Random Networks Nutshell

Last updated: 2022/08/28, 08:34:20 EDT

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394, 2022–2023 | @pocsvox

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Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont

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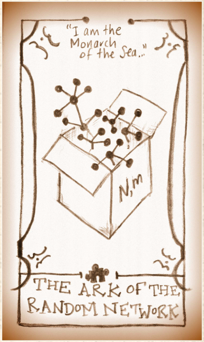
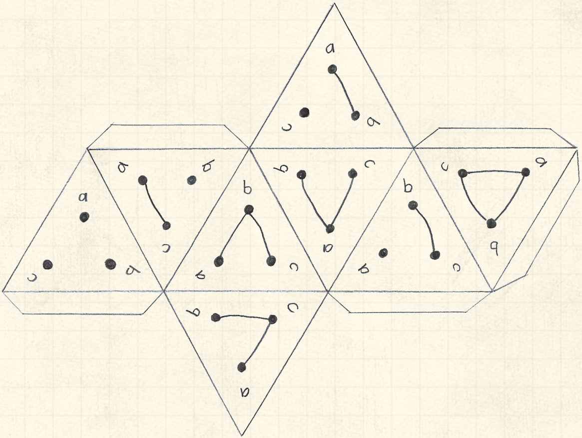
Outline


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


"I am the Monarch of the Sea.."
N,m
THE ARK OF THE RANDOM NETWORK

Random network generator for $N = 3$:



- Get your own exciting generator here.
- As $N \nearrow$, polyhedral die rapidly becomes a ball...

Outline

Pure random networks
Definitions
How to build theoretically
Some visual examples
Clustering
Degree distributions

Generalized Random Networks
Configuration model
How to build in practice
Motifs
Strange friends
Largest component

References

Random networks

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- Standard random network = one randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or ER graphs.

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- Real world: links are usually costly so real networks are almost always sparse.

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   - Best for adding relatively small numbers of links (most cases).
   - 1 and 2 are effectively equivalent for large N.

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- Which is what it should be...
- If we keep $\langle k \rangle$ constant then $p \propto 1/N \to 0$ as $N \to \infty$.

Outline

Random networks: examples

Next slides:

Example realizations of random networks

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Random networks: examples

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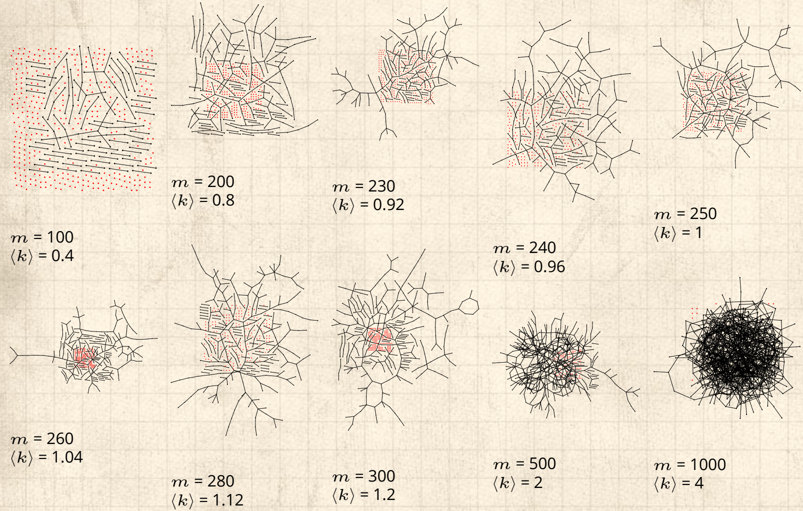
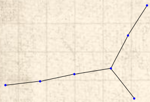
Example realizations of random networks

- $N = 500$
- Vary m, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.

Random networks: examples for N=500

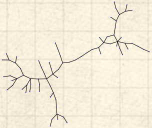


Random networks: largest components

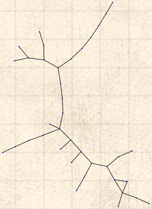


m = 100
$\langle k \rangle$ = 0.4

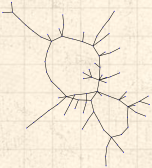
m = 200
$\langle k \rangle$ = 0.8



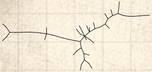
m = 230
$\langle k \rangle$ = 0.92



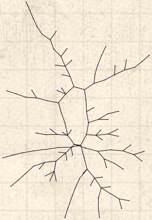
m = 240
$\langle k \rangle$ = 0.96



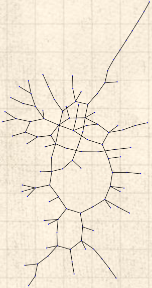
m = 250
$\langle k \rangle$ = 1



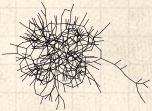
m = 260
$\langle k \rangle$ = 1.04



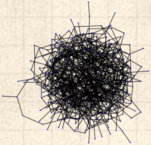
m = 280
$\langle k \rangle$ = 1.12



m = 300
$\langle k \rangle$ = 1.2



m = 500
$\langle k \rangle$ = 2



m = 1000
$\langle k \rangle$ = 4

Random networks: examples for N=500

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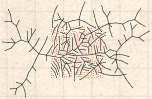

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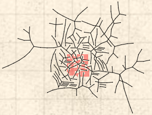

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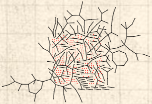

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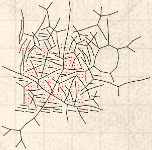

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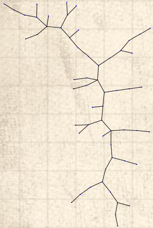
Random networks: largest components



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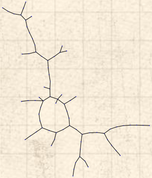

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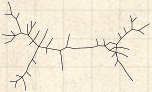

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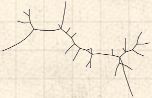

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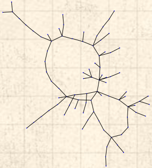

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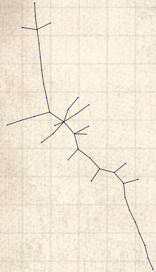

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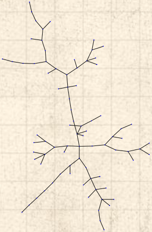

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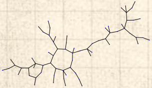

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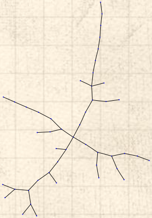

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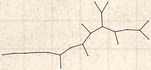

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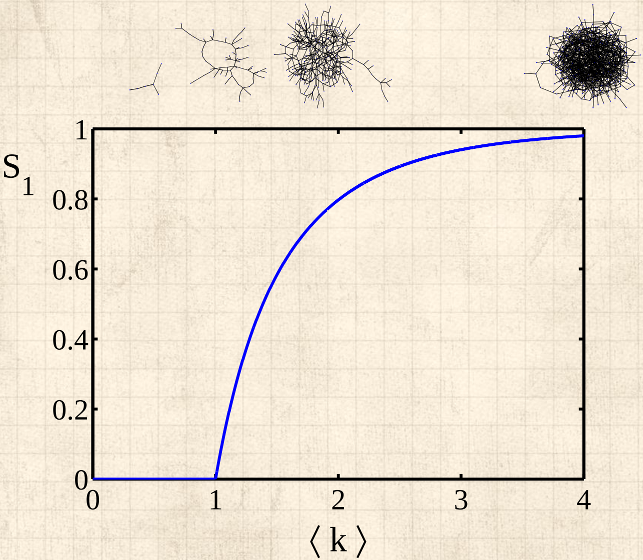
Giant component



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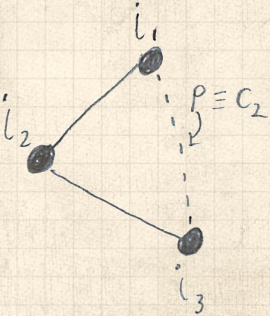

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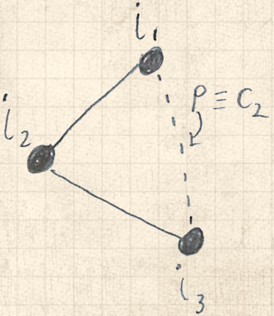

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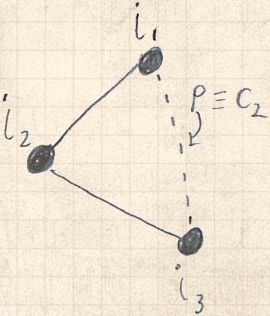

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- Or: C_2 = probability that a triple is part of a triangle.
- For standard random networks, we have simply that

$$C_2 = p.$$

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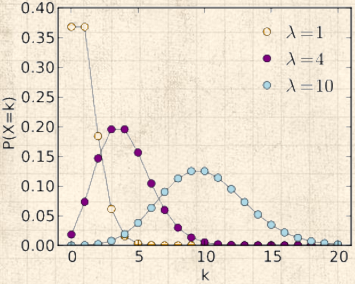
- This is a Poisson distribution with mean $\langle k \rangle$.

Poisson basics:

$$P(k;\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}$$



- $\lambda > 0$
- $k = 0, 1, 2, 3, \ldots$
- Classic use: probability that an event occurs k times in a given time period, given an average rate of occurrence.
- e.g.: phone calls/minute, horse-kick deaths.
- 'Law of small numbers'

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- Note: This is a special property of Poisson distribution and can trip us up...

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Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

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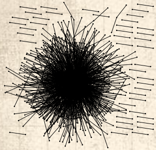
- Again, look at full network plus the largest component.
- Apart from degree distribution, wiring is random.

Random networks: examples for N=1000



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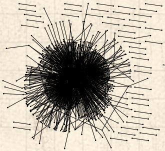

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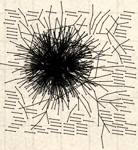

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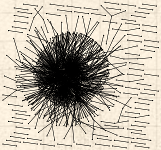

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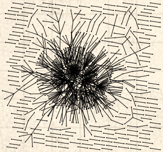

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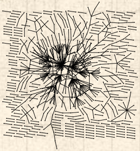

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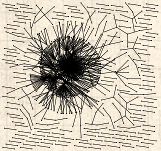

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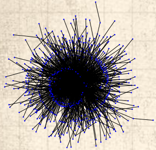
Random networks: largest components



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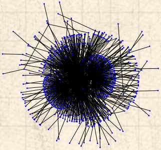

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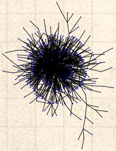

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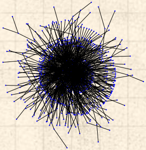

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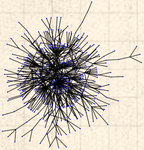

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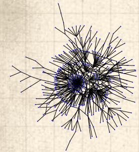

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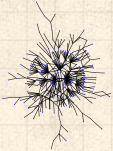

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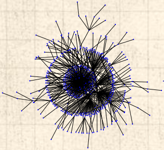

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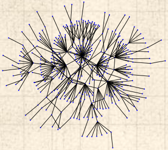

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- Create ensemble to test deviations from randomness.

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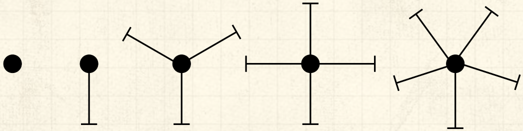
- Idea: start with a soup of unconnected nodes with stubs (half-edges):



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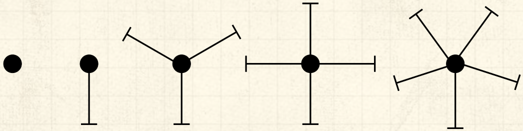

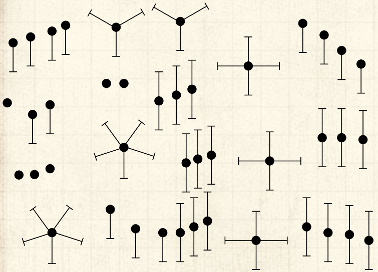

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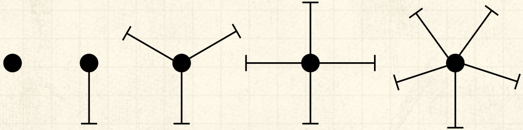

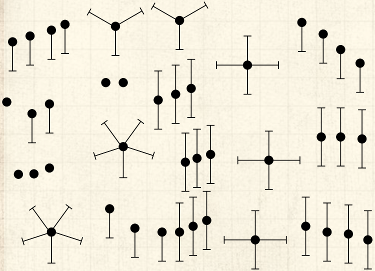

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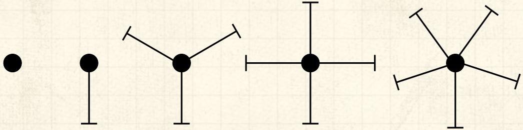

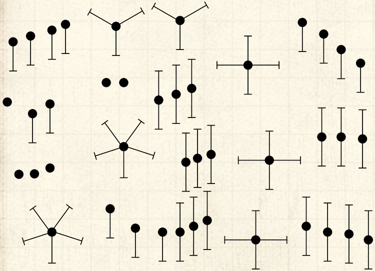

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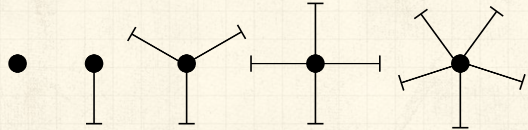

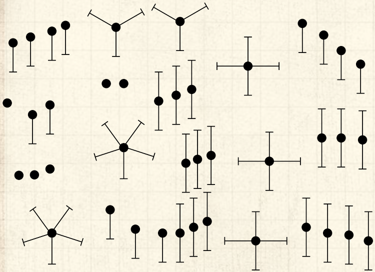

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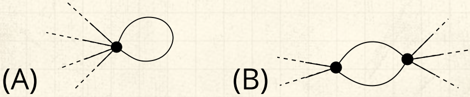
Building random networks: First rewiring

Phase 2:

- Now find any (A) self-loops and (B) repeat edges and randomly rewire them.



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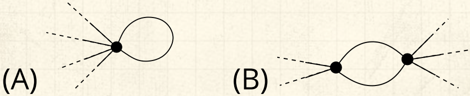

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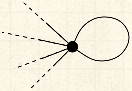
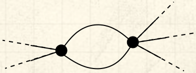
(A)  (B) 

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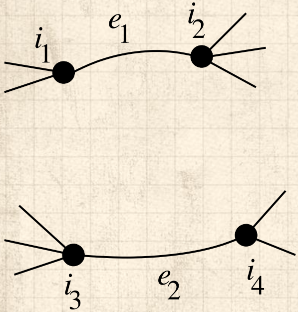
- Simplest solution: randomly rewire two edges at a time.

General random rewiring algorithm



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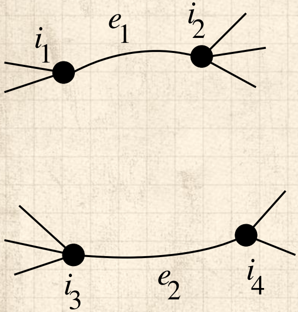

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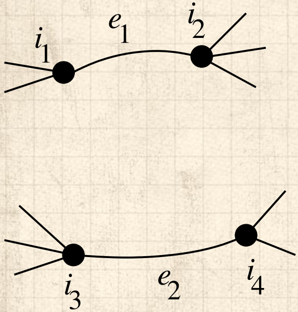

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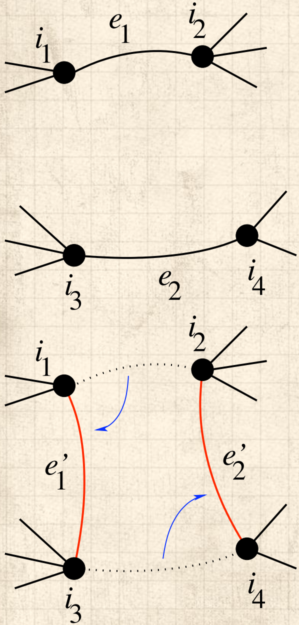

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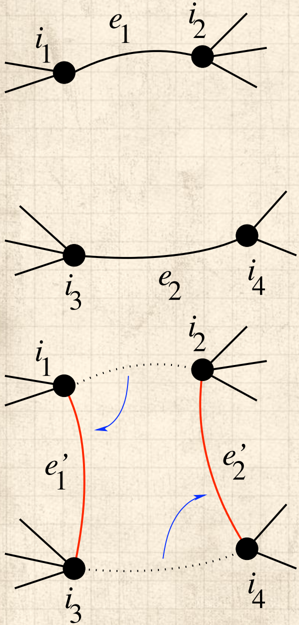

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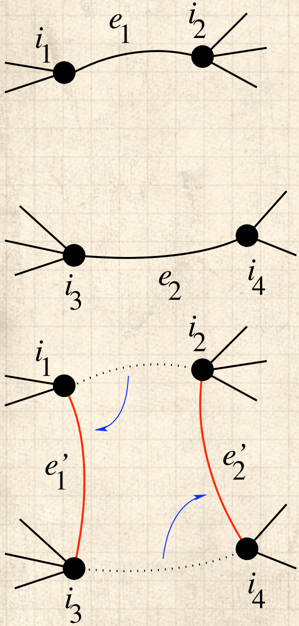

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- Same as finding on/off/on/off 4-cycles. and rotating them.

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- Rule of thumb: # Rewirings $\simeq 10 \times$ # edges [4].

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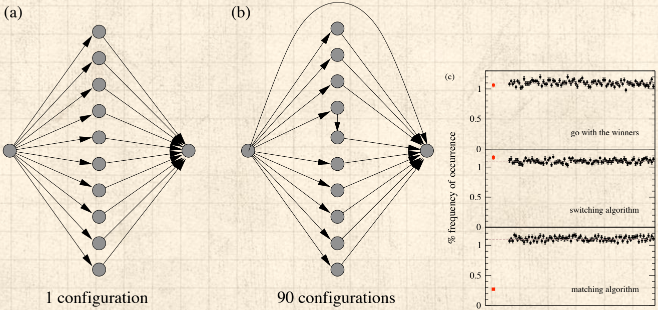

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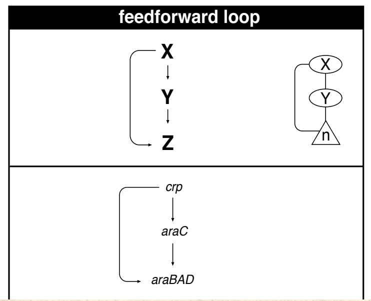

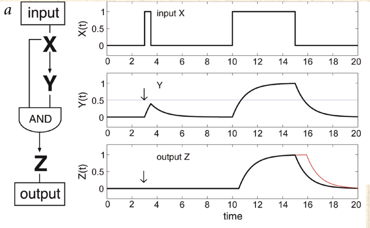

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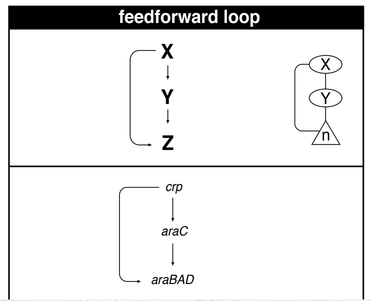

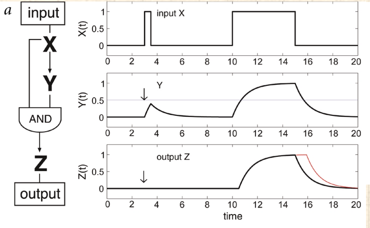

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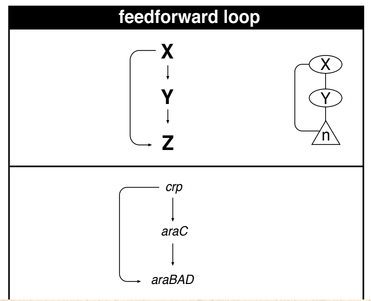

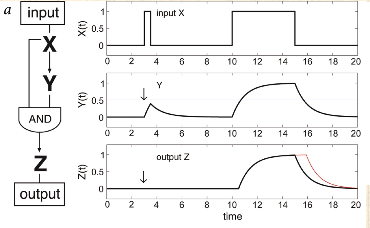

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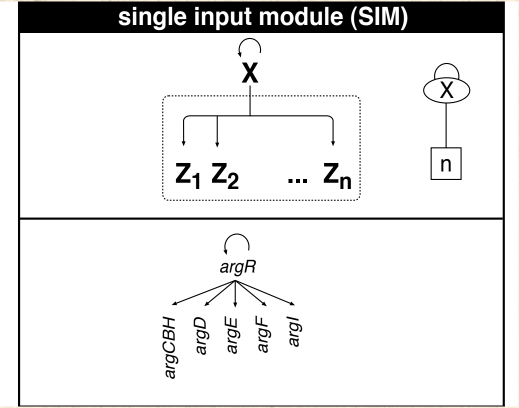
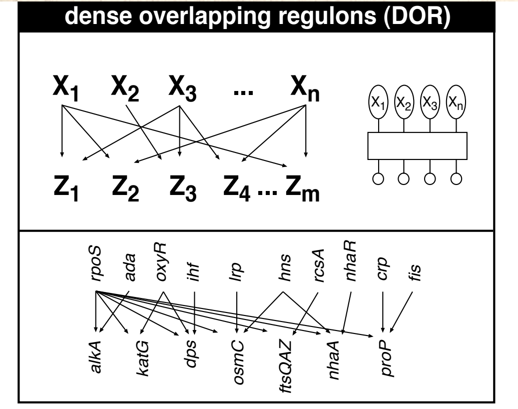
- Analogy to elevator doors.

Network motifs



- Master switch.

Network motifs



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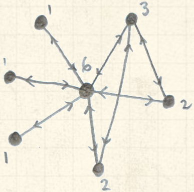
- Big deal: Rich-get-richer mechanism is built into this selection process.



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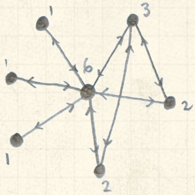

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- Natural question: what's the expected number of other friends that one friend has?

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  4. See also: class size paradoxes (nod to: Gelman)

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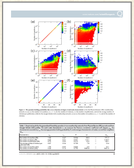
- So only if everyone has the same degree (variance$= \sigma^2 = 0$) can a node be the same as its friends.
- Intuition: for random networks, the more connected a node, the more likely it is to be chosen as a friend.



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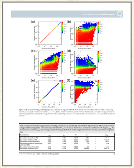

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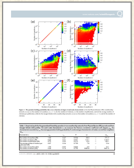

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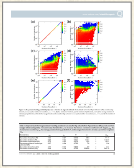

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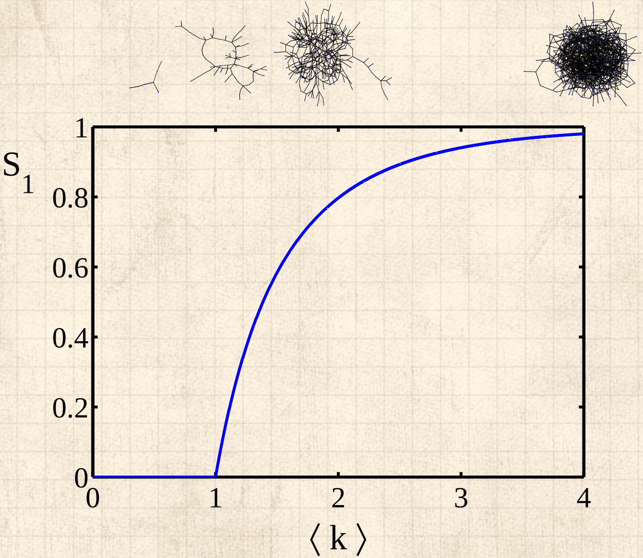
- Note: Component = Cluster

Outline

Giant component



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- Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$

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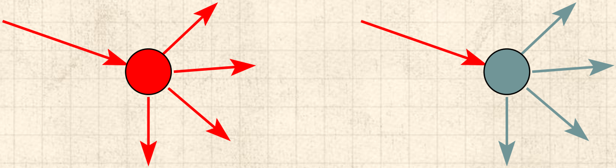

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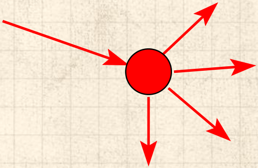

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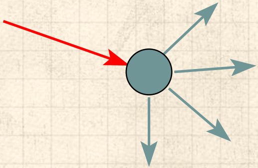

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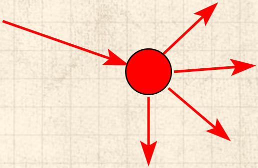

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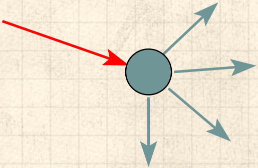

- Focus on binary case with edges and nodes either infected or not.
- First big question: for a given network and contagion process, can global spreading from a single seed occur?

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- Good: This is just our giant component condition again.

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- We say $\langle k \rangle = 1$ marks the critical point of the system.

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- How about $P_k = \delta_{k k_0}$?

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- Now substitute in $\delta = 1 - S_1$ and rearrange to obtain:

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- Really a transcritical bifurcation. [8]

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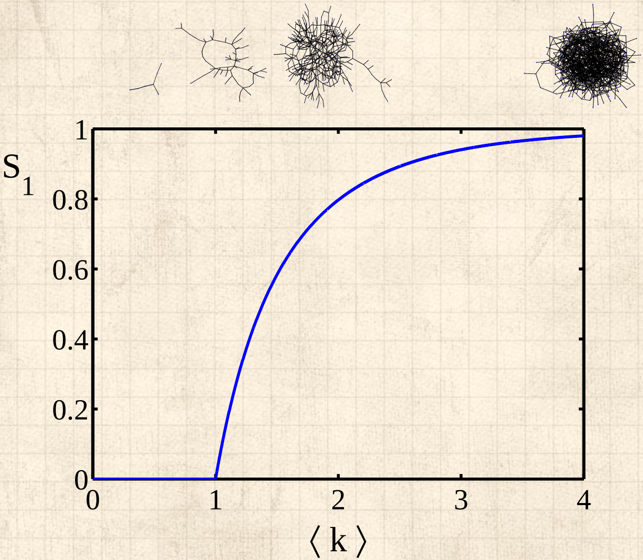

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- We can sort many things out with sensible probabilistic arguments...
- More detailed investigations will profit from a spot of Generatingfunctionology. [9]

References I

[1] P. S. Dodds, K. D. Harris, and J. L. Payne.
Direct, phyiscally motivated derivation of the contagion condition for spreading processes on generalized random networks.
Phys. Rev. E, 83:056122, 2011. pdf

[2] Y.-H. Eom and H.-H. Jo.
Generalized friendship paradox in complex networks: The case of scientific collaboration.
Nature Scientific Reports, 4:4603, 2014. pdf

[3] S. L. Feld.
Why your friends have more friends than you do.
Am. J. of Sociol., 96:1464–1477, 1991. pdf

References II

[4] R. Milo, N. Kashtan, S. Itzkovitz, M. E. J. Newman, and U. Alon.
On the uniform generation of random graphs with prescribed degree sequences, 2003. pdf

[5] M. E. J. Newman.
Ego-centered networks and the ripple effect,.
Social Networks, 25:83–95, 2003. pdf

[6] M. E. J. Newman.
The structure and function of complex networks.
SIAM Rev., 45(2):167–256, 2003. pdf

[7] S. S. Shen-Orr, R. Milo, S. Mangan, and U. Alon.
Network motifs in the transcriptional regulation network of *Escherichia coli*.
Nature Genetics, 31:64–68, 2002. pdf

References III

[8] S. H. Strogatz.
Nonlinear Dynamics and Chaos.
Addison Wesley, Reading, Massachusetts, 1994.

[9] H. S. Wilf.
Generatingfunctionology.
A K Peters, Natick, MA, 3rd edition, 2006. pdf