



Mechanisms for Generating Power-Law Size Distributions, Part 1

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Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394, 2022–2023 | @pocsvox

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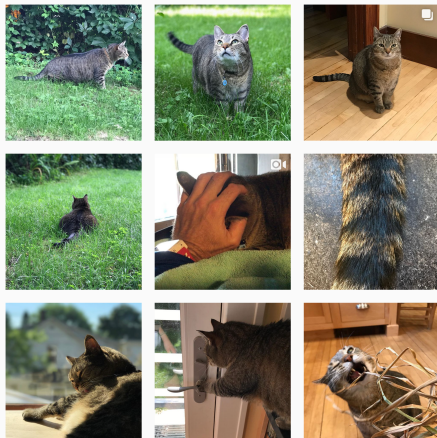
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Outline

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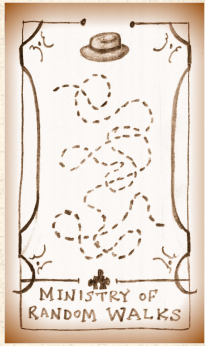
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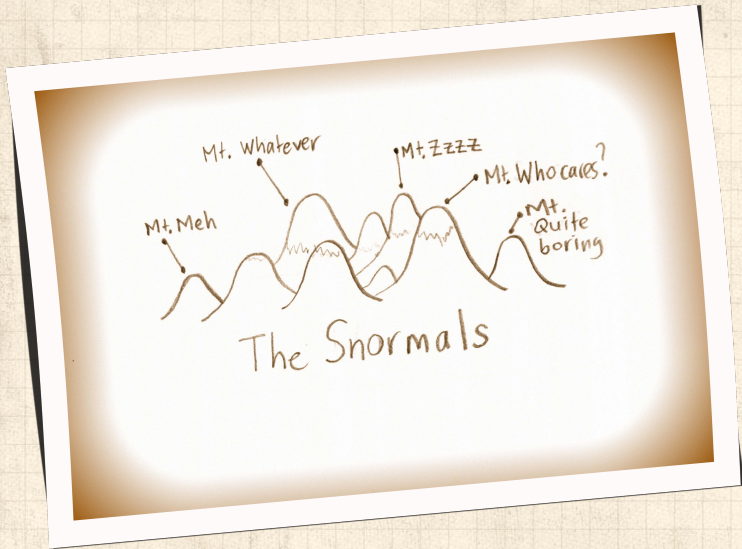
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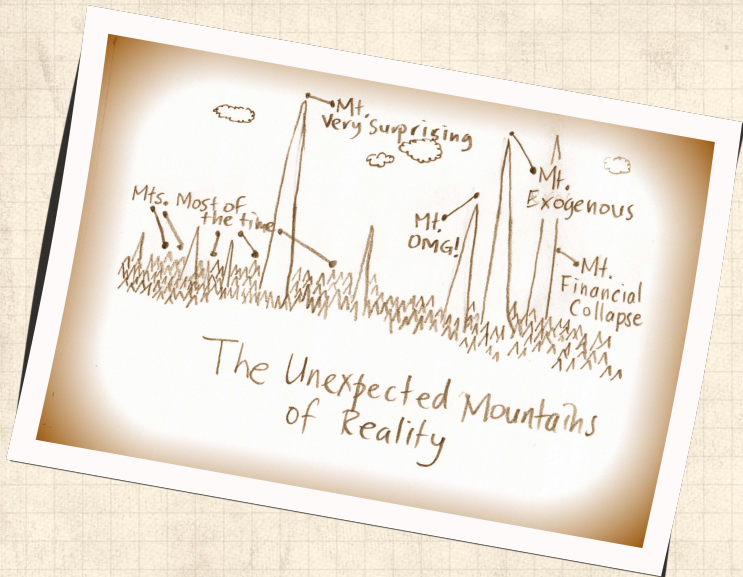






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- The First Return Problem
- Random River Networks
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A powerful story in the rise of complexity:

 structure arises out of randomness.

 **Exhibit A:** Random walks. 

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The essential random walk:

 One spatial dimension.

 Time and space are discrete

 Random walker (e.g., a zombie texter ) starts at origin $x = 0$.

 Step at time t is ϵ_t :

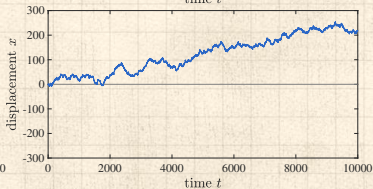
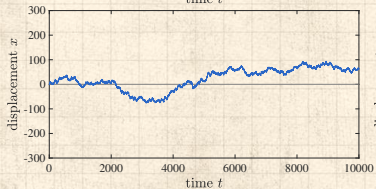
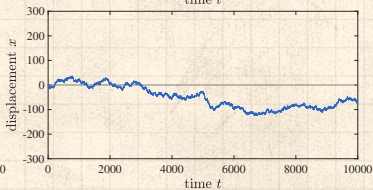
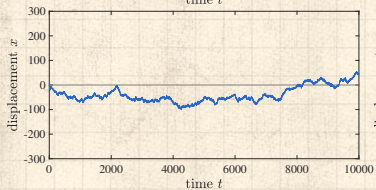
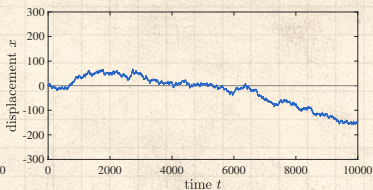
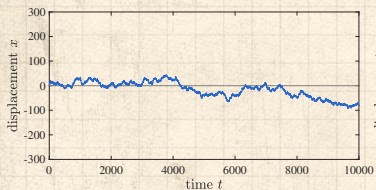
$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$



A few random random walks:

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Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^t \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \epsilon_i \right\rangle = \sum_{i=1}^t \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our zombie texter to be back at their starting place.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our texting undead friend lurching back to $x=0$ must diminish, right?

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Variations sum: *

$$\begin{aligned}\text{Var}(x_t) &= \text{Var}\left(\sum_{i=1}^t \epsilon_i\right) \\ &= \sum_{i=1}^t \text{Var}(\epsilon_i) = \sum_{i=1}^t 1 = t\end{aligned}$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

 A non-trivial scaling law arises out of additive aggregation or accumulation.

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Great moments in Televised Random Walks:

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<http://www.youtube.com/watch?v=05gqx6eSy00?rel=0>


[Plinko!](#) from the Price is Right.

 Also known as the [bean machine](#), the [quincunx \(simulation\)](#), and the Galton box.



Random walk basics:

Counting random walks:

- Each **specific** random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert question from assignment 5 

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

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How does $P(x_t)$ behave for large t ?

Take time $t = 2n$ to help ourselves.

$x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$


x_{2n} is even so set $x_{2n} = 2k$.

Using our expression $N(i, j, t)$ with $i = 0, j = 2k$, and $t = 2n$, we have


$$\Pr(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

For large n , the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\Pr(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

[Insert question from assignment 5](#) 

The whole is different from the parts. **#nutritious**

See also: [Stable Distributions](#) 

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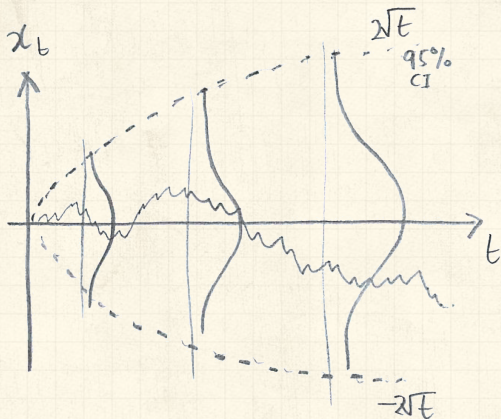
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Universality is also not left-handed:

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
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This is Diffusion : the most essential kind of spreading (more later).

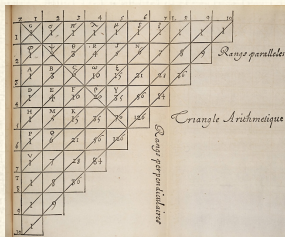



View as Random Additive Growth Mechanism.



So many things are connected:

Pascal's Triangle



Could have been the
Pyramid of Pingala ¹ or
the Triangle of Khayyam,
Jia Xian, Tartaglia, ...



Binomials tend towards the Normal.



Counting encoded in algebraic forms (and much more).



$(h + t)^n = \sum_{k=0}^n \binom{n}{k} h^k t^{n-k}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$



$(h + t)^3 = hhh + hht + hth + thh + htt + tht + tth + ttt$

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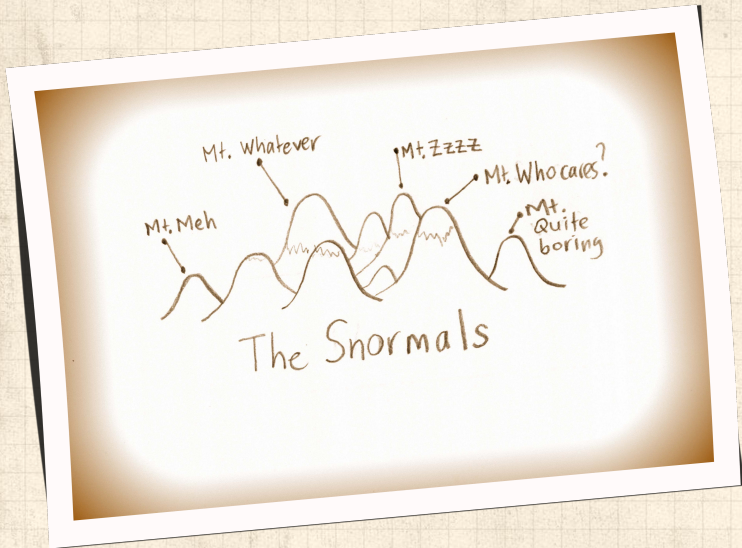
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¹Stigler's Law of Eponymy  showing excellent form again.



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The Random Road
through the Forests of Forgettable Events



Random walks are even weirder than you might think...

☇ $\xi_{r,t}$ = the probability that by time step t , a random walk has crossed the origin r times.

☇ Think of a coin flip game with ten thousand tosses.

☇ If you are behind early on, what are the chances you will make a comeback?

☇ The most likely number of lead changes is... 0.

☇ In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \dots$

☇ Even crazier:

The expected time between tied scores = ∞

See Feller, Intro to Probability Theory, Volume I ^[5]

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Applied knot theory:



“Designing tie knots by random walks” 

Fink and Mao,
Nature, **398**, 31–32, 1999. [6]

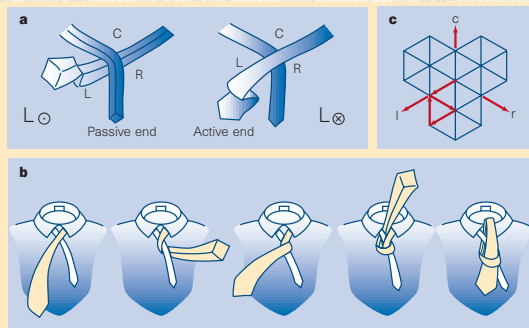


Figure 1 All diagrams are drawn in the frame of reference of the mirror image of the actual tie.
a. The two ways of beginning a knot, L_{\odot} and L_{\otimes} . For knots beginning with L_{\odot} , the tie must begin inside-out. **b.** The four-in-hand, denoted by the sequence $L_{\odot} R_{\odot} L_{\otimes} C_{\odot} T$. **c.** A knot may be represented by a persistent random walk on a triangular lattice. The example shown is the four-in-hand, indicated by the walk $\uparrow \uparrow \uparrow \&$.

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Table 1 **Aesthetic tie knots**

h	γ	γ/h	$K(h, \gamma)$	s	b	Name	Sequence
3	1	0.33	1	0	0		$L_0 R_0 C_0 T$
4	1	0.25	1	-1	1	Four-in-hand	$L_0 R_0 L_0 C_0 T$
5	2	0.40	2	-1	0	Pratt knot	$L_0 C_0 R_0 L_0 C_0 T$
6	2	0.33	4	0	0	Half-Windsor	$L_0 R_0 C_0 L_0 R_0 C_0 T$
7	2	0.29	6	-1	1		$L_0 R_0 L_0 C_0 R_0 L_0 C_0 T$
7	3	0.43	4	0	1		$L_0 C_0 R_0 C_0 L_0 R_0 C_0 T$
8	2	0.25	8	0	2		$L_0 R_0 L_0 C_0 R_0 L_0 R_0 C_0 T$
8	3	0.38	12	-1	0	Windsor	$L_0 C_0 R_0 L_0 C_0 R_0 L_0 C_0 T$
9	3	0.33	24	0	0		$L_0 R_0 C_0 L_0 R_0 C_0 L_0 R_0 C_0 T$
9	4	0.44	8	-1	2		$L_0 C_0 R_0 C_0 L_0 C_0 R_0 L_0 C_0 T$

Knots are characterized by half-winding number h , centre number γ , centre fraction γ/h , knots per class $K(h, \gamma)$, symmetry s , balance b , name and sequence.

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 h = number of moves

 γ = number of center moves

 $K(h, \gamma) = 2^{\gamma-1} \binom{h-\gamma-2}{\gamma-1}$

 $s = \sum_{i=1}^h x_i$ where $x = -1$ for L and $+1$ for R .

 $b = \frac{1}{2} \sum_{i=2}^{h-1} |\omega_i + \omega_{i-1}|$ where $\omega = \pm 1$ represents winding direction.



Random walks #crazytownbananapants

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The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our zombie texter always return to the origin?
- What about higher dimensions?

Reasons for caring:

- We will find a power-law size distribution with an interesting exponent.
- Some physical structures may result from random walks.
- We'll start to see how different scalings relate to each other.

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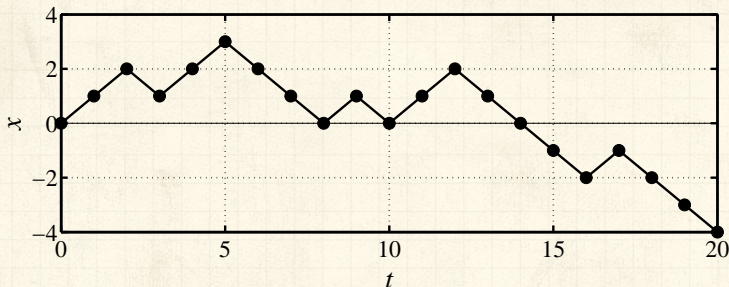
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For random walks in 1-d:



🧱 A **return** to origin can only happen when $t = 2n$.

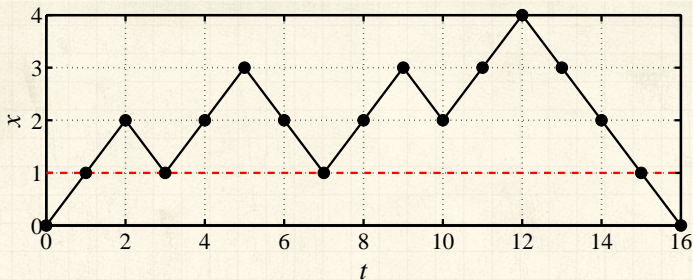
🧱 In example above, returns occur at $t = 8, 10,$ and 14 .

🧱 Call $P_{fr(2n)}$ the probability of **first return** at $t = 2n$.

🧱 Probability calculation \equiv Counting problem (combinatorics/statistical mechanics).

🧱 **Idea:** Transform first return problem into an easier return problem.





- Can assume zombie texter first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) = 2 \cdot \frac{1}{2} Pr(x_t \geq 1, 1 \leq t \leq 2n - 1, \text{ and } x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for texters that first lurch to $x = -1$.

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Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider **all paths** starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea:** If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ **excluded walks**.
- We'll use a method of images to identify these excluded walks.

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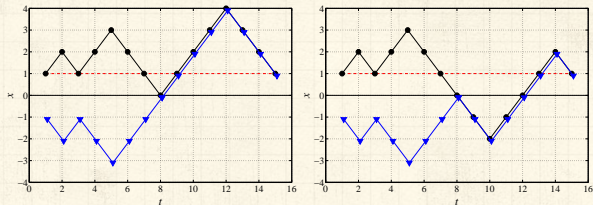
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Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t -step paths starting and ending at $x=1$ and hitting $x=0$ at least once
= # of t -step paths starting at $x=-1$ and ending at $x=1$ = $N(-1, 1, t)$
- So $N_{\text{first return}}(2n) = N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)$





Probability of first return:

Insert question from assignment 5 ↗ :

 Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}}.$$

 Normalized number of paths gives probability.

 Total number of possible paths = 2^{2n} .



$$\begin{aligned} P_{\text{fr}}(2n) &= \frac{1}{2^{2n}} N_{\text{fr}}(2n) \\ &\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi n^{3/2}}} \\ &= \frac{1}{\sqrt{2\pi}} (2n)^{-3/2} \propto t^{-3/2}. \end{aligned}$$

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🧱 We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.

🧱 Same scaling holds for continuous space/time walks.

🧱 $P(t)$ is normalizable.

🧱 **Recurrence:** Random walker always returns to origin

🧱 But mean, variance, and all higher moments are infinite. #totalmadness


🧱 Even though walker must return, expect a long wait...

🧱 **One moral:** Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions

🧱 Walker in $d = 2$ dimensions must also return

🧱 Walker may not return in $d \geq 3$ dimensions

🧱 Associated human genius: George Pólya 






Random walks



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On finite spaces:

-  In any finite homogeneous space, a random walker will visit every site with equal probability
-  Call this probability the **Invariant Density** of a dynamical system
-  Non-trivial Invariant Densities arise in chaotic systems.

On networks:

-  On networks, a random walker visits each node with frequency \propto node degree **#groovy**
-  Equal probability still present: walkers traverse **edges** with equal frequency. **#totallygroovy**

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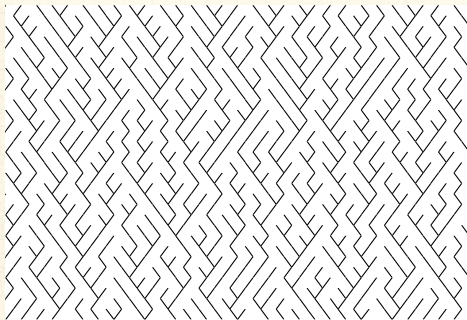
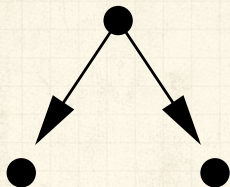
References






Scheidegger Networks [17, 4]

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-  Random directed network on triangular lattice.
-  Toy model of real networks.
-  'Flow' is southeast or southwest with equal probability.

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Scheidegger networks

- Creates basins with random walk boundaries.
- Observe** that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

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Connections between exponents:

 For a basin of length ℓ , width $\propto \ell^{1/2}$

 Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

 Invert: $\ell \propto a^{2/3}$

 $d\ell \propto d(a^{2/3}) = 2/3 a^{-1/3} da$

 $\Pr(\text{basin area} = a) da$
 $= \Pr(\text{basin length} = \ell) d\ell$
 $\propto \ell^{-3/2} d\ell$
 $\propto (a^{2/3})^{-3/2} a^{-1/3} da$
 $= a^{-4/3} da$
 $= a^{-\tau} da$

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Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:


- Hack's law^[10]:

$$\ell \propto a^h.$$


- For real, large networks^[13] $h \simeq 0.5$ (isometric scaling)
- Smaller basins possibly $h > 1/2$ (allometric scaling).
- Models exist with interesting values of h .
- Plan: Redo calc with γ , τ , and h .





Connections between exponents:

 Given

$$\ell \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$


 $d\ell \propto d(a^h) = ha^{h-1}da$

 Find τ in terms of γ and h .

 $\Pr(\text{basin area} = a)da$
 $= \Pr(\text{basin length} = \ell)d\ell$
 $\propto \ell^{-\gamma}d\ell$
 $\propto (a^h)^{-\gamma}a^{h-1}da$
 $= a^{-(1+h(\gamma-1))}da$



$$\tau = 1 + h(\gamma - 1)$$

 Excellent example of the **Scaling Relations** found between exponents describing power laws for many systems.

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Connections between exponents:

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies to:^[3]

$$\tau = 2 - h$$

and

$$\gamma = 1/h$$

-  Only one exponent is independent (take h).
-  Simplifies system description.
-  Expect Scaling Relations where power laws are found.
-  Need only characterize Universality  class with independent exponents.

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Failure:


 A very simple model of failure/death

 x_t = entity's 'health' at time t

 Start with $x_0 > 0$.

 Entity fails when x hits 0.



["Explaining mortality rate plateaus"](#) 

Weitz and Fraser,
Proc. Natl. Acad. Sci., **98**, 15383–15386,
2001. [18]

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... and the NBA:

Basketball and other sports [2]:

- Three arcsine laws ↗ (Lévy [12]) for continuous-time random walk last time T :

$$\frac{1}{\pi} \frac{1}{\sqrt{t(T-t)}}$$

The arcsine distribution ↗ applies for:
(1) fraction of time positive, (2) the last time the walk changes sign,
and (3) the time the maximum is achieved.

- Well approximated by basketball score lines [8, 2].
- Australian Rules Football has some differences [11].

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
Fractional
Brownian Motion


References




More than randomness

 Can generalize to Fractional Random Walks ^[15, 16, 14]

 Fractional Brownian Motion , Lévy flights 

 See Montroll and Shlesinger for example: ^[14]
"On $1/f$ noise and other distributions with long tails."

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 In 1-d, standard deviation σ scales as

$$\sigma \sim t^\alpha$$

$\alpha = 1/2$ — diffusive

$\alpha > 1/2$ — superdiffusive

$\alpha < 1/2$ — subdiffusive

 Extensive memory of path now matters...

Random Walks

The First Return Problem

Random River Networks

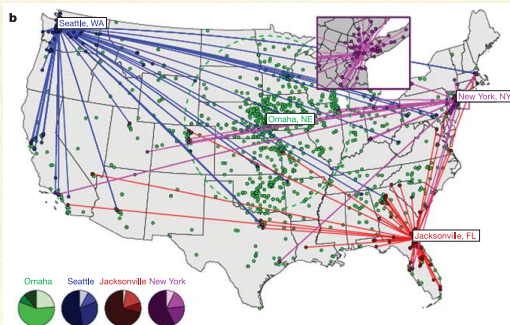
Scaling Relations





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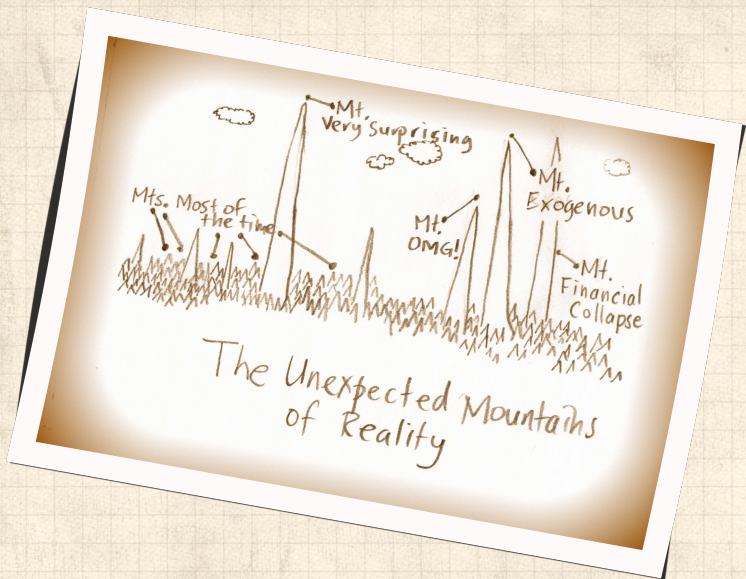
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-  First big studies of movement and interactions of people.
-  Brockmann *et al.* ^[1] “Where’s George” study.
-  Beyond Lévy: Superdiffusive in space but with long waiting times.
-  Tracking movement via cell phones ^[9] and Twitter ^[7].





- Random Walks
- The First Return Problem
- Random River Networks
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- Fractional Brownian Motion
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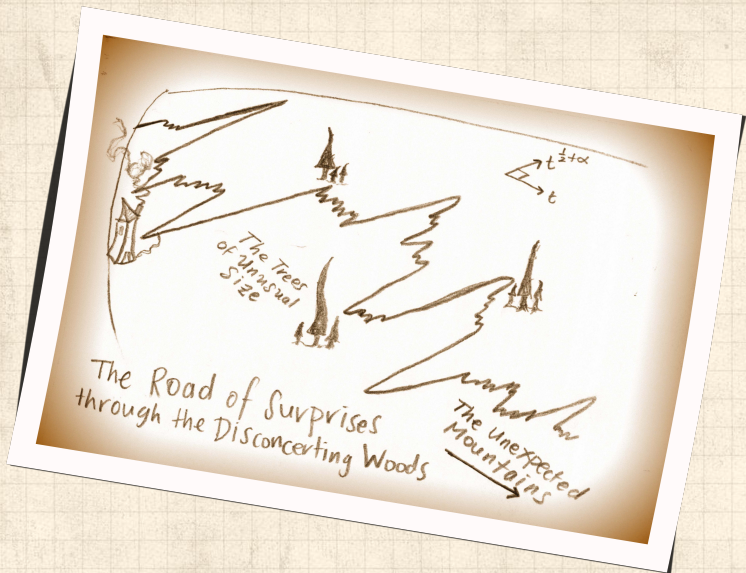
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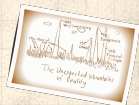
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