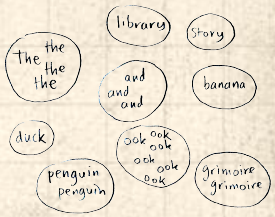
Mechanisms for Generating Power-Law Size Distributions, Part 3

Last updated: 2022/08/28, 08:34:20 EDT

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394, 2022–2023| @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont



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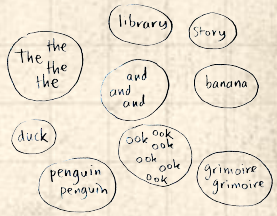

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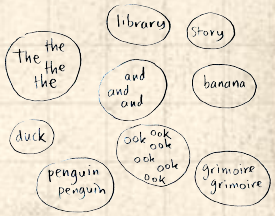
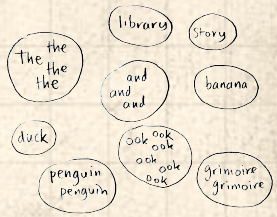


Outline

Rich-Get-Richer Mechanism

- Simon's Model
- Analysis
- Words
- Catchphrases
- First Mover Advantage

References



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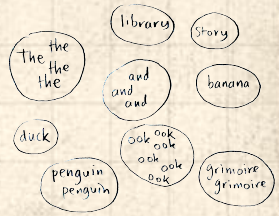
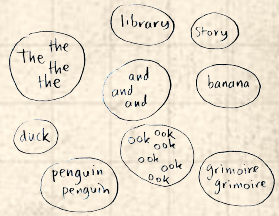


The Boggoracle Speaks:

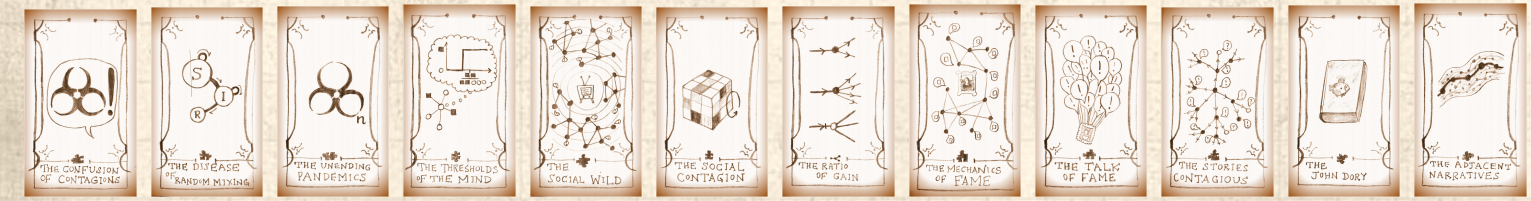



THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS


THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE


THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE BOOKS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MINDS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES

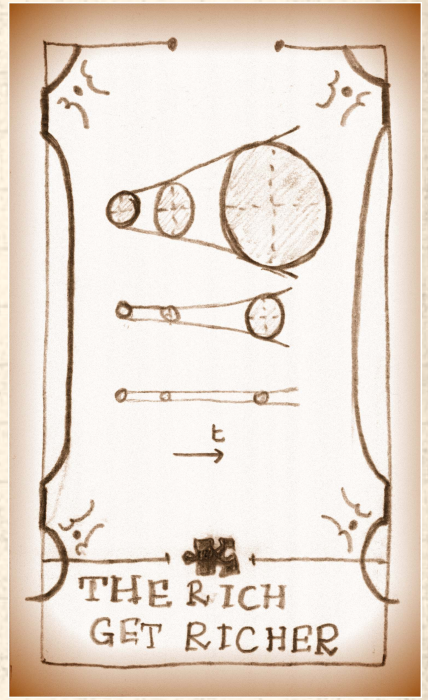

THE MECHANICS STATISTICAL
TWO THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED PHOENIX
THE CONTAGION


THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES


THE STORYTELLERS
THE FATED STEEL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUBLES OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPE OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES

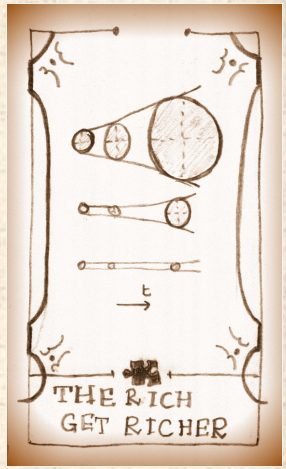

THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTERLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS


THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


t
THE RICH GET RICHER

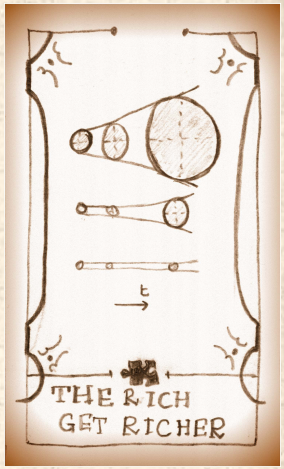
Aggregation:

- Random walks represent additive aggregation



Aggregation:

- Random walks represent additive aggregation
- Mechanism: Random addition and subtraction



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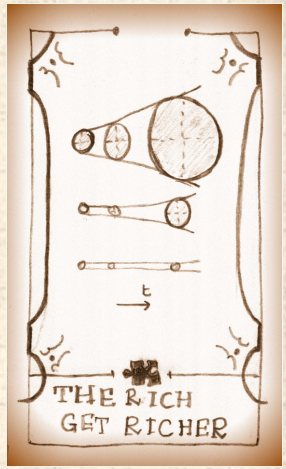
- Compare across realizations, no competition.



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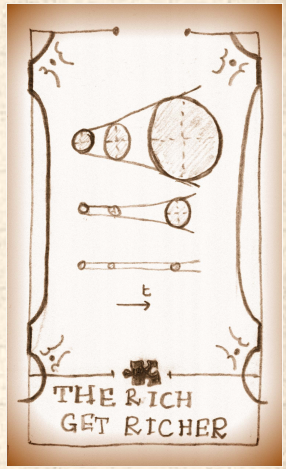
- Next: Random Additive/Copying Processes involving Competition.



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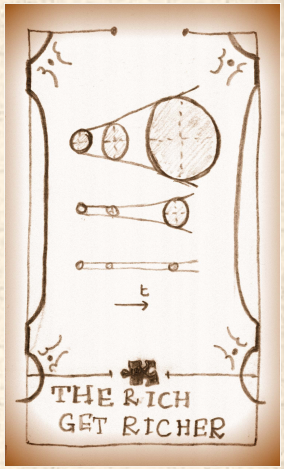
- Widespread: Words, Cities, the Web, Wealth, Productivity (Lotka), Popularity (Books, People, ...)



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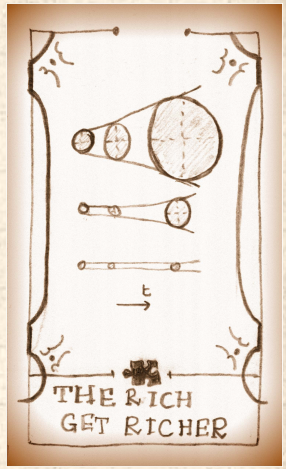
- Competing mechanisms (trickiness)



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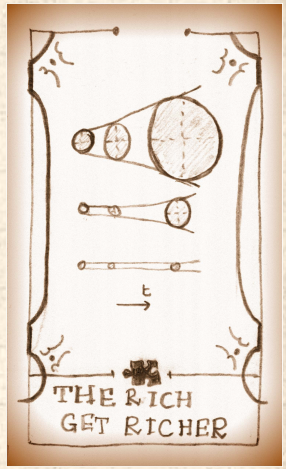
- 1910s: Word frequency examined re Stenography (or shorthand or brachygraphy or tachygraphy), Jean-Baptiste Estoup [6].



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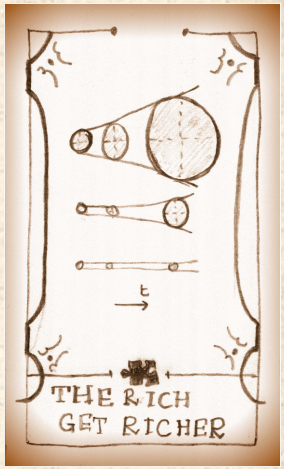

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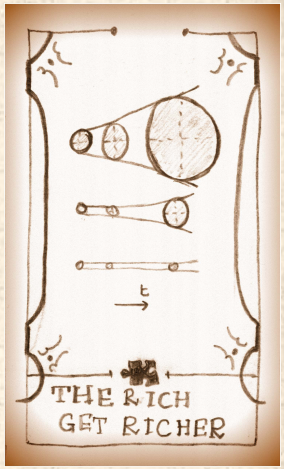

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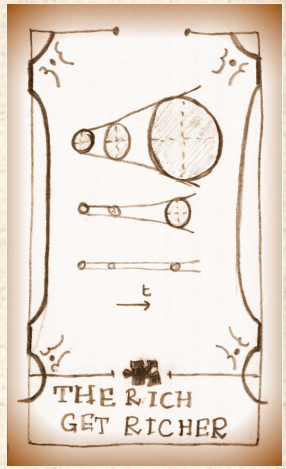
- 1926: Lotka [9]:
  # Scientific papers per author (Lotka's law)



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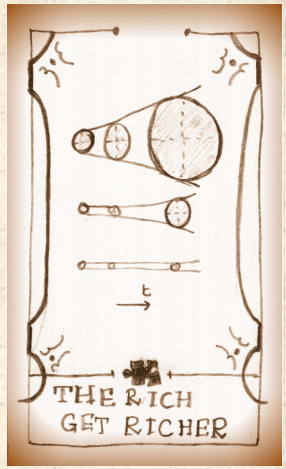

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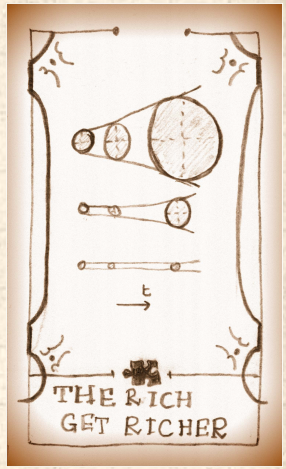

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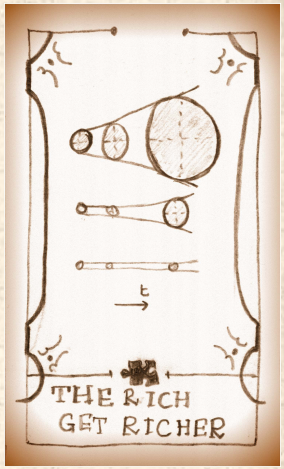

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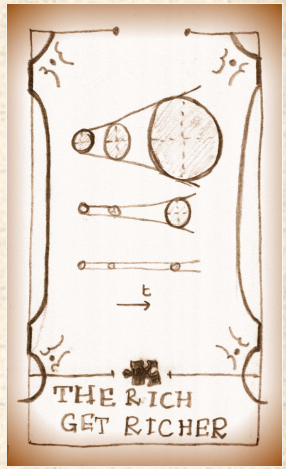

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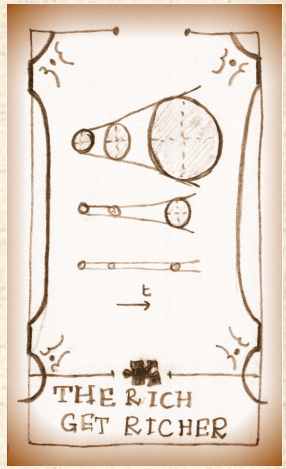
- 1999: Barabasi and Albert [2]:
  The World Wide Web, networks-at-large.



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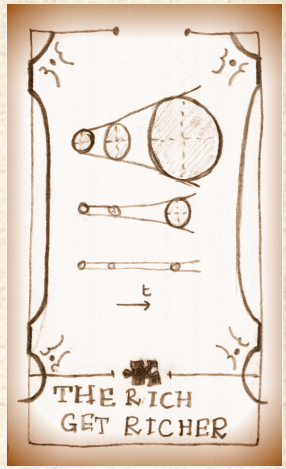

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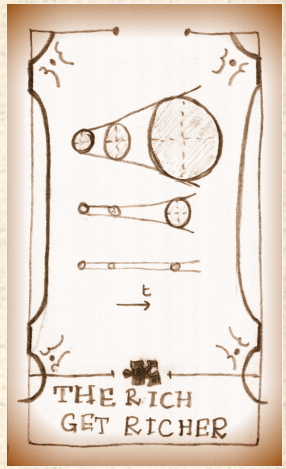

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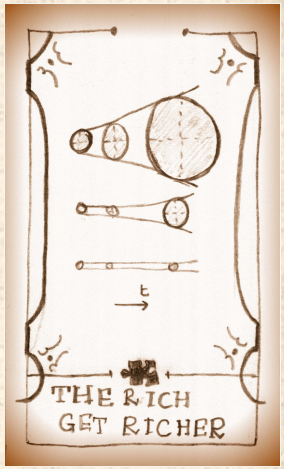

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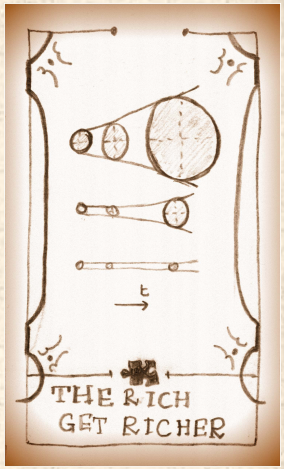

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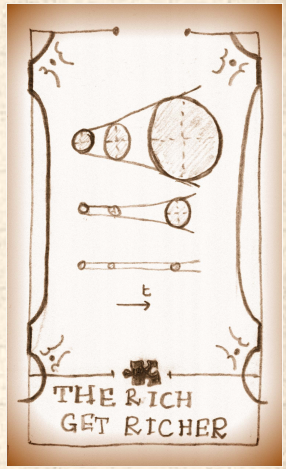

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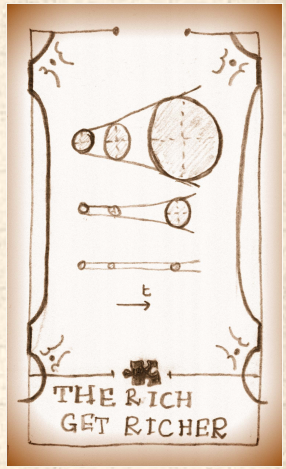
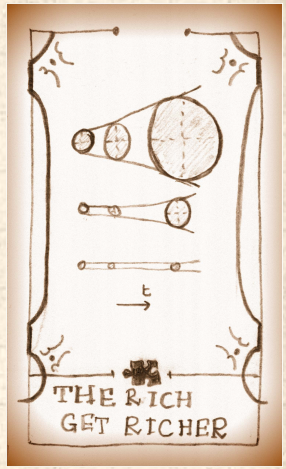
- 1978 Nobel Laureate in Economics (his Nobel bio is here).



Essential Extract of a Growth Model:

Random Competitive Replication (RCR):

1. Start with 1 elephant (or element) of a particular flavor at $t = 1$



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1. Start with 1 elephant (or element) of a particular flavor at $t = 1$
2. At time $t = 2, 3, 4, ...$, add a new elephant in one of two ways:
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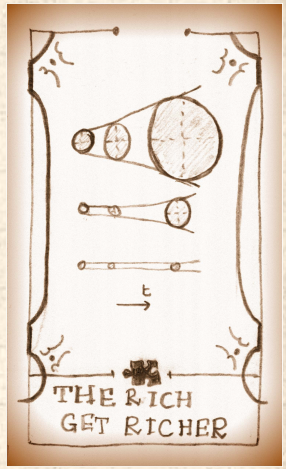

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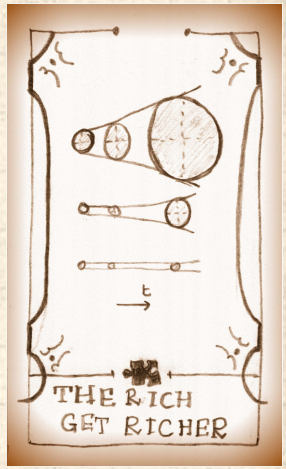

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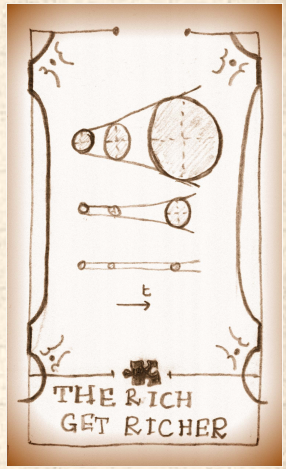
   - Elephants of the same flavor form a group



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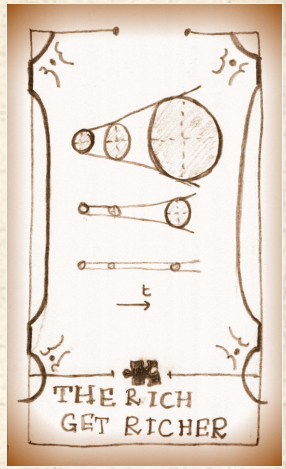

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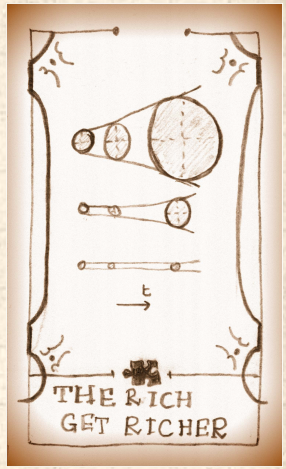
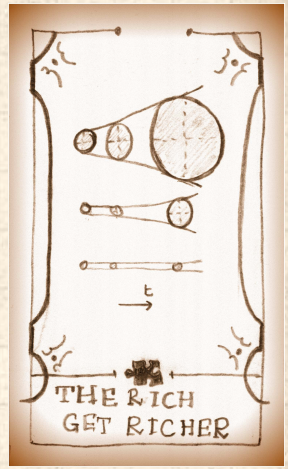


Random Competitive Replication:

Example: Words appearing in a language



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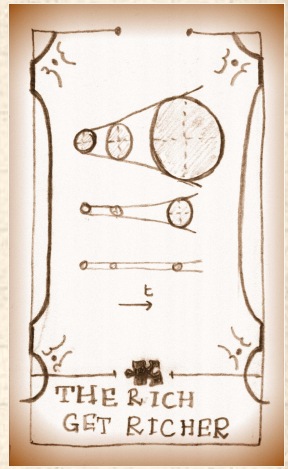

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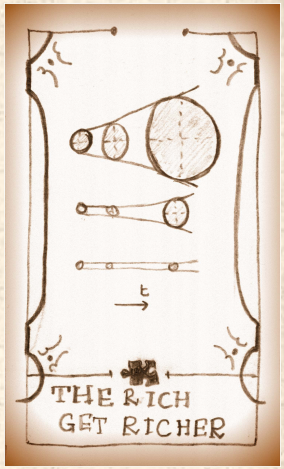

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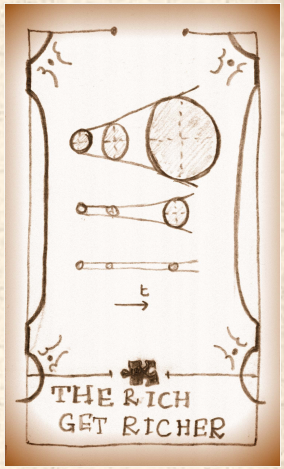

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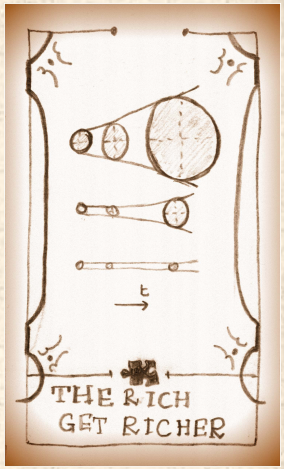

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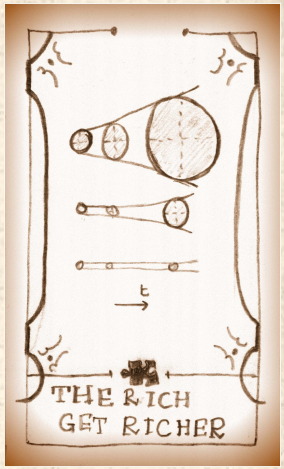

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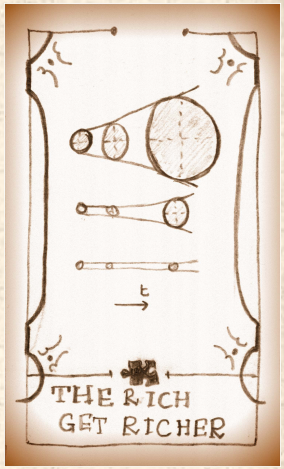
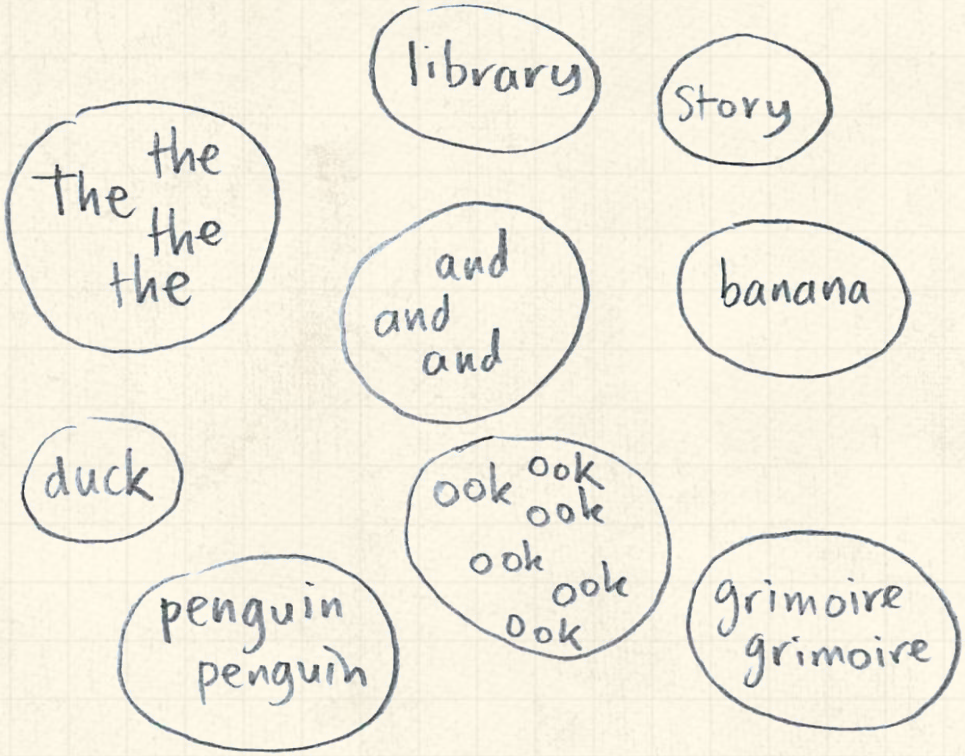


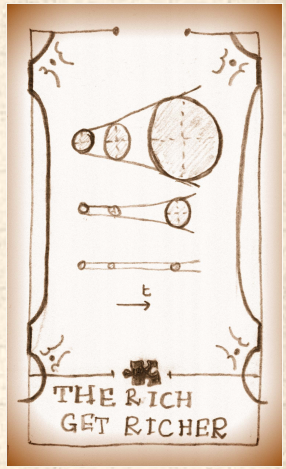
Note: This is a terrible way to write a novel.

For example:



- 21 words used
- next word is new with prob ρ
- next word is a copy with prob $1-\rho$

prob:	next word:
$6/21$	ook
$4/21$	the
$3/21$	and
$2/21$	penguin
$\vdots$	
$1/21$	library



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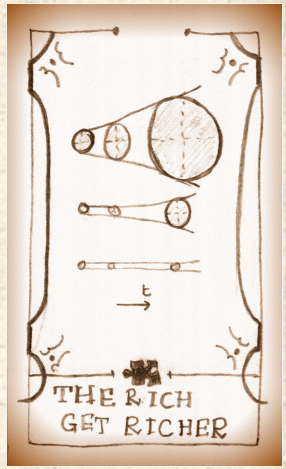
- Fundamental Rich-get-Richer story;



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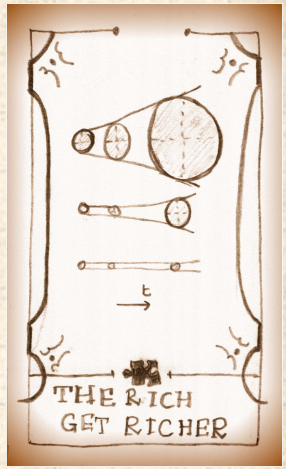

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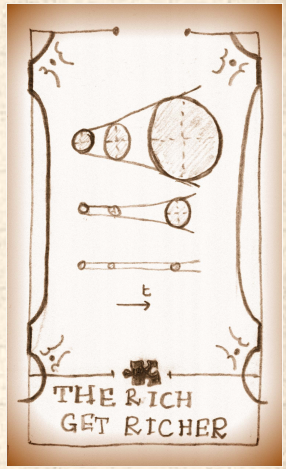

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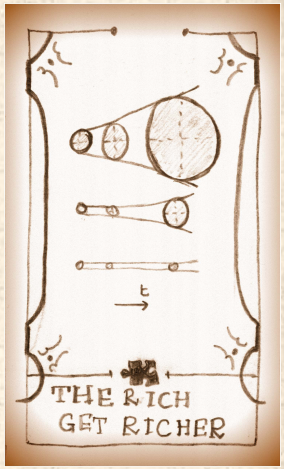

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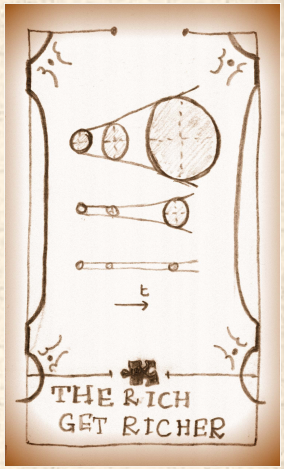

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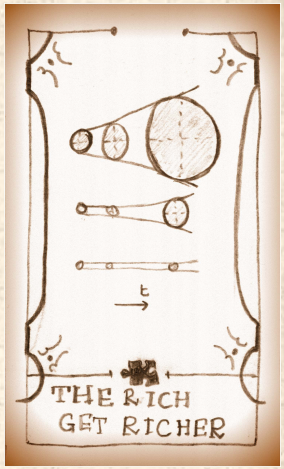

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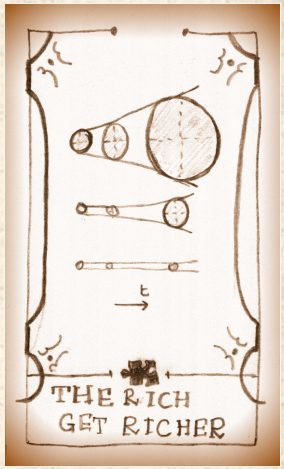
Your free set of tofu knives:



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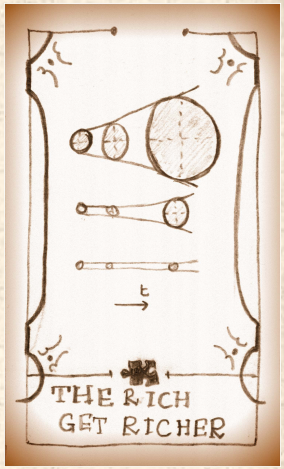

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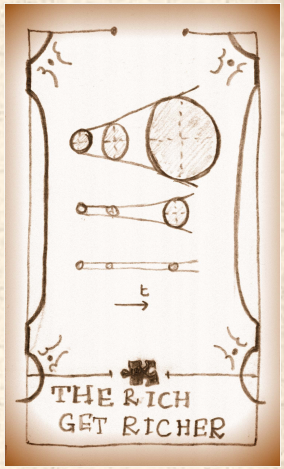

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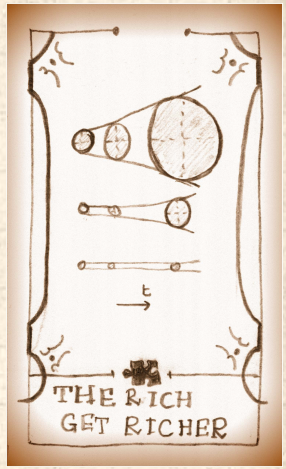

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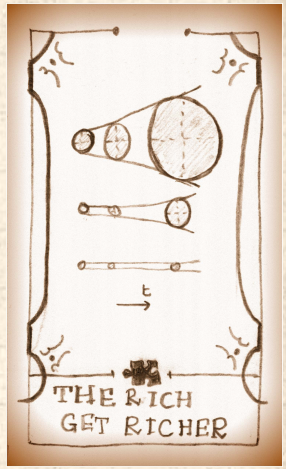

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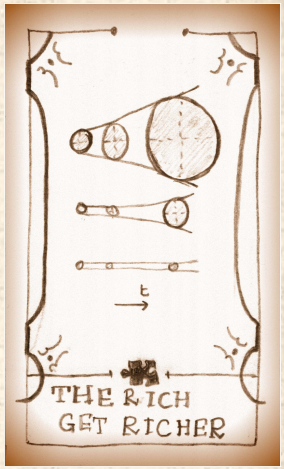

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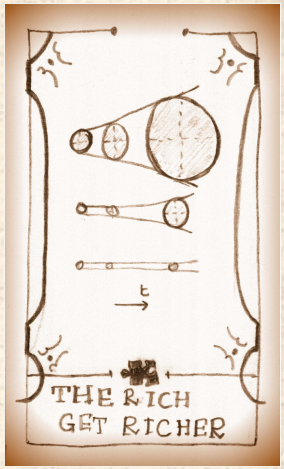

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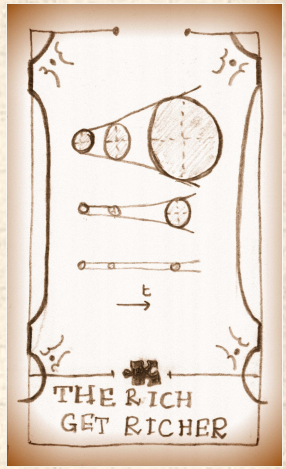

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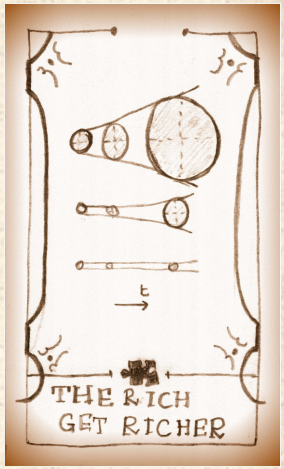

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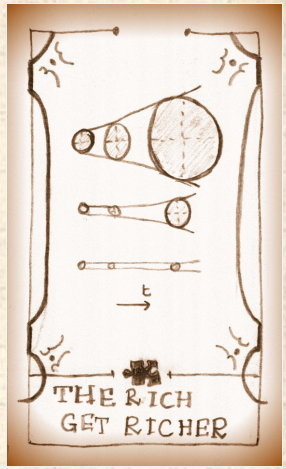

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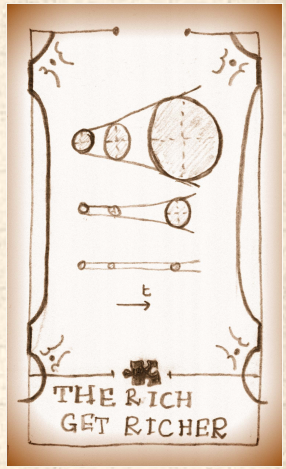
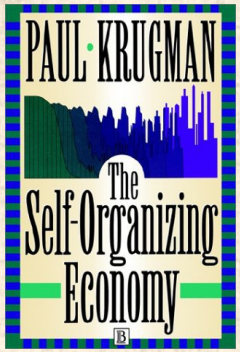
     (But mechanism for selection is not as simple...)





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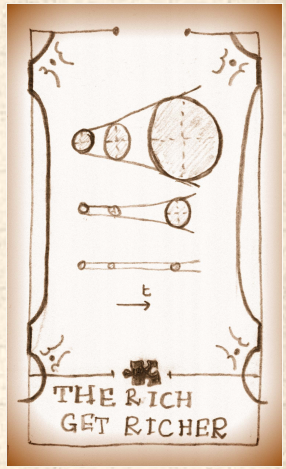

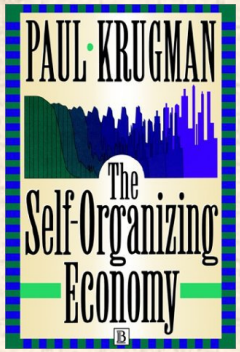

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Ch. 3: An Urban Mystery, p. 46

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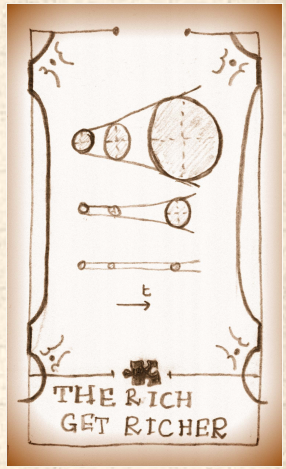

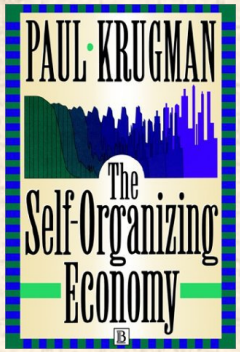

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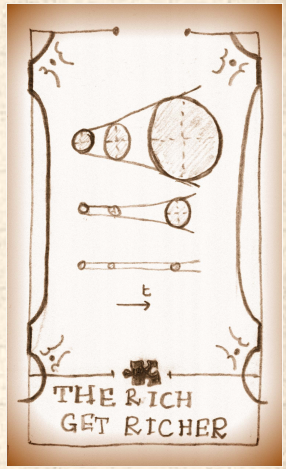

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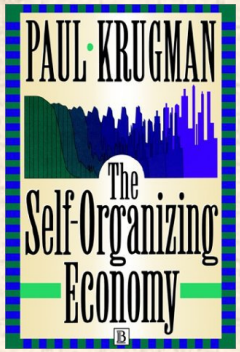

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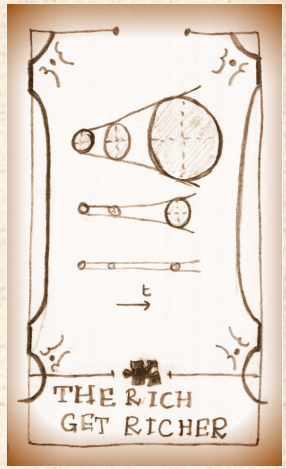

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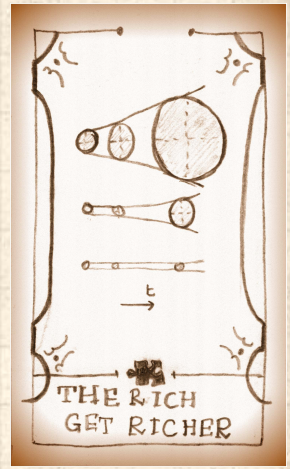
[2] Let's use π for probability because π's not special, right guys?

Outline

Rich-Get-Richer Mechanism

- Simon's Model
- Analysis
- Words
- Catchphrases
- First Mover Advantage

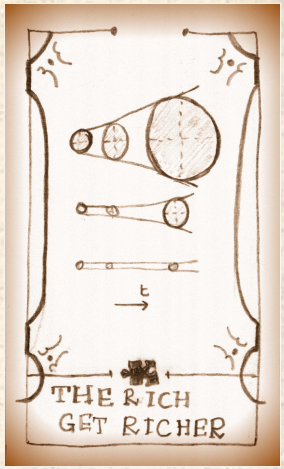
References



Random Competitive Replication:

Definitions:

- k_i = size of a group i



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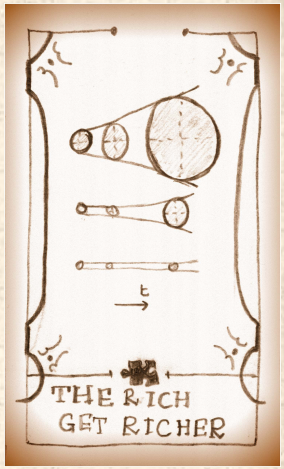

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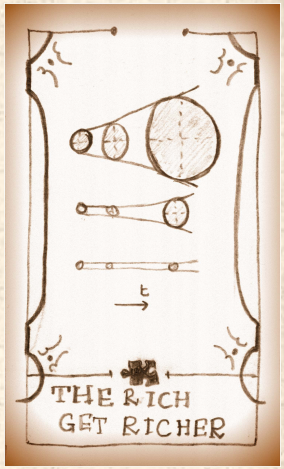

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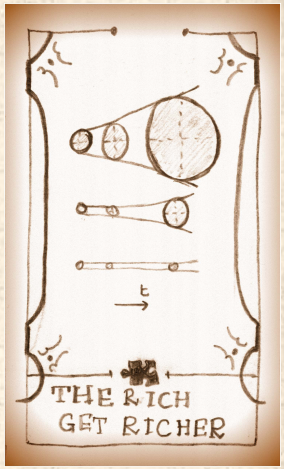
First: $\sum_k kN_{k,t} = t =$ number of elephants at time t



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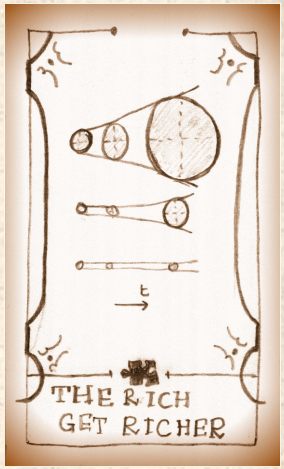

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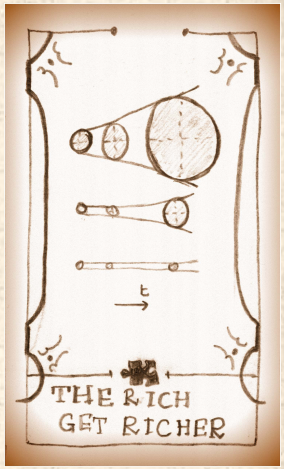

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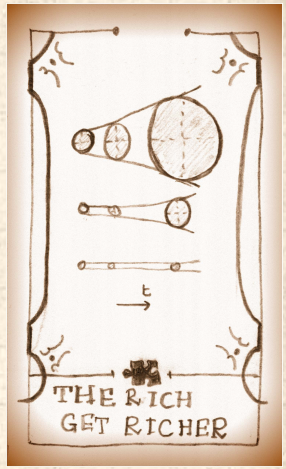

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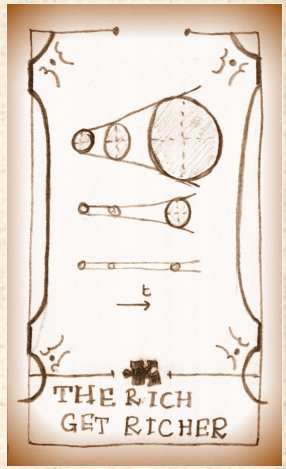
- $\Rightarrow kN_{k,t}$ elephants in size k groups
- t elephants overall

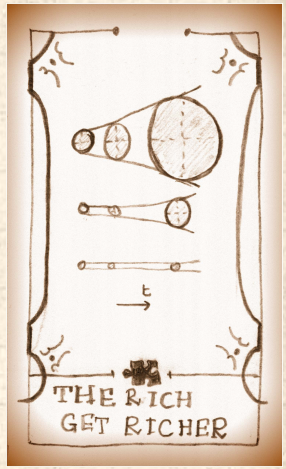


Random Competitive Replication:

$P_k(t)$ = Probability of choosing an elephant that belongs to a group of size k:

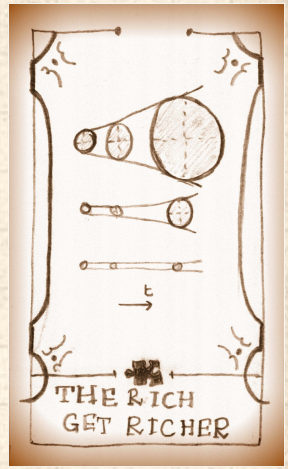
- $N_{k,t}$ size k groups
- $\Rightarrow kN_{k,t}$ elephants in size k groups
- t elephants overall

$$P_k(t) = \frac{kN_{k,t}}{t}.$$



Random Competitive Replication:

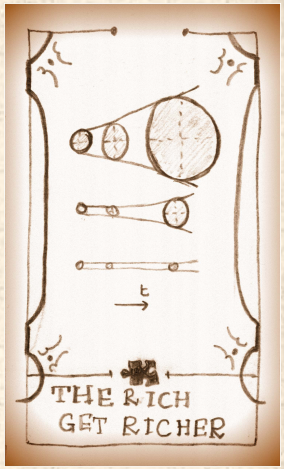
$N_{k,t}$, the number of groups with k elephants, changes at time t if



Random Competitive Replication:

$N_{k,t}$, the number of groups with k elephants, changes at time t if

1. An elephant belonging to a group with k elephants is replicated:



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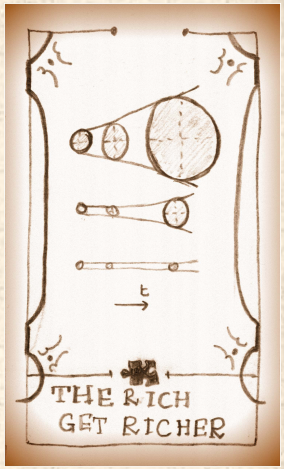

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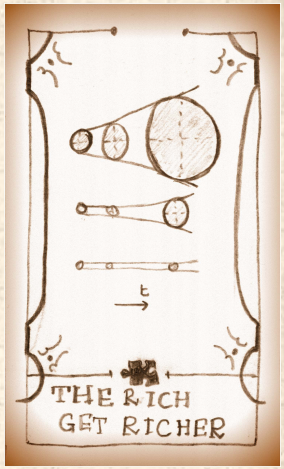

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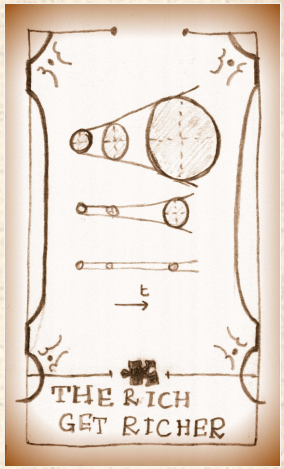

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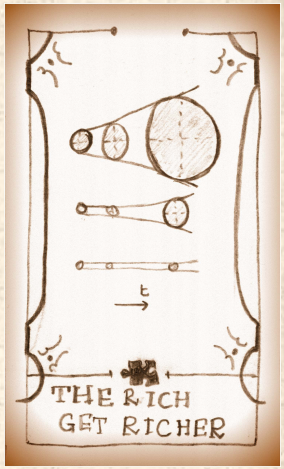

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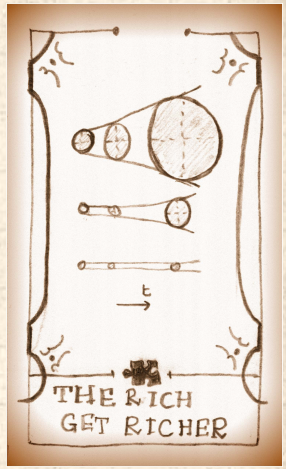
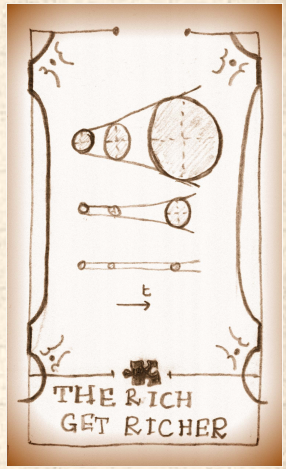


Random Competitive Replication:

Special case for $N_{1,t}$:



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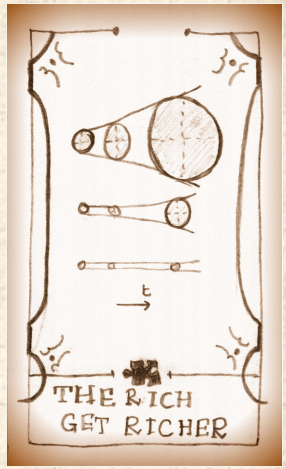
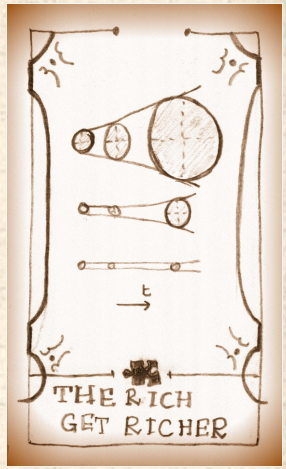


Random Competitive Replication:

Special case for $N_{1,t}$:

1. The new elephant is a new flavor:

2. A unique elephant is replicated:



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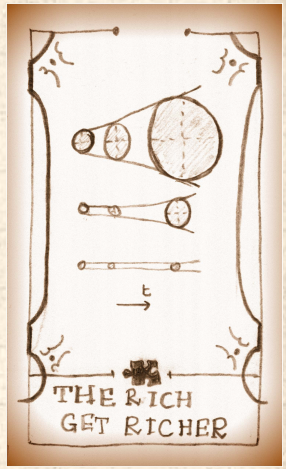

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Special case for $N_{1,t}$:

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   Happens with probability ρ

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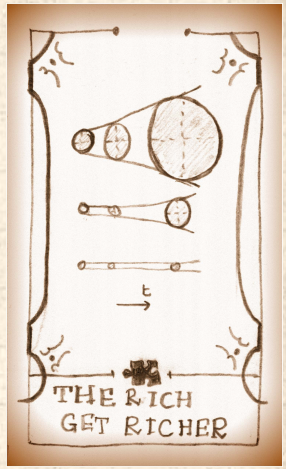

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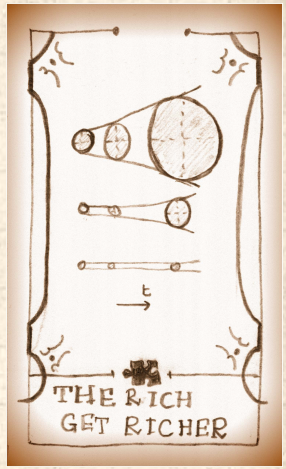

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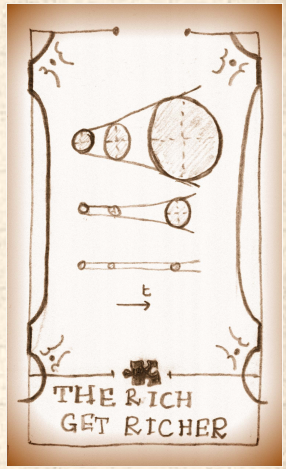
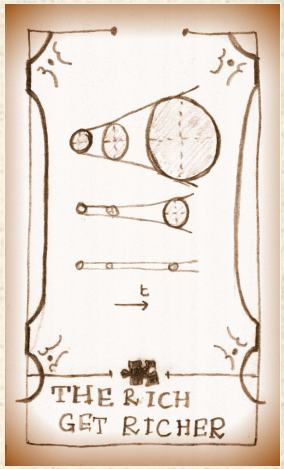


Random Competitive Replication:

Putting everything together:

For $k > 1$:

$$\langle N_{k,t+1} - N_{k,t} \rangle = (1-\rho) \left((+1)(k-1)\frac{N_{k-1,t}}{t} + (-1)k\frac{N_{k,t}}{t} \right)$$



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For $k > 1$:

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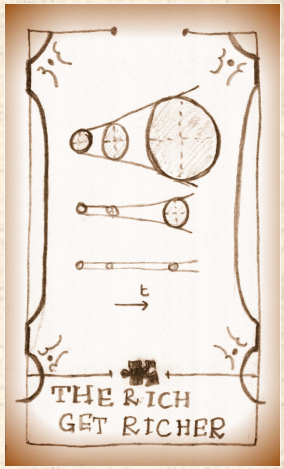
For $k = 1$:

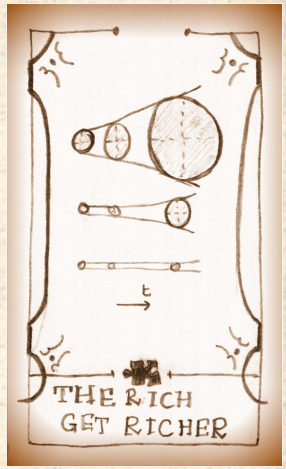
$$\left\langle N_{1,t+1} - N_{1,t} \right\rangle = (+1)\rho + (-1)(1-\rho)1 \cdot \frac{N_{1,t}}{t}$$



Random Competitive Replication:

Assume distribution stabilizes: $N_{k,t} = n_k t$

(Reasonable for t large)



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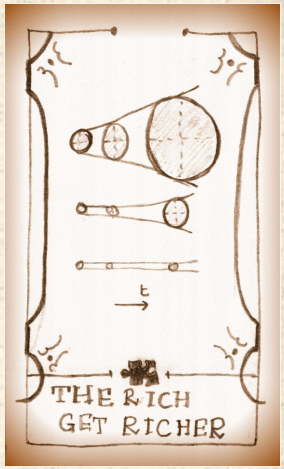

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(Reasonable for t large)

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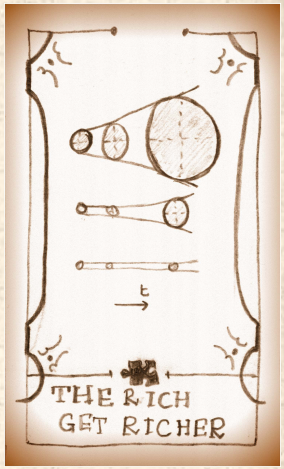
- Numbers of elephants now fractional



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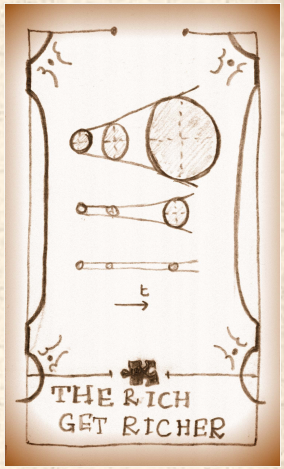

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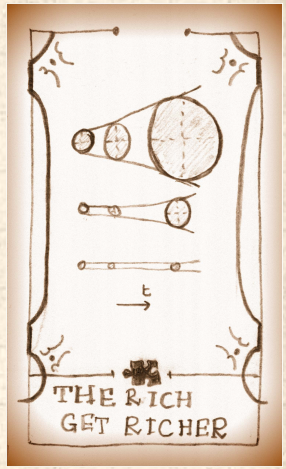
- For later: the fraction of groups that have size k is n_k/ρ since

$$\frac{N_{k,t}}{\rho t} = \frac{n_k \cancel{t}}{\rho \cancel{t}} = \frac{n_k}{\rho}.$$



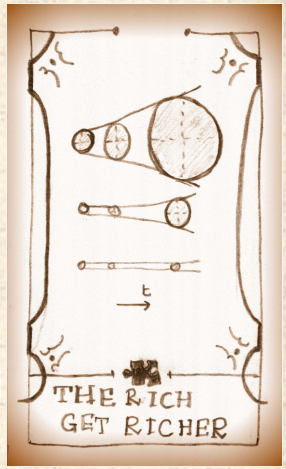
Random Competitive Replication:

Stochastic difference equation:

$$\left\langle N_{k,t+1} - N_{k,t} \right\rangle = (1-\rho)\left((k-1)\frac{N_{k-1,t}}{t} - k\frac{N_{k,t}}{t}\right)$$

becomes

$$n_k(t+1) - n_k t = (1-\rho)\left((k-1)\frac{n_{k-1}t}{t} - k\frac{n_k t}{t}\right)$$



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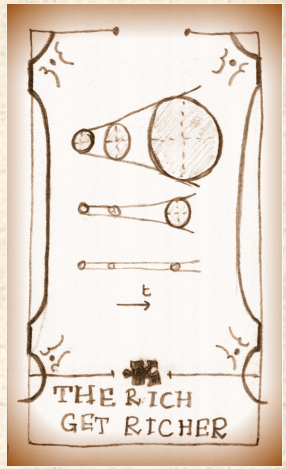

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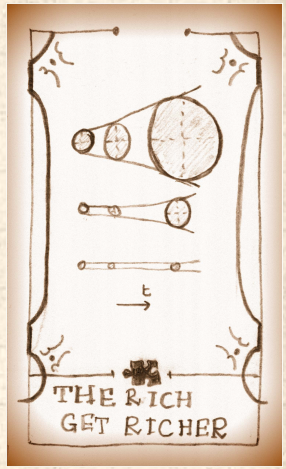

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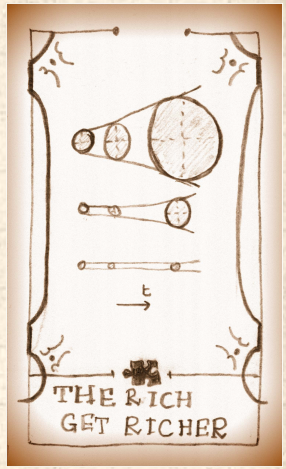
$$\Rightarrow n_k\left(1 + (1-\rho)k\right) = (1-\rho)(k-1)n_{k-1}$$



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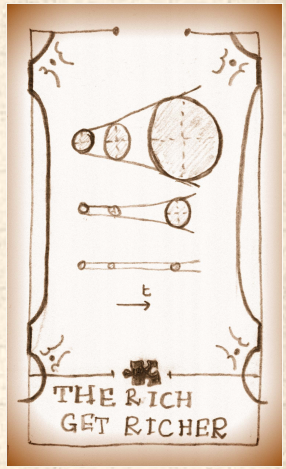
We have a simple recursion:

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}$$



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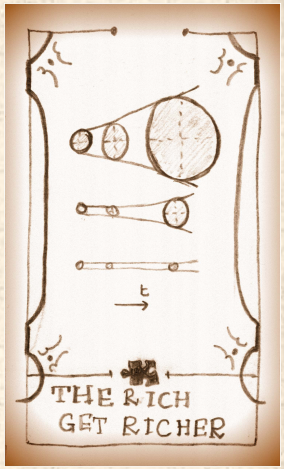

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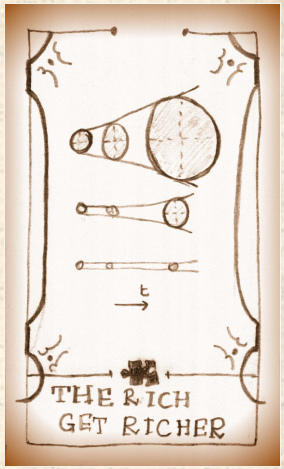

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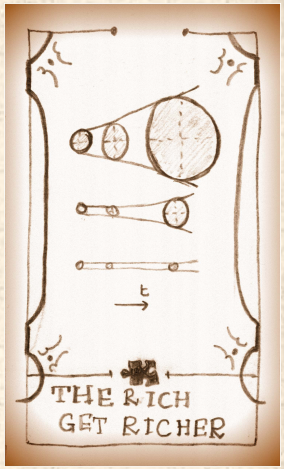

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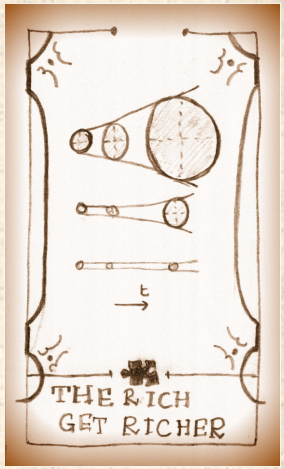

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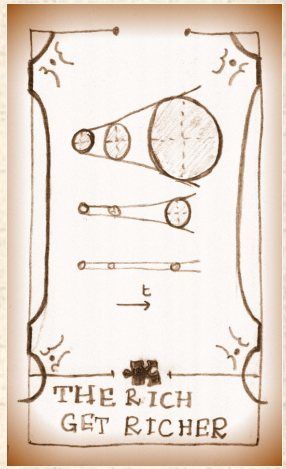

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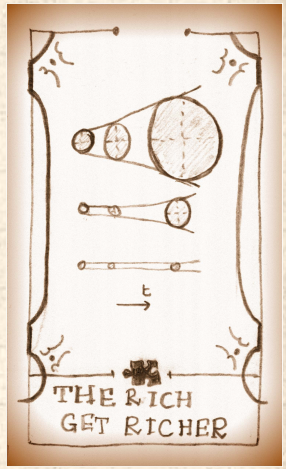
  We (okay, you) find

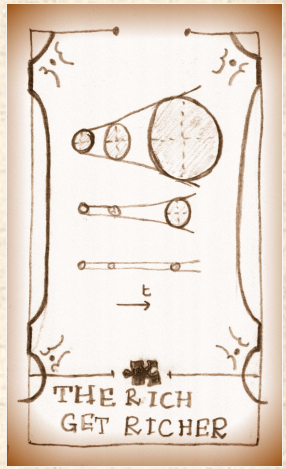
$$n_k \propto k^{-\frac{(2-\rho)}{(1-\rho)}} = k^{-\gamma}$$

$$\boxed{\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}}$$



Micro-to-Macro story with ρ and γ measurable.

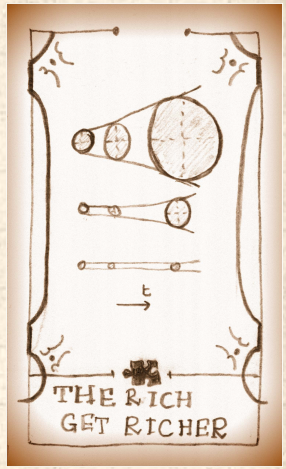
$$\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}$$



- Micro-to-Macro story with ρ and γ measurable.

$$\boxed{\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}}$$

- Observe $2 < \gamma < \infty$ for $0 < \rho < 1$.



- Micro-to-Macro story with ρ and γ measurable.

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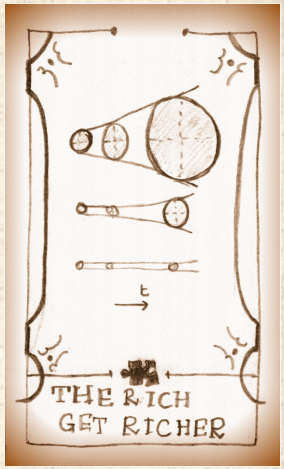

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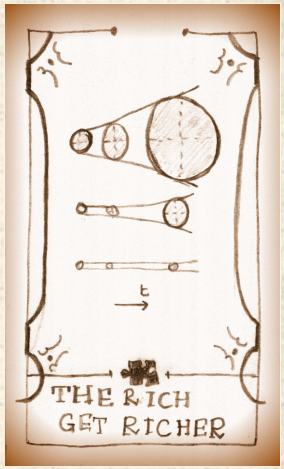

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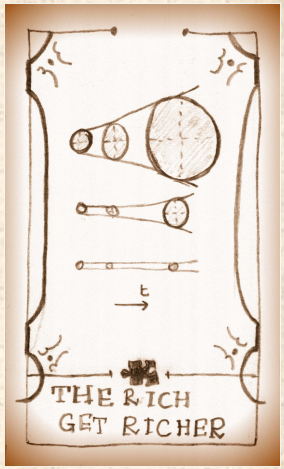

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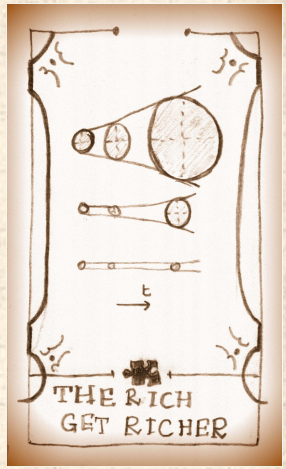

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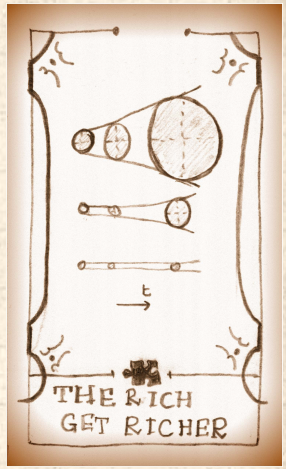
- All elephants have different flavors.
- Upshot: Tunable mechanism producing a family of universality classes.



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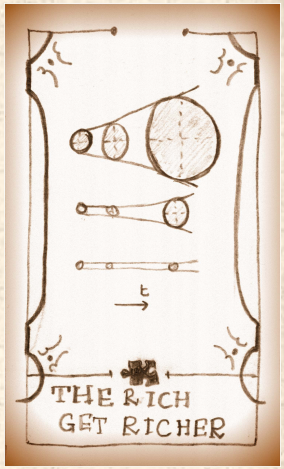

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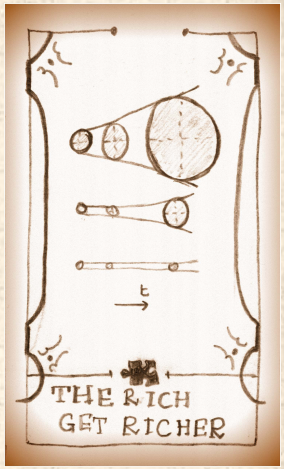
$$\boxed{\alpha = \frac{1}{\gamma - 1} = \frac{1}{\cancel{1} + \frac{1}{(1-\rho)} - \cancel{1}} = 1 - \rho.}$$



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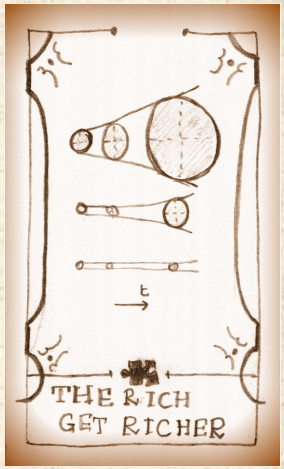

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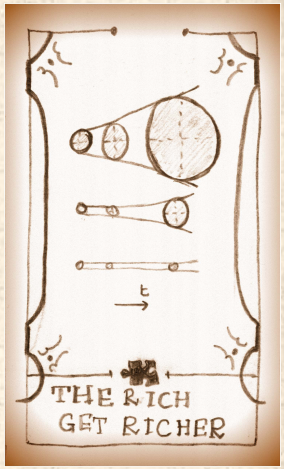

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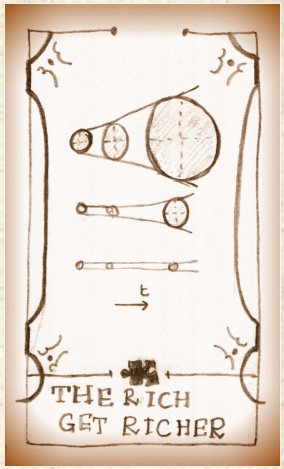

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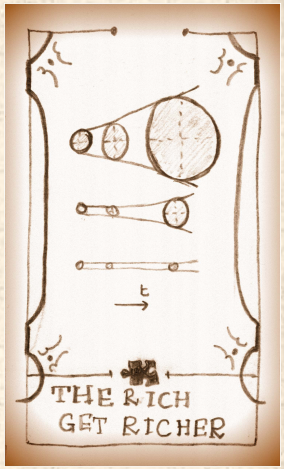

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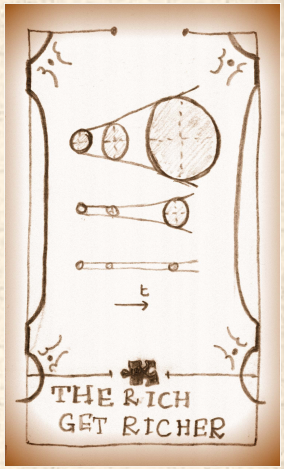
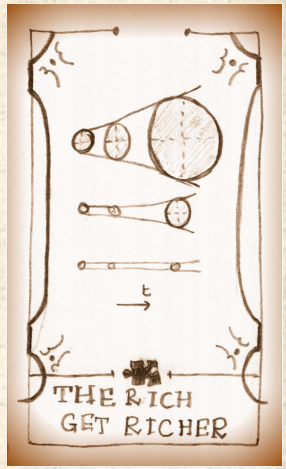
- Must look at the details to see if mechanism makes sense... more later.



What about small k?:

We had one other equation:

$$\left\langle N_{1,t+1} - N_{1,t} \right\rangle = \rho - (1-\rho)1 \cdot \frac{N_{1,t}}{t}$$



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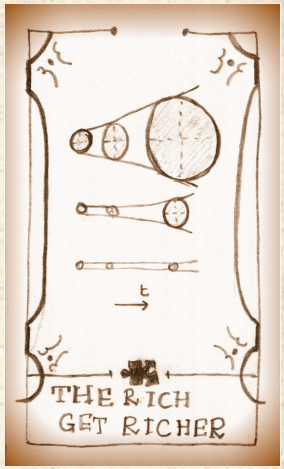

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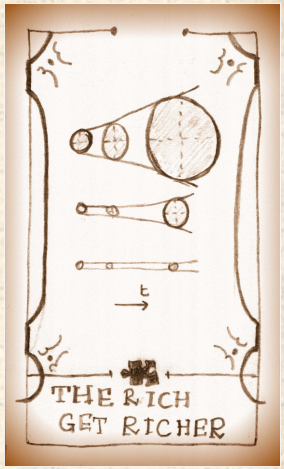

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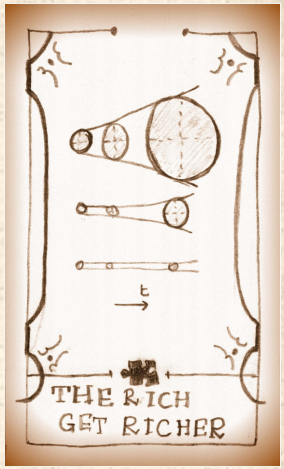

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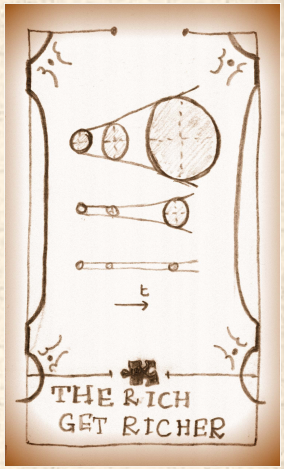

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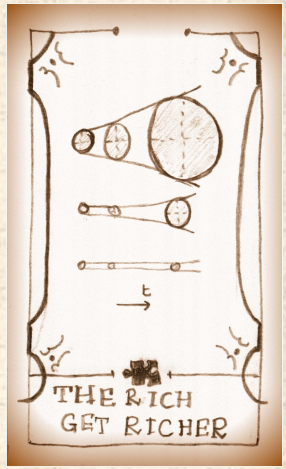

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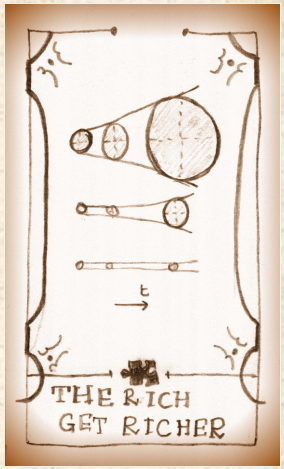

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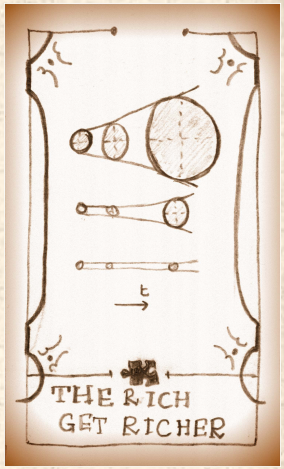

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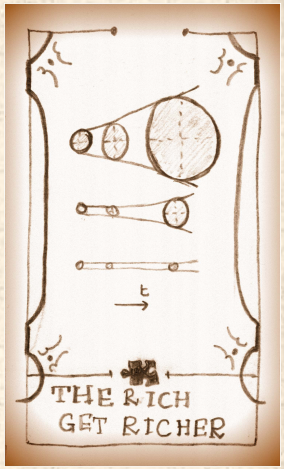

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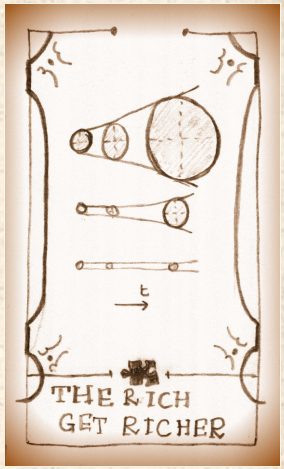

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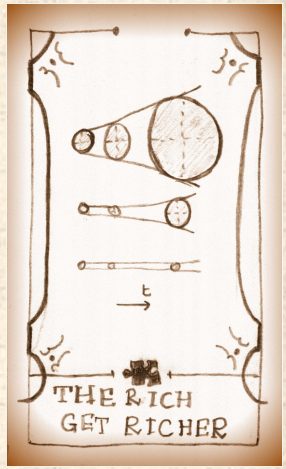

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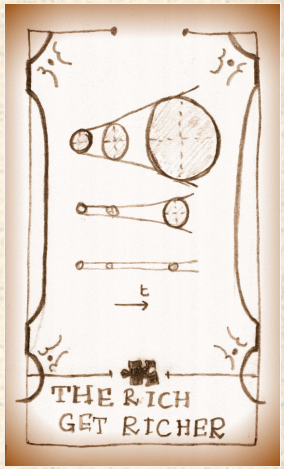

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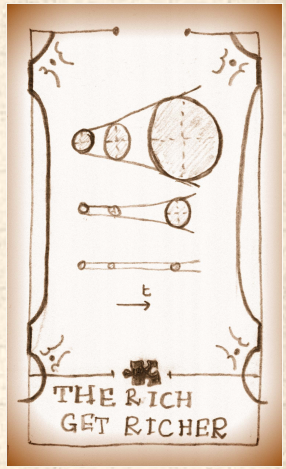

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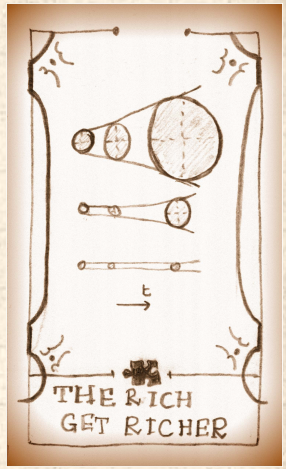
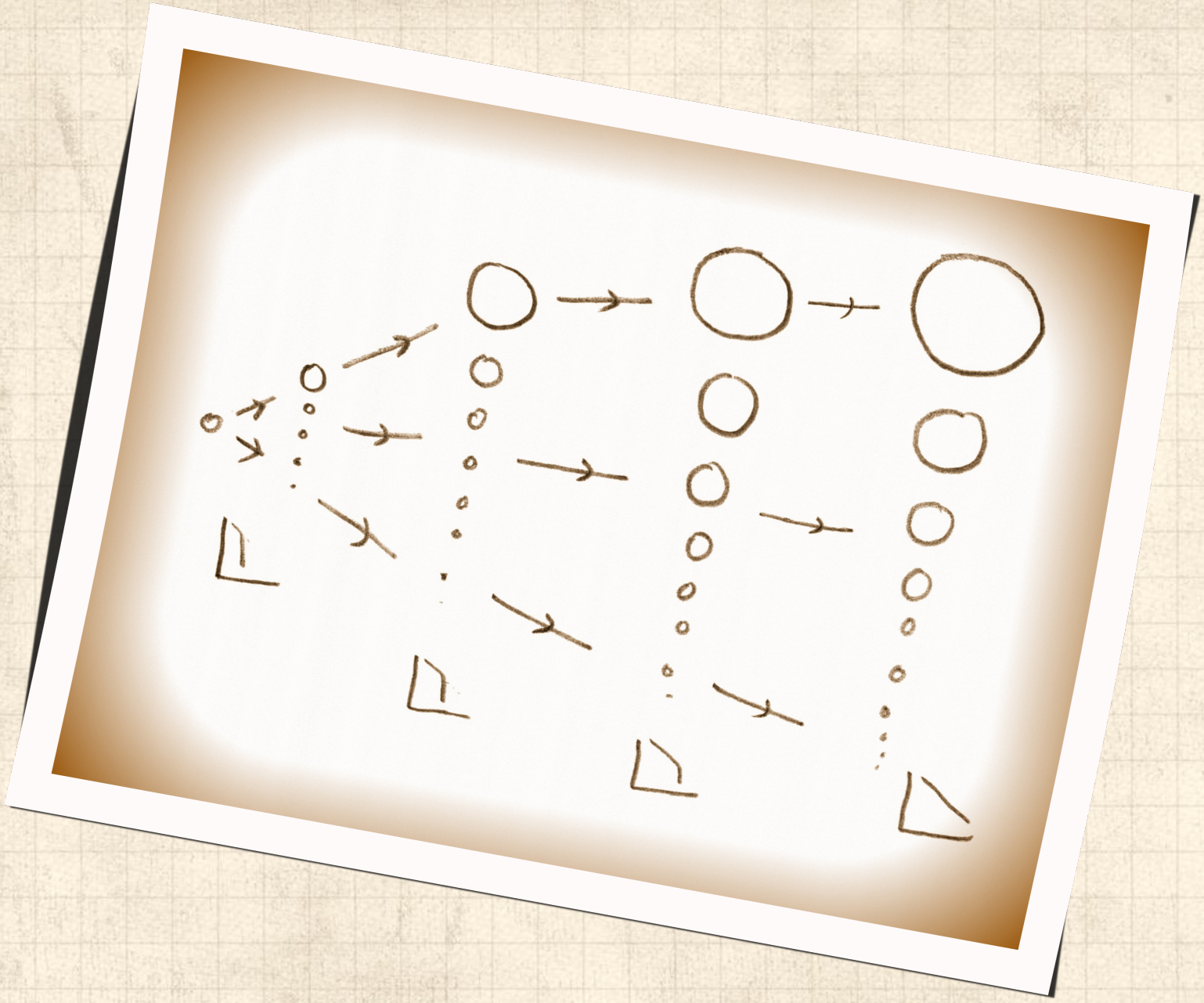
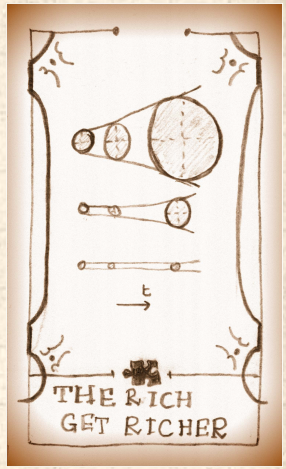




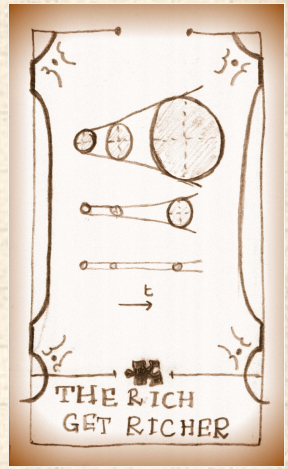


Outline

Rich-Get-Richer Mechanism

- Simon's Model
- Analysis
- Words
- Catchphrases
- First Mover Advantage

References



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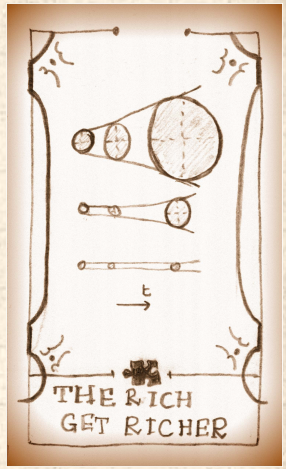
Estimate $\rho_{\text{est}} = \#\text{ unique words}/\#\text{ all words}$



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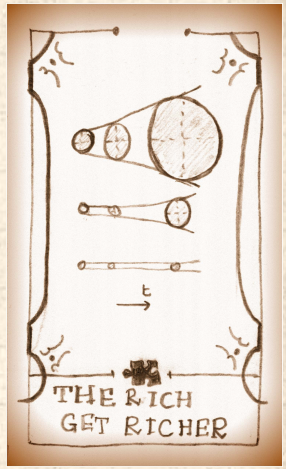

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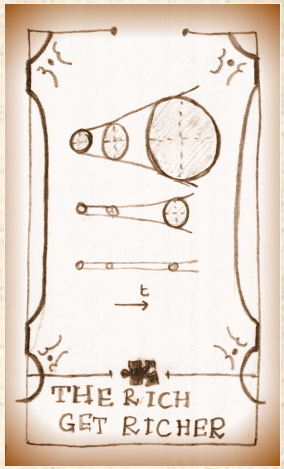
N_1 (real)	N_1 (est)	N_2 (real)	N_2 (est)
16,432	15,850	4,776	4,870



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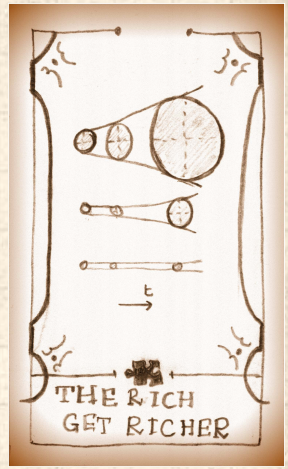

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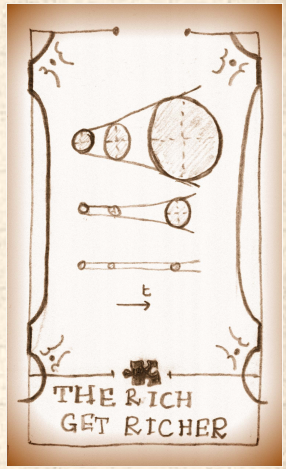

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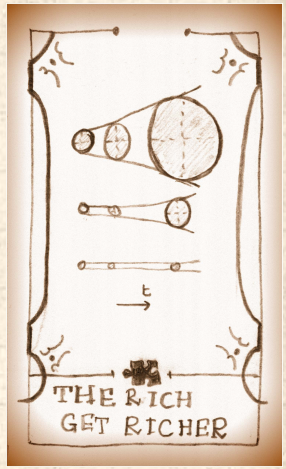

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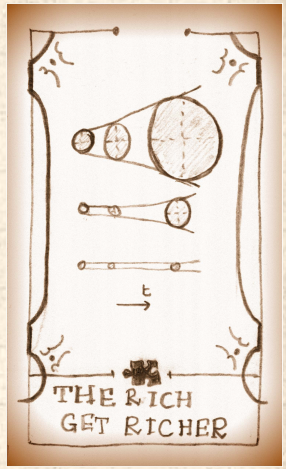

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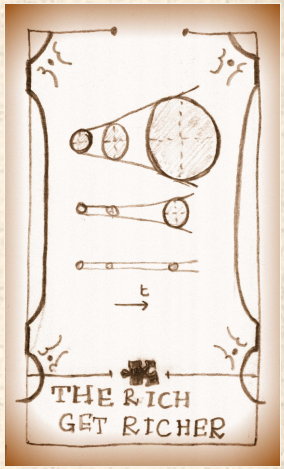

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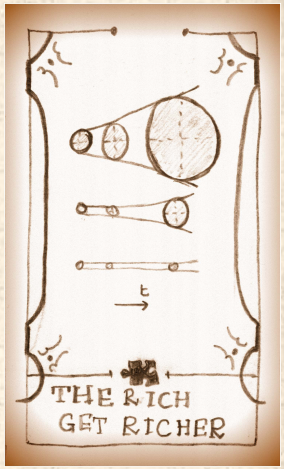

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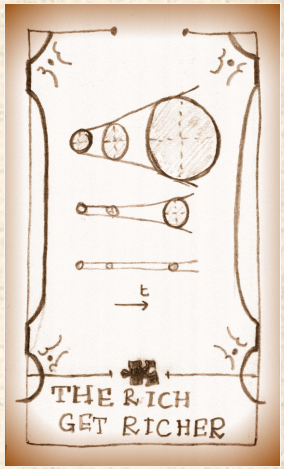

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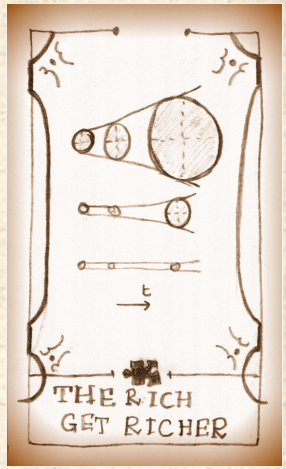

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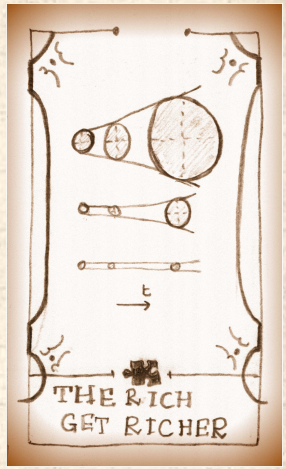

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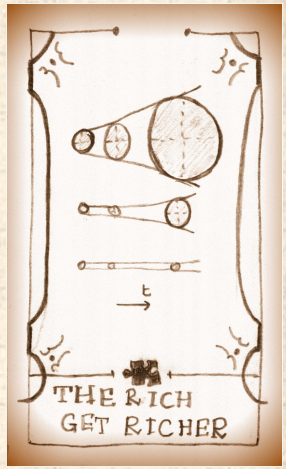

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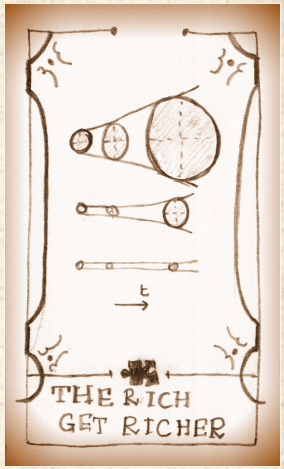

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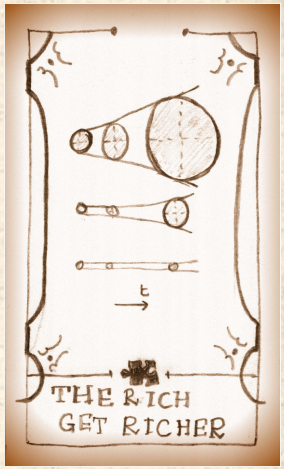

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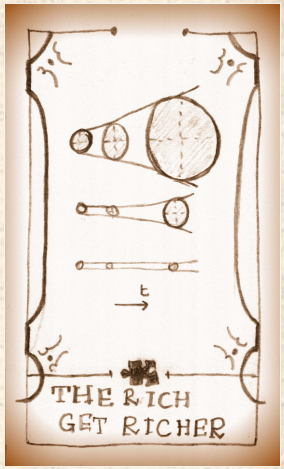

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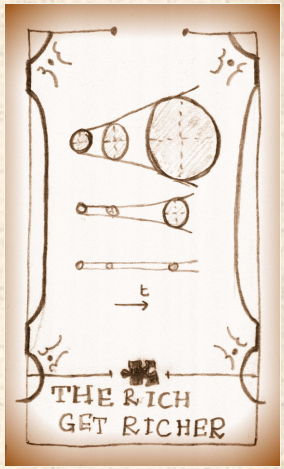

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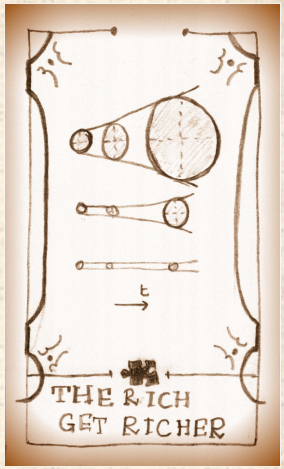
- Two (surmountable) problems:
  1. New papers have no citations
  2. Selection mechanism is more complicated



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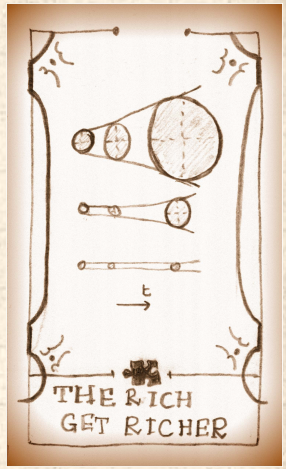

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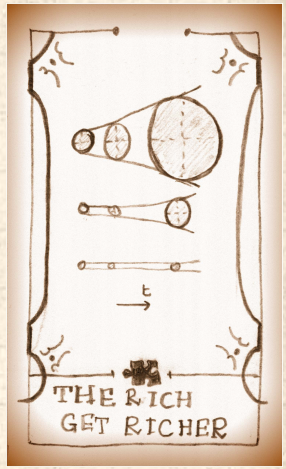

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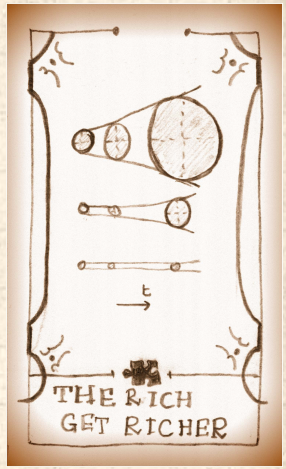

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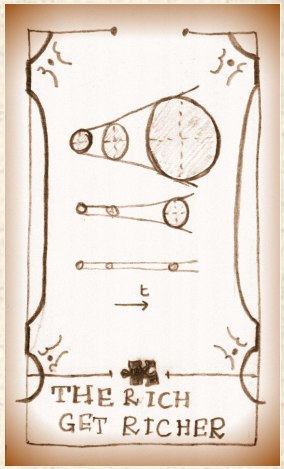

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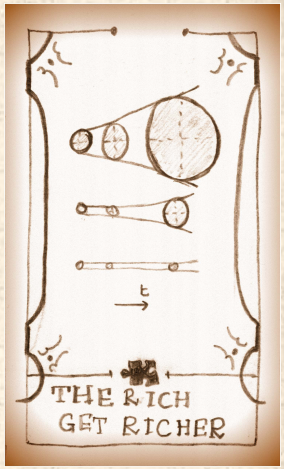

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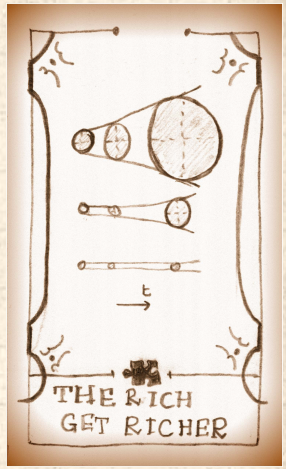

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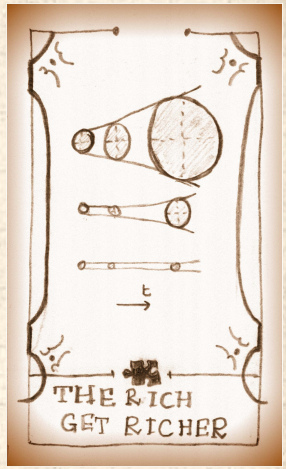
- Matilda effect: women's scientific achievements are often overlooked



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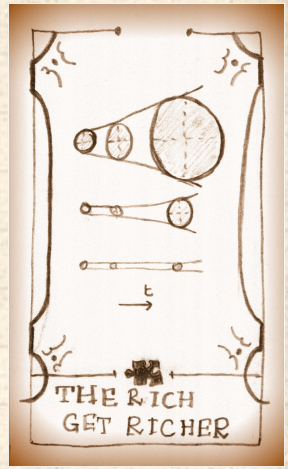

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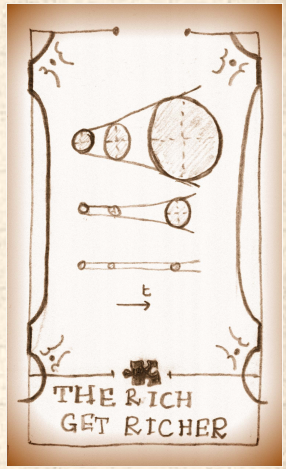

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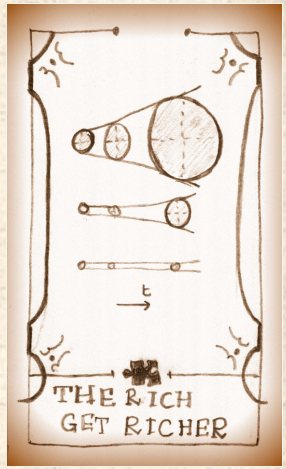

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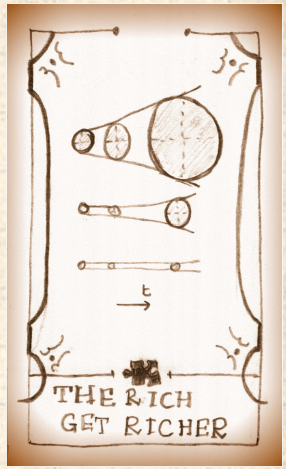

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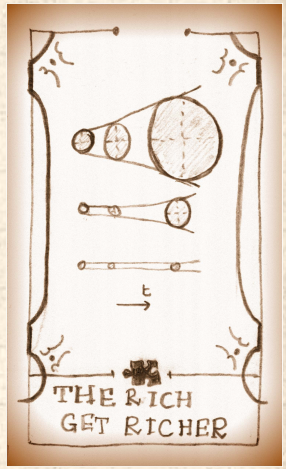

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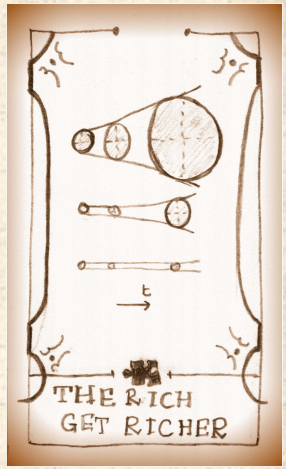

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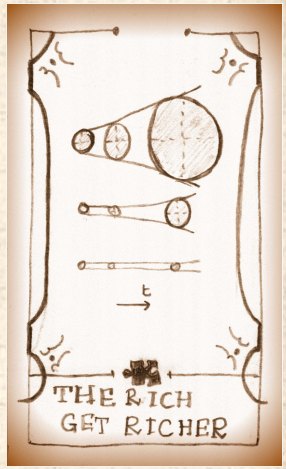

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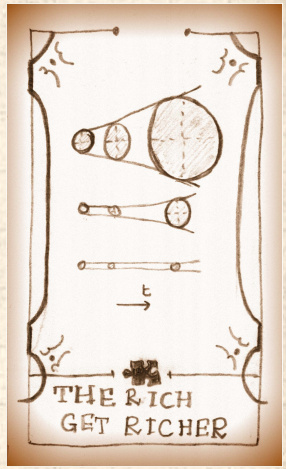

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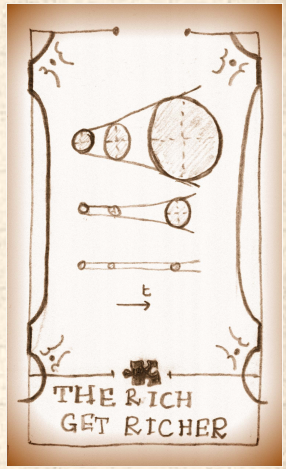

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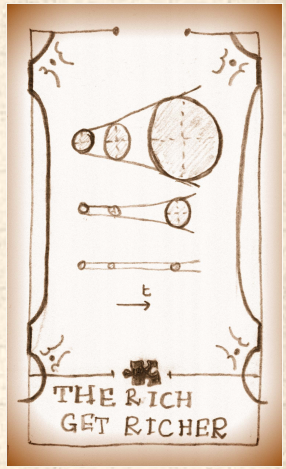

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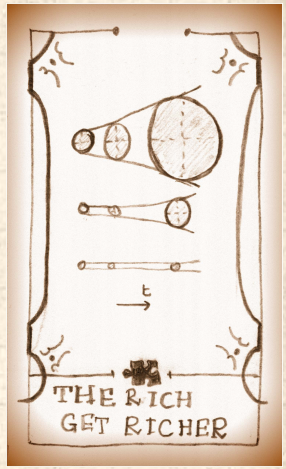

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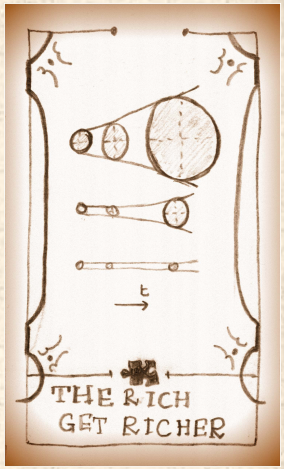

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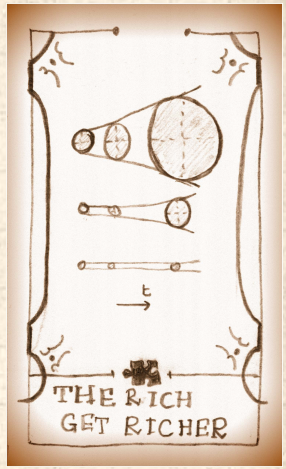

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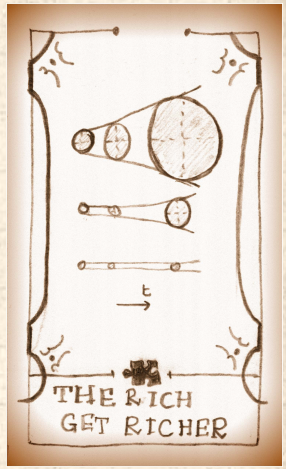

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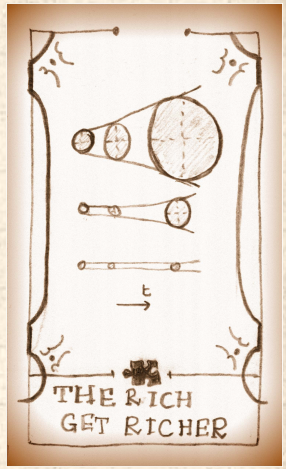

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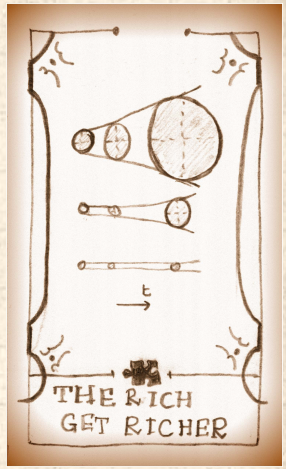

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- Analysis
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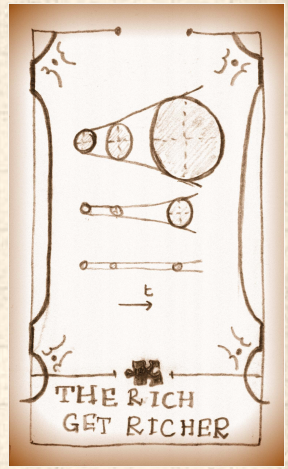
- Catchphrases
- First Mover Advantage

References



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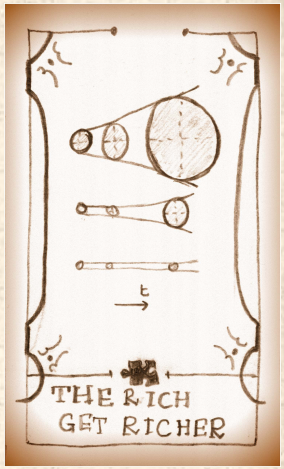

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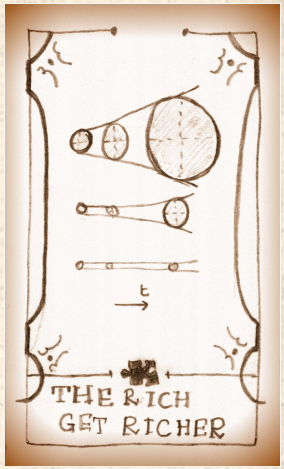

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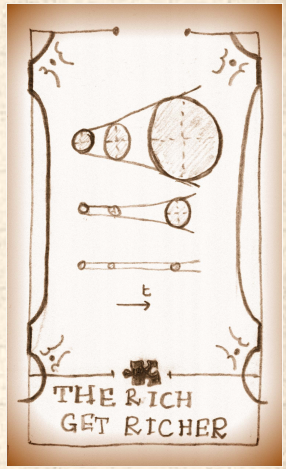

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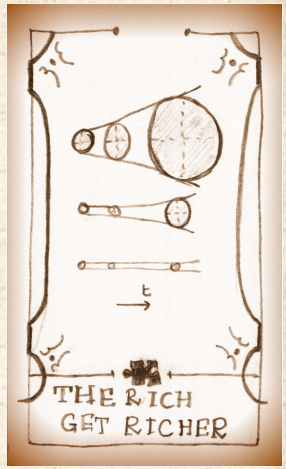
- Because ρ is usually close to 0, the first element is truly an elephant in the room.



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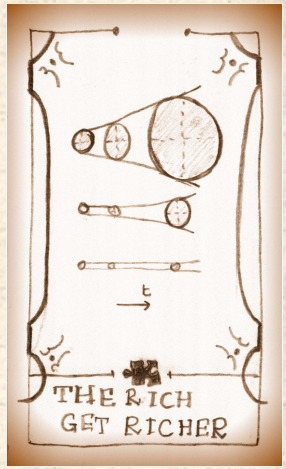
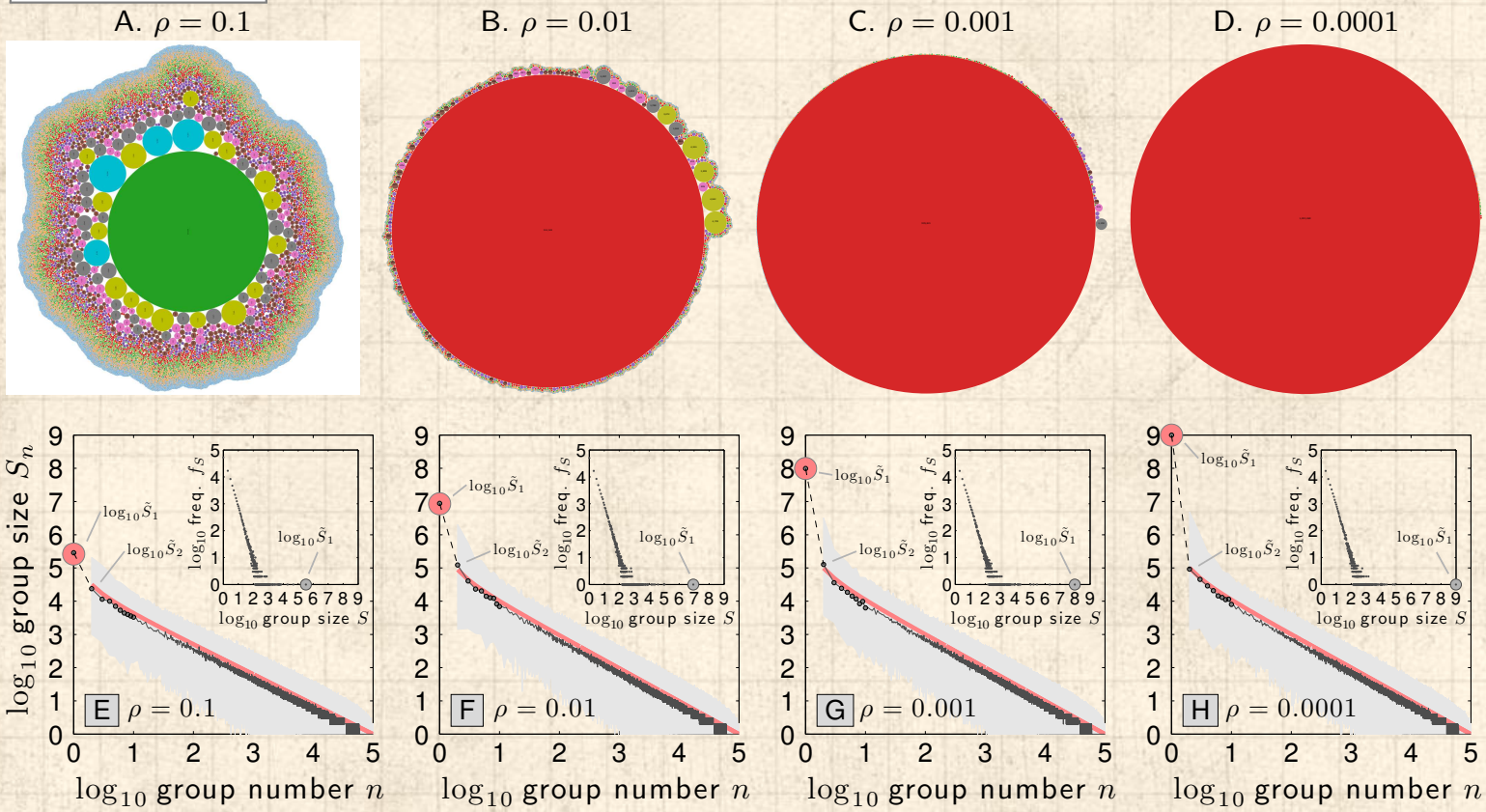
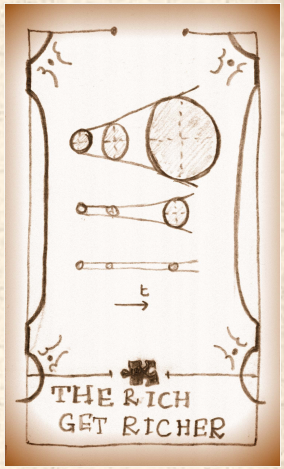
- First mover is a factor $1/\rho$ greater than expected.
- Because ρ is usually close to 0, the first element is truly an elephant in the room.
- Appears that this has been missed for 60 years ...



"Simon's fundamental rich-get-richer model entails a dominant first-mover advantage"

Dodds et al.,
Physical Review E, **95**, 052301, 2017. [5]



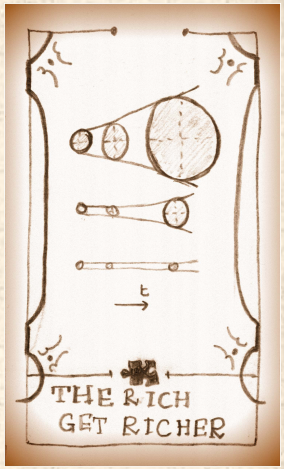


See visualization at paper's online app-endices

Alternate analysis:

- Evolution of the nth arriving group's size:

$$\left\langle S_{n,t+1} - S_{n,t} \right\rangle = (1 - \rho_t) \cdot \frac{S_{n,t}}{t} \cdot (+1).$$



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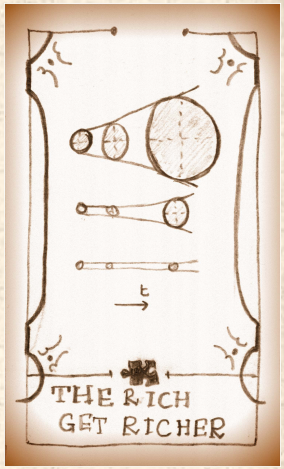
- For $t \geq t_n^{\text{init}}$, fix $\rho_t = \rho$ and shift t to $t-1$:

$$S_{n,t} = \left[1 + \frac{(1-\rho)}{t-1}\right] S_{n,t-1}.$$

where $S_{n,t_n^{\text{init}}} = 1$.



Betafication ensues:

$$
\begin{aligned}
S_{n,t} &= \left[1 + \frac{(1-\rho)}{t-1}\right]\left[1 + \frac{(1-\rho)}{t-2}\right] \cdots \left[1 + \frac{(1-\rho)}{t_n^{\text{init}}}\right] \cdot 1 \\
&= \left[\frac{t+1-\rho}{t-1}\right]\left[\frac{t-\rho}{t-2}\right] \cdots \left[\frac{t_n^{\text{init}}+1-\rho}{t_n^{\text{init}}}\right] \\
&= \frac{\Gamma(t+1-\rho)\Gamma(t_n^{\text{init}})}{\Gamma(t_n^{\text{init}}+1-\rho)\Gamma(t)} \\
&= \frac{\mathrm{B}(t_n^{\text{init}}, 1-\rho)}{\mathrm{B}(t, 1-\rho)}.
\end{aligned}
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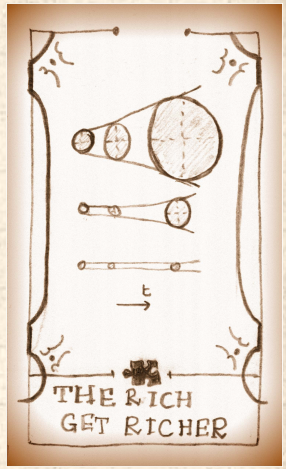
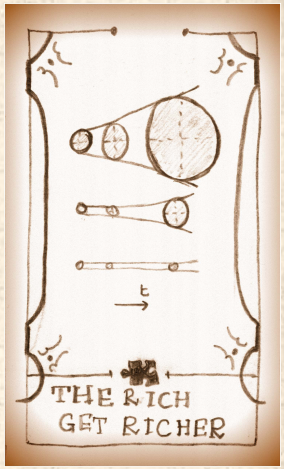


The first mover is really different:

- The issue is t_n^{init} in

$$S_{n,t} = \frac{\mathrm{B}(t_n^{\text{init}}, 1-\rho)}{\mathrm{B}(t, 1-\rho)}$$

- For $n \geq 2$ and $\rho \ll 1$, the nth group typically arrives at $t_n^{\text{init}} \simeq \left[\frac{n-1}{\rho}\right]$



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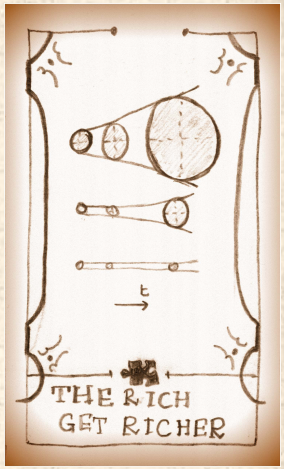

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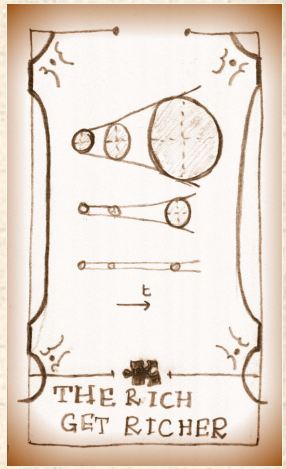
- Simon missed the first mover by working on the size distribution.



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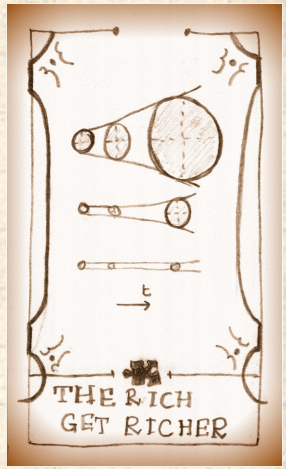

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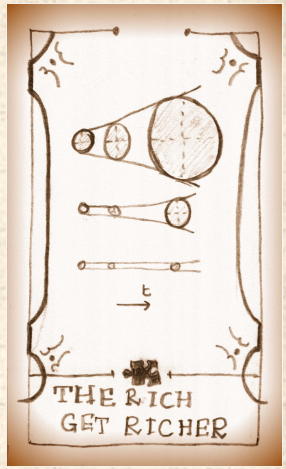
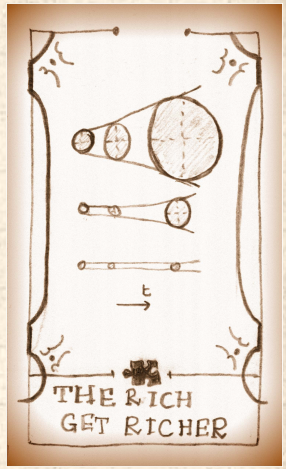
- Contribution to $P_{k,t}$ of the first element vanishes as $t \to \infty$.
- Note: Does not apply to Barabási-Albert model.



Variability:

- The probability that the nth arriving group, if of size $S_{n,t} = k$ at time t, first replicates at time $t + \tau$:



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&\quad = \prod_{i=0}^{\tau-1} \left[1 - (1-\rho)\frac{k}{t+i}\right] \cdot (1-\rho)\frac{k}{t+\tau} \\
&= k\frac{B(\tau, t)}{B\left(\tau, t-(1-\rho)\right)}\frac{1-\rho}{t+\tau} \propto \frac{\tau^{-(1-\rho)k}}{t+\tau} \sim \tau^{-(2-\rho)k}.
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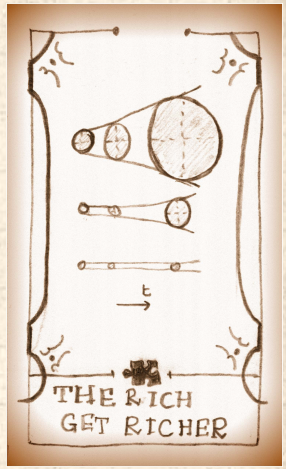

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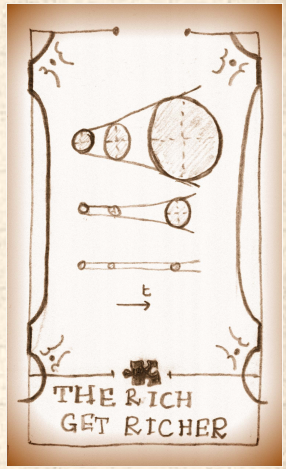
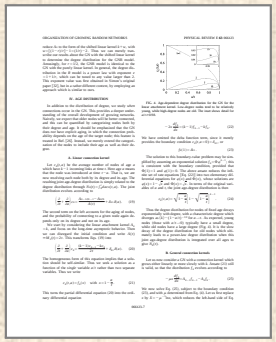


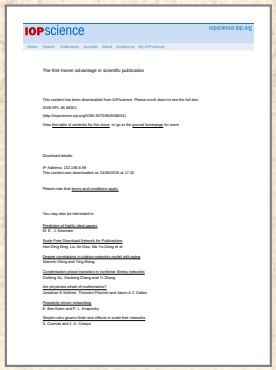
- Upshot: nth arriving group starting at size 1 will on average wait for an infinite time to replicate.

Related papers:



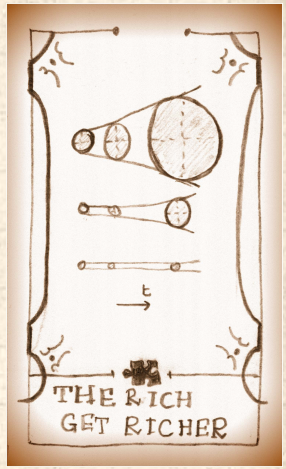
"Organization of Growing Random Networks"

Krapivsky and Redner,
Phys. Rev. E, **63**, 066123, 2001. [7]



"The first-mover advantage in scientific publication"

M. E. J. Newman,
Europhysics Letters, **86**, 68001, 2009. [11]



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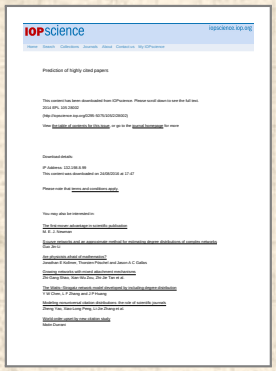

"Prediction of highly cited papers"
M. E. J. Newman,
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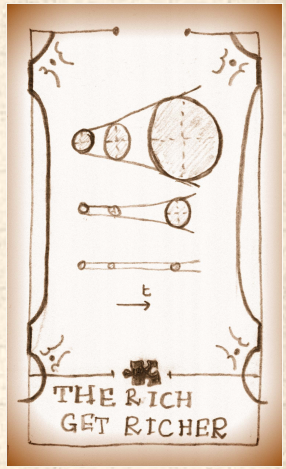
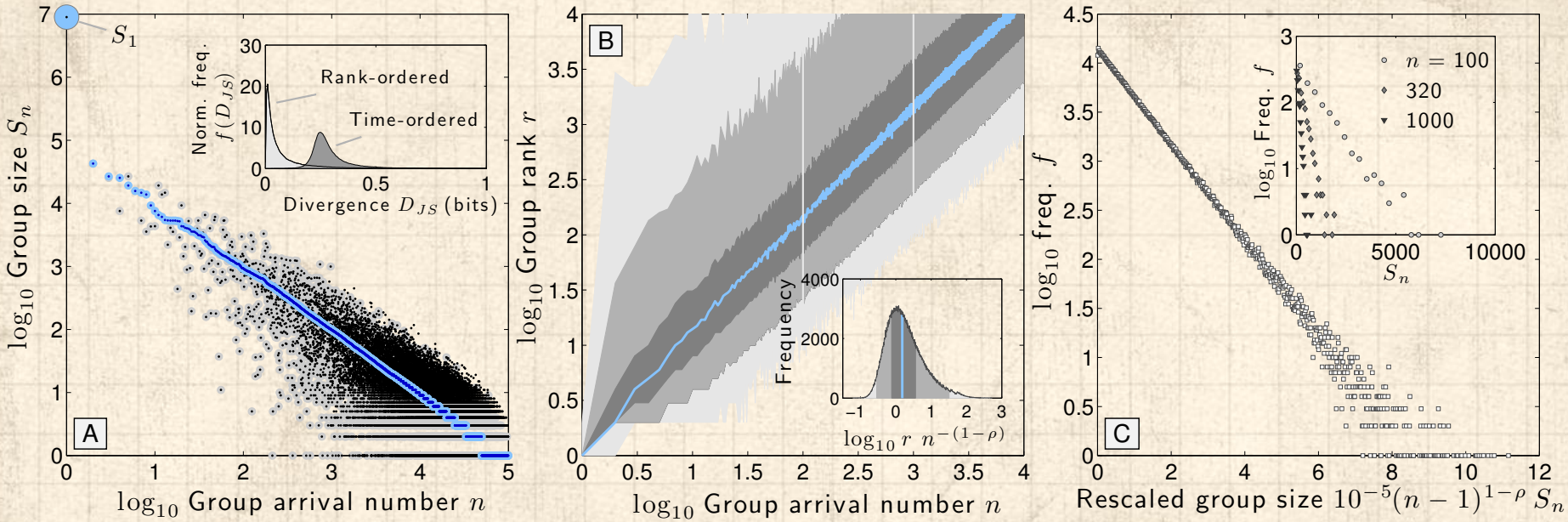
"The effect of the initial network configuration on preferential attachment"
Berset and Medo,
The European Physical Journal B, **86**, 1–7, 2013. [3]

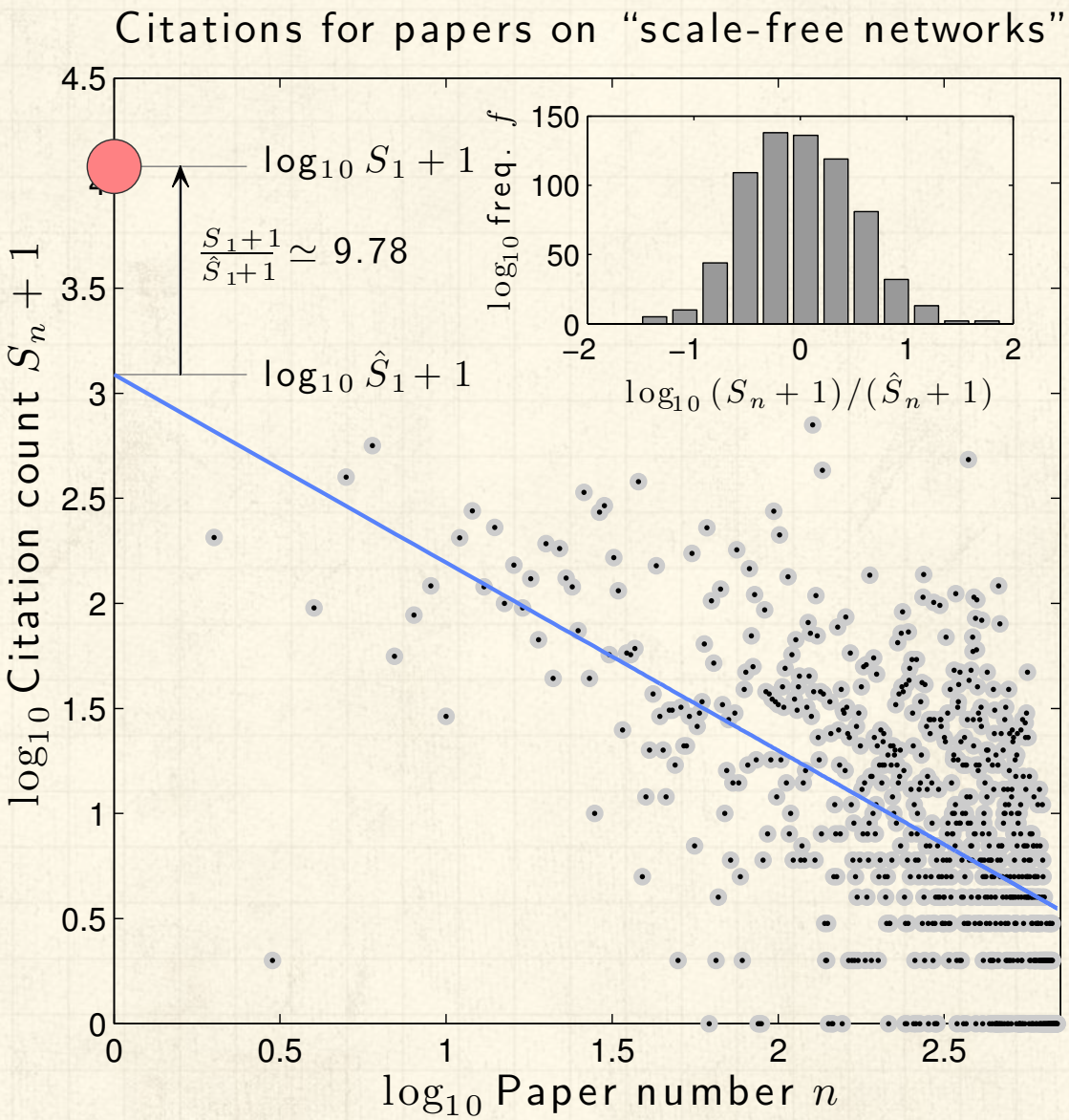


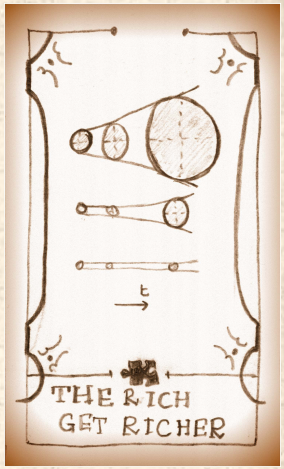
Arrival variability:



- Any one simulation shows a high amount of disorder.
- Two orders of magnitude variation in possible rank.
- Rank ordering creates a smooth Zipf distribution.
- Size distribution for the nth arriving group show exponential decay.

Self-referential citation data:

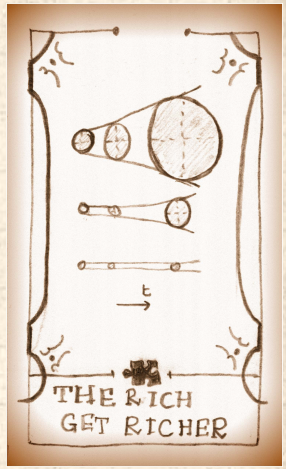




More mattering:

Rich-get-richerness in social contagion:

- We love to rank everyone, everything: Top n lists.



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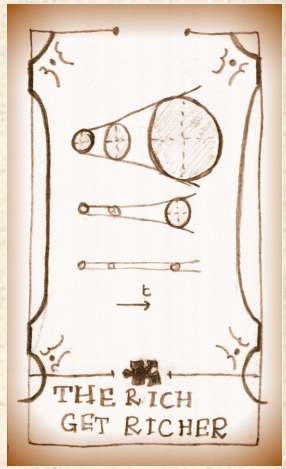

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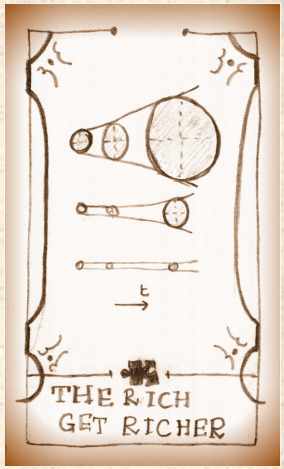

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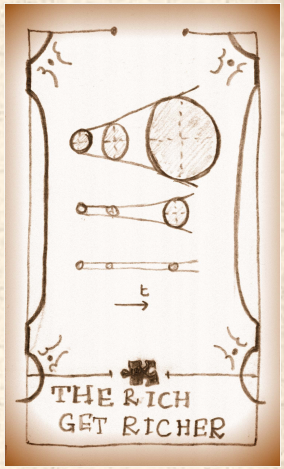

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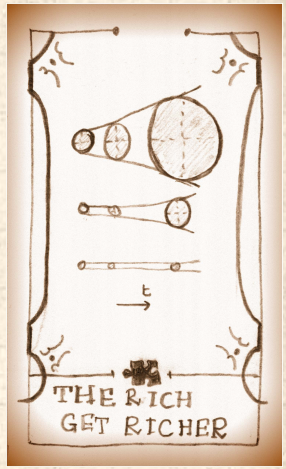

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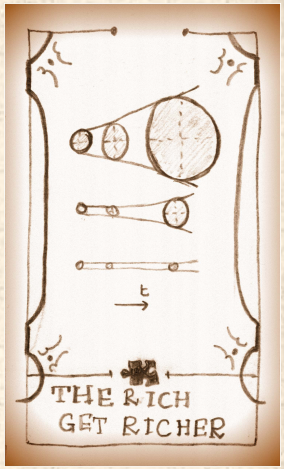

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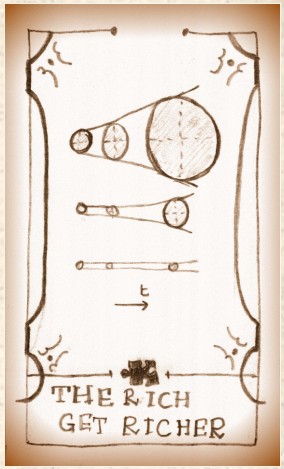

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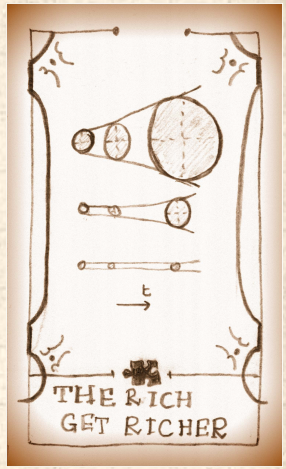
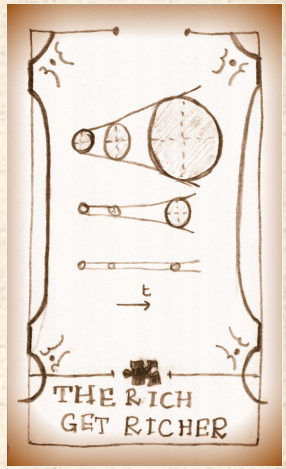
- No "regramming" is a positive feature of Instagram (also: Pratchett the Cat)



More mattering:

Rich-get-richerness in social contagion:

- We love to rank everyone, everything: Top n lists.
- People, wealth, sports, music, movies, books, schools, cities, countries, dogs (13/10), ...
- Gameable: payola, astroturfing, sockpuppetry, John Barron (the sockpuppet hype man), ...
- Black-box ranking algorithms make ranking opaque.
- Black boxes are gameable but takes money and commensurate skill.
- Black box algorithms can make things spread rampantly.[1]
- No "regramming" is a positive feature of Instagram (also: Pratchett the Cat)
- What if a healthier Facebook is just ...



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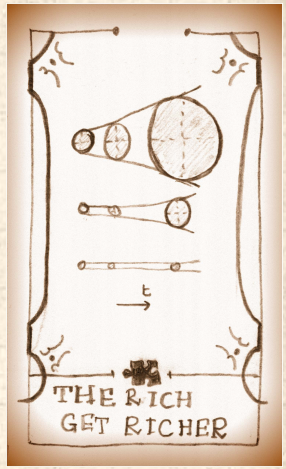

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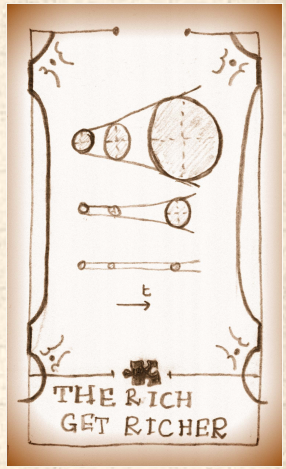

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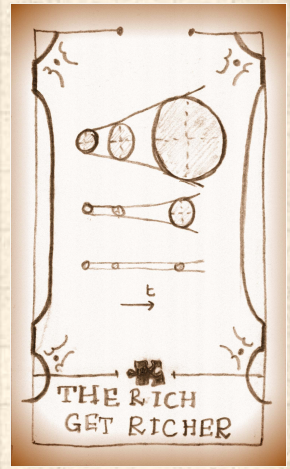

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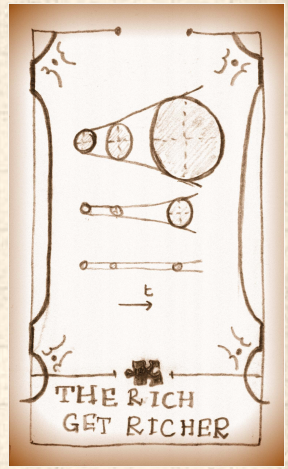

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