

Mechanisms for Generating Power-Law Size Distributions, Part 2

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Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394, 2022-2023 | @pocsvox

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Power-Law
Mechanisms, Pt. 2

Sealie & Lambie
Productions



Variable
transformation

Basics

Holtmark's Distribution

PLIPLO

References

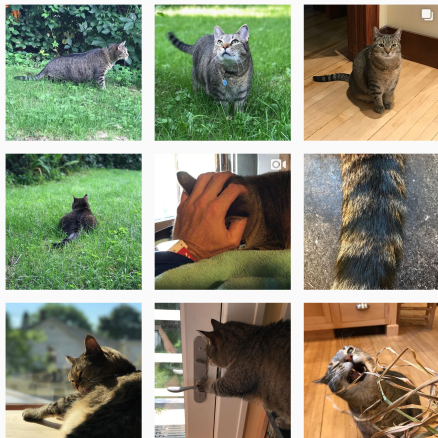


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Special Guest Executive Producer



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 On Instagram at [pratchett_the_cat](https://www.instagram.com/pratchett_the_cat) 



Outline

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Variable transformation

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The Boggoracle Speaks:

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Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

🧱 Random variable X with known distribution P_x

🧱 Second random variable Y with $y = f(x)$.

$$\begin{aligned} \text{🧱 } P_Y(y)dy &= \\ &= \sum_{x|f(x)=y} P_X(x)dx \\ &= \sum_{y|f(x)=y} P_X(f^{-1}(y)) \frac{dy}{|f'(f^{-1}(y))|} \end{aligned}$$

🧱 Often easier to do by hand...

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General Example

Assume relationship between x and y is 1-1.

Power-law relationship between variables:

$$y = cx^{-\alpha}, \alpha > 0$$

Look at y large and x small



$$dy = d(cx^{-\alpha})$$

$$= c(-\alpha)x^{-\alpha-1}dx$$

$$\text{invert: } dx = \frac{-1}{c\alpha} x^{\alpha+1} dy$$

$$dx = \frac{-1}{c\alpha} \left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha} dy$$

$$dx = \frac{-c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$



Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_y(y)dy = P_x \left(\overbrace{\left(\frac{y}{c}\right)^{-1/\alpha}}^{(x)} \right) \overbrace{\frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy}^{dx}$$

🧱 If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

🧱 If $P_x(x) \rightarrow x^\beta$ as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$



Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$$

- Exponentials arise from randomness (easy) ...
- More later when we cover robustness.

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Gravity

- ☰ Select a random point in the universe \vec{x} .
- ☰ Measure the force of gravity $F(\vec{x})$.
- ☰ Observe that $P_F(F) \sim F^{-5/2}$.
- ☰ Distribution named after Holtsmark who was thinking about electrostatics and plasma ^[1].
- ☰ Again, the humans naming things after humans, poorly.¹



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¹Stigler's Law of Eponymy

Matter is concentrated in stars: [2]

🧱 F is distributed unevenly

🧱 Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)dr \propto r^2 dr$$

🧱 Assume stars are distributed randomly in space (oops?)

🧱 Assume only one star has significant effect at \vec{x} .

🧱 Law of gravity:

$$F \propto r^{-2}$$

🧱 invert:

$$r \propto F^{-\frac{1}{2}}$$

🧱 Connect differentials: $dr \propto dF^{-\frac{1}{2}} \propto F^{-\frac{3}{2}} dF$



Transformation:

Using $r \propto F^{-1/2}$, $dr \propto F^{-3/2}dF$, and $P_r(r) \propto r^2$



$$P_F(F)dF = P_r(r)dr$$



$$\propto P_r(\text{const} \times F^{-1/2})F^{-3/2}dF$$



$$\propto (F^{-1/2})^2 F^{-3/2}dF$$



$$= F^{-1-3/2}dF$$



$$= F^{-5/2}dF.$$

Variable
transformation

Basics

Holtzmark's Distribution
PLIPLD

References



Gravity:

$$P_F(F) = F^{-5/2} dF$$

$$\gamma = 5/2$$



Mean is finite.



Variance = ∞ .



A wild distribution.



Upshot: Random sampling of space usually safe
but can end badly...

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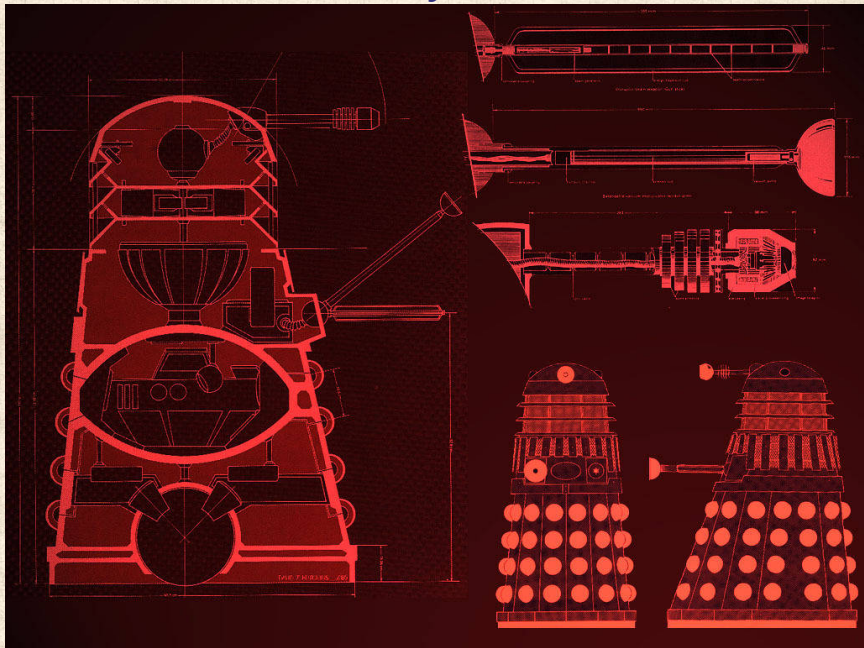
Holtsmark's Distribution

FLIPLD

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□ Todo: Build Dalek army.



Extreme Caution!

- PLIPLO = **Power law in, power law out**
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...
- Don't do this!!! (slap, slap)
- MIWO = **Mild in, Wild out** is the stuff.
- In general: We need mechanisms!



- [1] J. Holtsmark.
Über die verbreiterung von spektrallinien.
[Ann. Phys.](#), 58:577–630, 1919. pdf 
- [2] D. Sornette.
[Critical Phenomena in Natural Sciences.](#)
Springer-Verlag, Berlin, 1st edition, 2003.

