# Mechanisms for Generating Power-Law Size Distributions, Part 2

Last updated: 2022/08/27, 23:54:10 EDT

Principles of Complex Systems, Vols. 1, 2, & 3D CSYS/MATH 300, 303, & 394, 2022-2023 | @pocsvox

#### Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center Santa Fe Institute | University of Vermont



Licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License

## Outline

### Variable transformation

Basics Holtsmark's Distribution PLIPLO

References

# Variable Transformation

## Understand power laws as arising from

- 1. Elementary distributions (e.g., exponentials).
- 2. Variables connected by power relationships.
- $\mathbb{R}$  Random variable X with known distribution  $P_x$
- Second random variable Y with y = f(x).

$$\begin{split} \bigotimes & P_Y(y) \mathrm{d}y = \\ & \sum_{x \mid f(x) = y} P_X(x) \mathrm{d}x \\ &= \\ & \sum_{y \mid f(x) = y} P_X(f^{-1}(y)) \frac{\mathrm{d}y}{\mid f'(f^{-1}(y)) \mid} \end{split}$$

Often easier to do by hand...

PoCS @pocsvox

Power-Law Mechanisms, Pt. 2

transformation

References

UM O

PoCS

@pocsvox

Variable

Power-Law

Mechanisms, Pt. 2

transformation

少Q (~ 1 of 18

Variable

General Example

Assume relationship between x and y is 1-1.

Power-law relationship between variables:  $y = cx^{-\alpha}, \alpha > 0$ 

& Look at y large and x small

8

 $dy = d(cx^{-\alpha})$ 

 $=c(-\alpha)x^{-\alpha-1}\mathrm{d}x$ 

invert:  $dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$ 

 $dx = \frac{-1}{c\alpha} \left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha} dy$ 

 $dx = \frac{-c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$ 

#### Now make transformation:

$$P_y(y)\mathsf{d} y = P_x(x)\mathsf{d} x$$

 $P_y(y) \mathrm{d} y = P_x \underbrace{\left( \left( \frac{y}{c} \right)^{-1/\alpha} \right)}_{(x)} \underbrace{\frac{\mathrm{d} x}{c^{1/\alpha} y^{-1-1/\alpha} \mathrm{d} y}}_{(x)}$ 

 $\Re$  If  $P_m(x) \to \text{non-zero constant as } x \to 0 \text{ then}$ 

$$P_y(y) \propto y^{-1-1/\alpha}$$
 as  $y \to \infty$ .

 $\Re$  If  $P_x(x) \to x^{\beta}$  as  $x \to 0$  then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \to \infty.$$



•9 q (№ 2 of 18

PoCS @pocsvox Power-Law Mechanisms, Pt. 2

Example

III |

少 Q (~ 6 of 18

# **Exponential distribution**

Given  $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$  and  $y = cx^{-\alpha}$ , then

 $P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$ 

- Exponentials arise from randomness (easy) ...
- More later when we cover robustness.

#### Gravity @pocsvox Power-Law

PoCS

Variable

References

.... |S

@pocsvox

Power-Law

Variable

References

.... |S

PoCS

@pocsvox

Variable

PLIPLO

transformation

Power-Law

Mechanisms, Pt. 2

少 q (→ 8 of 18

transformation

少<a>♠ 7 of 18</a>

Mechanisms, Pt. 2

transformation

Select a random point in the universe  $\vec{x}$ .

Measure the force of gravity  $F(\vec{x})$ .

Observe that  $P_F(F) \sim F^{-5/2}.$ 

Distribution named after Holtsmark who was thinking about electrostatics and plasma [1].

Again, the humans naming things after humans, poorly.1

<sup>1</sup>Stigler's Law of Eponymy ☑.

 $\Re$  F is distributed unevenly

Matter is concentrated in stars: [2]



Variable transformation Holtsmark's Distribution References

Mechanisms, Pt. 2

PoCS

@pocsvox

Power-Law

W |S

୬९℃ 11 of 18

@pocsvox Power-Law

Variable transformation Holtsmark's Distribution

References

 $P_{\infty}(r) dr \propto r^2 dr$ Assume stars are distributed randomly in space

 $\clubsuit$  Probability of being a distance r from a single star

& Assume only one star has significant effect at  $\vec{x}$ .

Law of gravity:

at  $\vec{x} = \vec{0}$ :

 $F \propto r^{-2}$ 

a invert:

**Transformation:** 

8

8

8

8

 $r \propto F^{-\frac{1}{2}}$ 

 $\Leftrightarrow$  Connect differentials:  $dr \propto dF^{-\frac{1}{2}} \propto F^{-\frac{3}{2}} dF$ 



夕 Q № 12 of 18

PoCS @pocsvox

Power-Law

Variable transformation Holtsmark's Distribution

 $\propto (F^{-1/2})^2 F^{-3/2} dF$ 

 $P_{\mathbf{F}}(F) dF = P_{\mathbf{F}}(r) dr$ 

 $\propto P_r({\rm const}\times F^{-1/2})F^{-3/2}{\rm d}F$ 

 ${
m d} r \propto F^{-3/2} {
m d} F$  , and  $P_r(r) \propto r^2$ 

 $= F^{-1-3/2} dF$ 







少 q (~ 13 of 18

少∢(~ 9 of 18

## Gravity:

 $P_F(F) = F^{-5/2} \mathrm{d}F$ 

 $\gamma = 5/2$ 

Mean is finite.

& Variance =  $\infty$ .

& A wild distribution.

& Upshot: Random sampling of space usually safe but can end badly...



Variable transformation Basics Holtsmark's Distribution PLIPLO

References

(III)

少 Q ← 14 of 18

PLIPLO = Power law in, power law out

& Explain a power law as resulting from another unexplained power law.

Don't do this!!! (slap, slap)

MIWO = Mild in, Wild out is the stuff.

In general: We need mechanisms!

PoCS @pocsvox Power-Law

References I

Variable transformation Basics Holtsmark's Distribution PLIPLO

References

Mechanisms, Pt. 2

[1] J. Holtsmark.

Über die verbreiterung von spektrallinien.

Ann. Phys., 58:577–630, 1919. pdf♂

[2] D. Sornette. Critical Phenomena in Natural Sciences. Springer-Verlag, Berlin, 1st edition, 2003. References

PoCS @pocsvox

Power-Law

Variable

transformation

Mechanisms, Pt. 2

UM | | | |

少<a>へ 18 of 18</a>

୍ଥା | <mark>ଚ୍ଚ</mark> ୬ବନ 17 of 18

☐ Todo: Build Dalek army.

