1/2} = i^{-\alpha}.$$

We are already at the Zipf distribution:

- Degree of node i is the size of the ith ranked node:

$$k_i(t) = m\left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$

- From before:

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

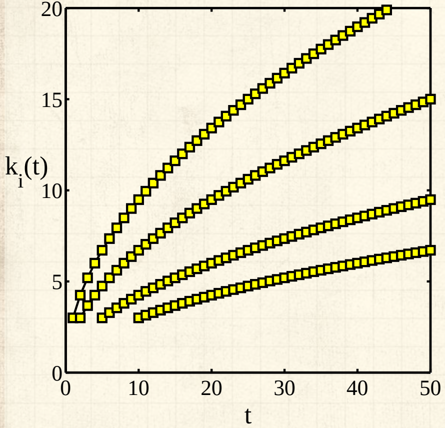
so $t_{i,\text{start}} \sim i$ which is the rank.

- We then have:

$$k_i \propto i^{-1/2} = i^{-\alpha}.$$

- Our connection $\alpha = 1/(\gamma - 1)$ or $\gamma = 1 + 1/\alpha$ then gives

$$\boxed{\gamma = 1 + 1/(1/2) = 3.}$$



- $m = 3$
- $t_{i,\text{start}} = 1, 2, 5,$ and $10.$

Degree distribution

- So what's the degree distribution at time t?

Degree distribution

- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly between time 0 and t:

$$\mathbf{Pr}(t_{i,\text{start}})\mathrm{d}t_{i,\text{start}} \simeq \frac{\mathrm{d}t_{i,\text{start}}}{t}$$

Degree distribution

- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly between time 0 and t:

$$\mathbf{Pr}(t_{i,\mathrm{start}})\mathrm{d}t_{i,\mathrm{start}} \simeq \frac{\mathrm{d}t_{i,\mathrm{start}}}{t}$$

- Also use

$$k_i(t) = m\left(\frac{t}{t_{i,\mathrm{start}}}\right)^{1/2} \Rightarrow t_{i,\mathrm{start}} = \frac{m^2 t}{k_i(t)^2}.$$

Degree distribution

- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly between time 0 and t:

$$\mathbf{Pr}(t_{i,\mathrm{start}})\mathrm{d}t_{i,\mathrm{start}} \simeq \frac{\mathrm{d}t_{i,\mathrm{start}}}{t}$$

- Also use

$$k_i(t) = m\left(\frac{t}{t_{i,\mathrm{start}}}\right)^{1/2} \Rightarrow t_{i,\mathrm{start}} = \frac{m^2 t}{k_i(t)^2}.$$

Transform variables—Jacobian:

$$\frac{\mathrm{d}t_{i,\mathrm{start}}}{\mathrm{d}k_i} = -2\frac{m^2 t}{k_i(t)^3}.$$

Degree distribution

$$\mathbf{Pr}(k_i)\mathrm{d}k_i = \mathbf{Pr}(t_{i,\mathrm{start}})\mathrm{d}t_{i,\mathrm{start}}$$

Degree distribution

$$\mathbf{Pr}(k_i)\mathrm{d}k_i = \mathbf{Pr}(t_{i,\text{start}})\mathrm{d}t_{i,\text{start}}$$

$$= \mathbf{Pr}(t_{i,\text{start}})\mathrm{d}k_i \left| \frac{\mathrm{d}t_{i,\text{start}}}{\mathrm{d}k_i} \right|$$

Degree distribution

$$\mathbf{Pr}(k_i)\mathrm{d}k_i = \mathbf{Pr}(t_{i,\text{start}})\mathrm{d}t_{i,\text{start}}$$

$$= \mathbf{Pr}(t_{i,\text{start}})\mathrm{d}k_i \left| \frac{\mathrm{d}t_{i,\text{start}}}{\mathrm{d}k_i} \right|$$

$$= \frac{1}{t}\mathrm{d}k_i \, 2\frac{m^2 t}{k_i(t)^3}$$

Degree distribution

$$\mathbf{Pr}(k_i)\mathrm{d}k_i = \mathbf{Pr}(t_{i,\text{start}})\mathrm{d}t_{i,\text{start}}$$

$$= \mathbf{Pr}(t_{i,\text{start}})\mathrm{d}k_i \left| \frac{\mathrm{d}t_{i,\text{start}}}{\mathrm{d}k_i} \right|$$

$$= \frac{1}{t}\mathrm{d}k_i \, 2\frac{m^2 t}{k_i(t)^3}$$

$$= 2\frac{m^2}{k_i(t)^3}\mathrm{d}k_i$$

Degree distribution

$$\mathbf{Pr}(k_i)\mathrm{d}k_i = \mathbf{Pr}(t_{i,\mathrm{start}})\mathrm{d}t_{i,\mathrm{start}}$$

$$= \mathbf{Pr}(t_{i,\mathrm{start}})\mathrm{d}k_i \left| \frac{\mathrm{d}t_{i,\mathrm{start}}}{\mathrm{d}k_i} \right|$$

$$= \frac{1}{t}\mathrm{d}k_i \, 2\frac{m^2 t}{k_i(t)^3}$$

$$= 2\frac{m^2}{k_i(t)^3}\mathrm{d}k_i$$

$$\propto k_i^{-3}\mathrm{d}k_i \,.$$

Degree distribution

- We thus have a very specific prediction of $\mathbf{Pr}(k) \sim k^{-\gamma}$ with $\gamma = 3$.

Degree distribution

- We thus have a very specific prediction of $\mathbf{Pr}(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- Typical for real networks: $2 < \gamma < 3$.

Degree distribution

- We thus have a very specific prediction of $\mathbf{Pr}(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.

Degree distribution

- We thus have a very specific prediction of **Pr**$(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- $2 < \gamma < 3$: finite mean and ‘infinite’ variance

Degree distribution

- We thus have a very specific prediction of $\mathbf{Pr}(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- $2 < \gamma < 3$: finite mean and 'infinite' variance
- In practice, $\gamma < 3$ means variance is governed by upper cutoff.

Degree distribution

- We thus have a very specific prediction of $\mathbf{Pr}(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- $2 < \gamma < 3$: finite mean and 'infinite' variance
- In practice, $\gamma < 3$ means variance is governed by upper cutoff.
- $\gamma > 3$: finite mean and variance

Degree distribution

- We thus have a very specific prediction of **Pr**$(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- $2 < \gamma < 3$: finite mean and 'infinite' variance (wild)
- In practice, $\gamma < 3$ means variance is governed by upper cutoff.
- $\gamma > 3$: finite mean and variance

Degree distribution

- We thus have a very specific prediction of **Pr**$(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- $2 < \gamma < 3$: finite mean and 'infinite' variance (wild)
- In practice, $\gamma < 3$ means variance is governed by upper cutoff.
- $\gamma > 3$: finite mean and variance (mild)

Back to that real data:

From Barabási and Albert's original paper [2]:

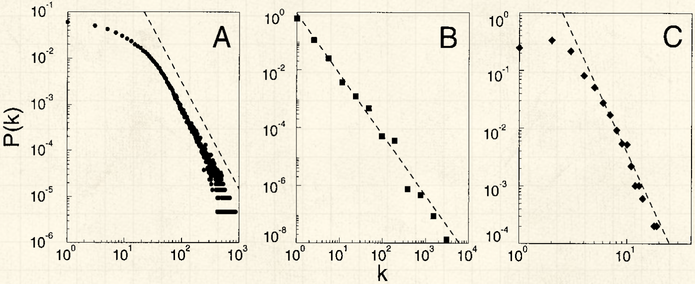


Fig. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\text{actor}} = 2.3$, (B) $\gamma_{\text{www}} = 2.1$ and (C) $\gamma_{\text{power}} = 4$.

Examples

Web	$\gamma \simeq 2.1$ for in-degree
Web	$\gamma \simeq 2.45$ for out-degree
Movie actors	$\gamma \simeq 2.3$
Words (synonyms)	$\gamma \simeq 2.8$

Examples

Web	$\gamma \simeq 2.1$ for in-degree
Web	$\gamma \simeq 2.45$ for out-degree
Movie actors	$\gamma \simeq 2.3$
Words (synonyms)	$\gamma \simeq 2.8$

The Internets is a different business...

Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
  1. Add edge deletion
  2. Add node deletion
  3. Add edge rewiring
- Deal with directed versus undirected networks.

Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
  1. Add edge deletion
  2. Add node deletion
  3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?

Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
  1. Add edge deletion
  2. Add node deletion
  3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?

Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
  1. Add edge deletion
  2. Add node deletion
  3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
- Q.: Do we need preferential attachment and growth?

Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
  1. Add edge deletion
  2. Add node deletion
  3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter?

Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
  1. Add edge deletion
  2. Add node deletion
  3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter? Maybe …

Outline

Scale-free networks
Main story
Model details
Analysis
A more plausible mechanism
Robustness
Krapivsky & Redner's model
Generalized model
Analysis
Universality?
Sublinear attachment kernels
Superlinear attachment kernels
Nutshell
References

Preferential attachment

- Let's look at preferential attachment (PA) a little more closely.

Preferential attachment

- Let’s look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network’s degree distribution.

Preferential attachment

- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- For example: If $P_{\mathrm{attach}}(k) \propto k$, we need to determine the constant of proportionality.

Preferential attachment

- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- We need to know what everyone's degree is...

Preferential attachment

- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- For example: If $P_{\textrm{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- We need to know what everyone's degree is...
- PA is $\therefore$ an outrageous assumption of node capability.

Preferential attachment

- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- For example: If $P_{\textrm{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- We need to know what everyone's degree is...
- PA is ∴ an outrageous assumption of node capability.
- But a very simple mechanism saves the day...

Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.

Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.

Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.

Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- Assuming the existing network is random, we know probability of a random friend having degree k is

$$Q_k \propto kP_k$$

Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- Assuming the existing network is random, we know probability of a random friend having degree k is

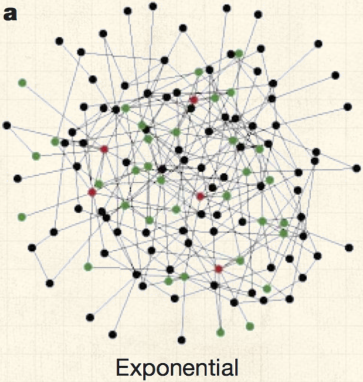
$$Q_k \propto kP_k$$

- So rich-gets-richer scheme can now be seen to work in a natural way.

Outline

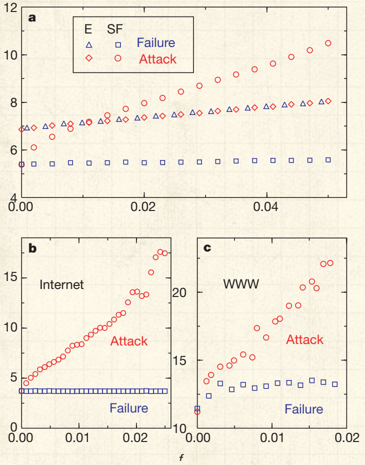
Robustness

- Albert et al., Nature, 2000:
  "Error and attack tolerance of complex networks" [1]
- Standard random networks (Erdős-Rényi) versus Scale-free networks:



from Albert et al., 2000

Robustness



from Albert et al., 2000

- Plots of network diameter as a function of fraction of nodes removed
- Erdős-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  1. Physically larger nodes that may be harder to 'target'

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  1. Physically larger nodes that may be harder to 'target'
  2. or subnetworks of smaller, normal-sized nodes.

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  1. Physically larger nodes that may be harder to 'target'
  2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

Robustness

Not a robust paper:

"The "Robust yet Fragile" nature of the Internet"
Doyle et al.,
Proc. Natl. Acad. Sci., **2005**, 14497–14502, 2005. [3]

- HOT networks versus scale-free networks
- Same degree distributions, different arrangements.
- Doyle *et al.* take a look at the actual Internet.

Outline

Outline

Generalized model

Fooling with the mechanism:

- 2001: Krapivsky & Redner (KR) [4] explored the general attachment kernel:

Generalized model

Fooling with the mechanism:

- 2001: Krapivsky & Redner (KR) [4] explored the general attachment kernel:

$$\mathbf{Pr}(\text{attach to node } i) \propto A_k = k_i^{\nu}$$

where A_k is the attachment kernel and $\nu > 0$.

Generalized model

Fooling with the mechanism:

- 2001: Krapivsky & Redner (KR) [4] explored the general attachment kernel:

$$\textbf{Pr}(\text{attach to node } i) \propto A_k = k_i^{\nu}$$

  where A_k is the attachment kernel and $\nu > 0$.
- KR also looked at changing the details of the attachment kernel.

Generalized model

Fooling with the mechanism:

- 2001: Krapivsky & Redner (KR) [4] explored the general attachment kernel:

$$\textbf{Pr}(\text{attach to node } i) \propto A_k = k_i^{\nu}$$

  where A_k is the attachment kernel and $\nu > 0$.
- KR also looked at changing the details of the attachment kernel.
- KR model will be fully studied in CoNKS.

Generalized model

- We'll follow KR's approach using rate equations.

Generalized model

- We'll follow KR's approach using rate equations.
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

Generalized model

- We'll follow KR's approach using rate equations.
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

1. One node with one link is added per unit time.

Generalized model

- We'll follow KR's approach using rate equations.
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

1. One node with one link is added per unit time.
2. The first term corresponds to degree $k-1$ nodes becoming degree k nodes.

Generalized model

- We'll follow KR's approach using rate equations.
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

1. One node with one link is added per unit time.
2. The first term corresponds to degree $k-1$ nodes becoming degree k nodes.
3. The second term corresponds to degree k nodes becoming degree $k-1$ nodes.

Generalized model

- We'll follow KR's approach using rate equations.
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

1. One node with one link is added per unit time.
2. The first term corresponds to degree $k-1$ nodes becoming degree k nodes.
3. The second term corresponds to degree k nodes becoming degree $k-1$ nodes.
4. A is the correct normalization (coming up).

Generalized model

- We'll follow KR's approach using rate equations.
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

1. One node with one link is added per unit time.
2. The first term corresponds to degree $k-1$ nodes becoming degree k nodes.
3. The second term corresponds to degree k nodes becoming degree $k-1$ nodes.
4. A is the correct normalization (coming up).
5. Seed with some initial network

Generalized model

- We'll follow KR's approach using rate equations.
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

1. One node with one link is added per unit time.
2. The first term corresponds to degree $k-1$ nodes becoming degree k nodes.
3. The second term corresponds to degree k nodes becoming degree $k-1$ nodes.
4. A is the correct normalization (coming up).
5. Seed with some initial network (e.g., a connected pair)

Generalized model

- We'll follow KR's approach using rate equations.
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

1. One node with one link is added per unit time.
2. The first term corresponds to degree $k-1$ nodes becoming degree k nodes.
3. The second term corresponds to degree k nodes becoming degree $k-1$ nodes.
4. A is the correct normalization (coming up).
5. Seed with some initial network (e.g., a connected pair)
6. Detail: $A_0 = 0$

Outline

Generalized model

- In general, probability of attaching to a specific node of degree k at time t is

Generalized model

- In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

Generalized model

- In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

Generalized model

- In general, probability of attaching to a specific node of degree k at time t is

$$\textbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

- E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.

Generalized model

- In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

- E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.
- For $A_k = k$, we have

Generalized model

- In general, probability of attaching to a specific node of degree k at time t is

$$\textbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

- E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.
- For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t)$$

Generalized model

- In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

- E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.
- For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

Generalized model

- In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

- E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.
- For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

Generalized model

- In general, probability of attaching to a specific node of degree k at time t is

$$\textbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

- E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.
- For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

- Detail: we are ignoring initial seed network's edges.

Generalized model

- So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

Generalized model

- So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution:

Generalized model

- So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution: $N_k = n_k t$.

Generalized model

- So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution: $N_k = n_k t$.
- We replace $\mathrm{d}N_k/\mathrm{d}t$ with $\mathrm{d}n_k t/\mathrm{d}t = n_k$.

Generalized model

- So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution: $N_k = n_k t$.
- We replace $\mathrm{d}N_k/\mathrm{d}t$ with $\mathrm{d}n_k t/\mathrm{d}t = n_k$.
- We arrive at a difference equation:

$$n_k = \frac{1}{2\cancel{t}}\left[(k-1)n_{k-1}\cancel{t} - kn_k\cancel{t}\right] + \delta_{k1}$$

Outline

Universality?

- As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t \text{ for large } k.$$

Universality?

- As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t \text{ for large } k.$$

- Now: what happens if we start playing around with the attachment kernel A_k?

Universality?

- As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t \text{ for large } k.$$

- Now: what happens if we start playing around with the attachment kernel A_k?
- Again, we're asking if the result $\gamma = 3$ universal?

Universality?

- As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t \text{ for large } k.$$

- Now: what happens if we start playing around with the attachment kernel A_k?
- Again, we're asking if the result $\gamma = 3$ universal?
- KR's natural modification: $A_k = k^{\nu}$ with $\nu \neq 1$.

Universality?

- As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t \text{ for large } k.$$

- Now: what happens if we start playing around with the attachment kernel A_k?
- Again, we're asking if the result $\gamma = 3$ universal?
- KR's natural modification: $A_k = k^{\nu}$ with $\nu \neq 1$.
- But we'll first explore a more subtle modification of A_k made by Krapivsky/Redner [4]

Universality?

- As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t \text{ for large } k.$$

- Now: what happens if we start playing around with the attachment kernel A_k?
- Again, we're asking if the result $\gamma = 3$ universal?
- KR's natural modification: $A_k = k^\nu$ with $\nu \neq 1$.
- But we'll first explore a more subtle modification of A_k made by Krapivsky/Redner [4]
- Keep A_k linear in k but tweak details.

Universality?

- As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t \text{ for large } k.$$

- Now: what happens if we start playing around with the attachment kernel A_k?
- Again, we're asking if the result $\gamma = 3$ universal?
- KR's natural modification: $A_k = k^\nu$ with $\nu \neq 1$.
- But we'll first explore a more subtle modification of A_k made by Krapivsky/Redner [4]
- Keep A_k linear in k but tweak details.
- Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

Universality?

- Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

Universality?

- Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

- We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

Universality?

- Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

- We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k.

Universality?

- Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

- We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k.

- We assume that $A = \mu t$

Universality?

- Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

- We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k.

- We assume that $A = \mu t$
- We'll find μ later and make sure that our assumption is consistent.

Universality?

- Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

- We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

  where we only know the asymptotic behavior of A_k.

- We assume that $A = \mu t$
- We'll find μ later and make sure that our assumption is consistent.
- As before, also assume $N_k(t) = n_k t$.

Universality?

- For $A_k = k$ we had

$$n_k = \frac{1}{2}\left[(k-1)n_{k-1} - kn_k\right] + \delta_{k1}$$

Universality?

- For $A_k = k$ we had

$$n_k = \frac{1}{2}\left[(k-1)n_{k-1} - kn_k\right] + \delta_{k1}$$

- This now becomes

$$n_k = \frac{1}{\mu}\left[A_{k-1}n_{k-1} - A_k n_k\right] + \delta_{k1}$$

Universality?

- For $A_k = k$ we had

$$n_k = \frac{1}{2}\left[(k-1)n_{k-1} - kn_k\right] + \delta_{k1}$$

- This now becomes

$$n_k = \frac{1}{\mu}\left[A_{k-1}n_{k-1} - A_k n_k\right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

Universality?

- For $A_k = k$ we had

$$n_k = \frac{1}{2}\left[(k-1)n_{k-1} - kn_k\right] + \delta_{k1}$$

- This now becomes

$$n_k = \frac{1}{\mu}\left[A_{k-1}n_{k-1} - A_k n_k\right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

- Again two cases:

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1};$$

Universality?

- For $A_k = k$ we had

$$n_k = \frac{1}{2}\left[(k-1)n_{k-1} - kn_k\right] + \delta_{k1}$$

- This now becomes

$$n_k = \frac{1}{\mu}\left[A_{k-1}n_{k-1} - A_k n_k\right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

- Again two cases:

$$k = 1: n_1 = \frac{\mu}{\mu + A_1}; \qquad k > 1: n_k = n_{k-1}\frac{A_{k-1}}{\mu + A_k}.$$

Universality?

Time for pure excitement: Find asymptotic behavior of n_k given $A_k \rightarrow k$ as $k \rightarrow \infty$.

Universality?

- Time for pure excitement: Find asymptotic behavior of n_k given $A_k \to k$ as $k \to \infty$.
- For large k, we find:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} \propto k^{-\mu-1}$$

Universality?

- Time for pure excitement: Find asymptotic behavior of n_k given $A_k \to k$ as $k \to \infty$.
- For large k, we find:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} \propto k^{-\mu-1}$$

- Since μ depends on A_k, details matter...

Universality?

- Now we need to find μ.

Universality?

- Now we need to find μ.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$

Universality?

- Now we need to find μ.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$

Universality?

- Now we need to find μ.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now subsitute in our expression for n_k:

Universality?

- Now we need to find μ.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now subsitute in our expression for n_k:

$$\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

Universality?

- Now we need to find μ.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now subsitute in our expression for n_k:

$$\mu = \sum_{k=1}^{\infty} \frac{\mu}{\cancel{A_k}} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} \cancel{A_k}$$

Universality?

- Now we need to find μ.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now subsitute in our expression for n_k:

$$1\cancel{\mu} = \sum_{k=1}^{\infty} \frac{\cancel{\mu}}{\cancel{A_k}} \prod_{j=1}^{k} \frac{1}{1+\frac{\mu}{A_j}} \cancel{A_k}$$

Universality?

- Now we need to find μ.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now subsitute in our expression for n_k:

$$1\cancel{\mu} = \sum_{k=1}^{\infty} \frac{\cancel{\mu}}{\cancel{A_k}} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} \cancel{A_k}$$

- Closed form expression for μ.

Universality?

- Now we need to find μ.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now subsitute in our expression for n_k:

$$1\cancel{\mu} = \sum_{k=1}^{\infty} \frac{\cancel{\mu}}{\cancel{A_k}} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} \cancel{A_k}$$

- Closed form expression for μ.
- We can solve for μ in some cases.

Universality?

- Now we need to find μ.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now subsitute in our expression for n_k:

$$1\cancel{\mu} = \sum_{k=1}^{\infty} \frac{\cancel{\mu}}{\cancel{A_k}} \prod_{j=1}^{k} \frac{1}{1+\frac{\mu}{A_j}} \cancel{A_k}$$

- Closed form expression for μ.
- We can solve for μ in some cases.
- Our assumption that $A = \mu t$ looks to be not too horrible.

Universality?

- Consider tunable $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.

Universality?

- Consider tunable $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
- Again, we can find $\gamma = \mu + 1$ by finding μ.

Universality?

- Consider tunable $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
- Again, we can find $\gamma = \mu + 1$ by finding μ.
- Closed form expression for μ:

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

#mathisfun

Universality?

- Consider tunable $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
- Again, we can find $\gamma = \mu + 1$ by finding μ.
- Closed form expression for μ:

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

#mathisfun

-

$$\mu(\mu-1) = 2\alpha \Rightarrow \mu = \frac{1+\sqrt{1+8\alpha}}{2}.$$

Universality?

- Consider tunable $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
- Again, we can find $\gamma = \mu + 1$ by finding μ.
- Closed form expression for μ:

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

#mathisfun

-
$$\mu(\mu-1) = 2\alpha \Rightarrow \mu = \frac{1+\sqrt{1+8\alpha}}{2}.$$

- Since $\gamma = \mu + 1$, we have

$$0 \leq \alpha < \infty \Rightarrow 2 \leq \gamma < \infty$$

Universality?

- Consider tunable $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
- Again, we can find $\gamma = \mu + 1$ by finding μ.
- Closed form expression for μ:

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

#mathisfun

-

$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}.$$

- Since $\gamma = \mu + 1$, we have

$$0 \leq \alpha < \infty \Rightarrow 2 \leq \gamma < \infty$$

- Craziness...

Outline

Sublinear attachment kernels

- Rich-get-somewhat-richer:

$$A_k \sim k^{\nu} \text{ with } 0 < \nu < 1.$$

Sublinear attachment kernels

- Rich-get-somewhat-richer:

$$A_k \sim k^{\nu} \text{ with } 0 < \nu < 1.$$

- General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}.$$

Sublinear attachment kernels

- Rich-get-somewhat-richer:

$$A_k \sim k^{\nu} \text{ with } 0 < \nu < 1.$$

- General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}.$$

- Stretched exponentials (truncated power laws).

Sublinear attachment kernels

- Rich-get-somewhat-richer:

$$A_k \sim k^{\nu} \text{ with } 0 < \nu < 1.$$

- General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}.$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.

Sublinear attachment kernels

- Rich-get-somewhat-richer:

$$A_k \sim k^{\nu} \text{ with } 0 < \nu < 1.$$

- General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}.$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality: now details of kernel do not matter.

Sublinear attachment kernels

- Rich-get-somewhat-richer:

$$A_k \sim k^{\nu} \text{ with } 0 < \nu < 1.$$

- General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}.$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality: now details of kernel do not matter.
- Distribution of degree is universal providing $\nu < 1$.

Sublinear attachment kernels

Details:

- For $1/2 < \nu < 1$:

$$n_k \sim k^{-\nu} e^{-\mu\left(\frac{k^{1-\nu}-2^{1-\nu}}{1-\nu}\right)}$$

Sublinear attachment kernels

Details:

- For $1/2 < \nu < 1$:

$$n_k \sim k^{-\nu} e^{-\mu\left(\frac{k^{1-\nu} - 2^{1-\nu}}{1-\nu}\right)}$$

- For $1/3 < \nu < 1/2$:

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

Sublinear attachment kernels

Details:

- For $1/2 < \nu < 1$:

$$n_k \sim k^{-\nu} e^{-\mu \left(\frac{k^{1-\nu} - 2^{1-\nu}}{1-\nu} \right)}$$

- For $1/3 < \nu < 1/2$:

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

- And for $1/(r+1) < \nu < 1/r$, we have r pieces in exponential.

Outline

Scale-free networks
- Main story
- Model details
- Analysis
- A more plausible mechanism
- Robustness
- Krapivsky & Redner's model
- Generalized model
- Analysis
- Universality?
- Sublinear attachment kernels
- Superlinear attachment kernels
- Nutshell

References

Superlinear attachment kernels

- Rich-get-much-richer:

$$A_k \sim k^{\nu} \text{ with } \nu > 1.$$

Superlinear attachment kernels

- Rich-get-much-richer:

$$A_k \sim k^{\nu} \text{ with } \nu > 1.$$

- Now a winner-take-all mechanism.

Superlinear attachment kernels

- Rich-get-much-richer:

$$A_k \sim k^{\nu} \text{ with } \nu > 1.$$

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes.

Superlinear attachment kernels

- Rich-get-much-richer:

$$A_k \sim k^{\nu} \text{ with } \nu > 1.$$

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes.
- For $\nu > 2$, all but a finite # of nodes connect to one node.

Outline

Scale-free networks
Main story
Model details
Analysis
A more plausible mechanism
Robustness
Krapivsky & Redner's model
Generalized model
Analysis
Universality?
Sublinear attachment kernels
Superlinear attachment kernels
Nutshell
References

Nutshell:

Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.

Nutshell:

Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.

Nutshell:

Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
  1. Description: Characterizing very large networks
  2. Explanation: Micro story $\Rightarrow$ Macro features

Nutshell:

Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
    1. Description: Characterizing very large networks
    2. Explanation: Micro story $\Rightarrow$ Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...

Nutshell:

Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
  1. Description: Characterizing very large networks
  2. Explanation: Micro story $\Rightarrow$ Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- Still much work to be done, especially with respect to dynamics...

Nutshell:

Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
  1. Description: Characterizing very large networks
  2. Explanation: Micro story $\Rightarrow$ Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- Still much work to be done, especially with respect to dynamics... #excitement

Neural reboot (NR):

Turning the corner:

https://www.youtube.com/watch?v=axrTxEVQqN4?rel=0

References I

[1] R. Albert, H. Jeong, and A.-L. Barabási.
Error and attack tolerance of complex networks.
Nature, 406:378–382, 2000. pdf

[2] A.-L. Barabási and R. Albert.
Emergence of scaling in random networks.
Science, 286:509–511, 1999. pdf

[3] J. Doyle, D. Alderson, L. Li, S. Low, M. Roughan, S. S., R. Tanaka, and W. Willinger.
The "Robust yet Fragile" nature of the Internet.
Proc. Natl. Acad. Sci., 2005:14497–14502, 2005. pdf

[4] P. L. Krapivsky and S. Redner.
Organization of growing random networks.
Phys. Rev. E, 63:066123, 2001. pdf