

These slides are brought to you by:

PoCS
@pocsvox

Properties of
Complex
Networks

Sealie & Lambie
Productions



Properties of
Complex
Networks

A problem
Degree distributions
Assortativity
Clustering
Motifs
Concurrency
Branching ratios
Network distances
Interconnectedness

Nutshell

References



These slides are also brought to you by:

Special Guest Executive Producer



PoCS
@pocsvox

Properties of
Complex
Networks



Properties of
Complex
Networks

- A problem
- Degree distributions
- Assortativity
- Clustering
- Motifs
- Concurrency
- Branching ratios
- Network distances
- Interconnectedness

Nutshell

References



 On Instagram at [pratchett_the_cat](https://www.instagram.com/pratchett_the_cat) 



Outline

PoCS
@pocsvox

Properties of
Complex
Networks

Properties of Complex Networks

- A problem
- Degree distributions
- Assortativity
- Clustering
- Motifs
- Concurrency
- Branching ratios
- Network distances
- Interconnectedness

Properties of
Complex
Networks

- A problem
- Degree distributions
- Assortativity
- Clustering
- Motifs
- Concurrency
- Branching ratios
- Network distances
- Interconnectedness

Nutshell


References

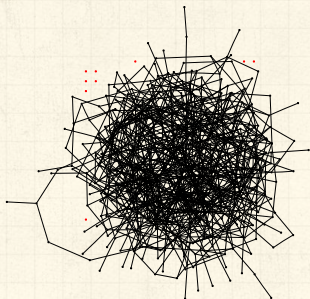
Nutshell

References







A notable feature of large-scale networks:


 Graphical renderings are often just a big mess.



⇐ Typical hairball

-  number of nodes $N = 500$
-  number of edges $m = 1000$
-  average degree $\langle k \rangle = 4$

 And even when renderings somehow look good:
“That is a very graphic analogy which aids understanding wonderfully while being, strictly speaking, wrong in every possible way”
said Ponder [Stibbons] —*Making Money*, T. Pratchett.

 We need to extract **digestible, meaningful aspects**.

A problem


Degree distributions
Assortativity
Clustering
Motifs
Concurrency
Branching ratios
Network distances
Interconnectedness


Nutshell


References





Some key aspects of real complex networks:


 degree distribution*

 assortativity


 homophily


 clustering


 motifs


 modularity




 hierarchical scaling


 concurrency


 network distances

 centrality

 multilayeriness

 efficiency


 robustness


 Plus coevolution of network structure and processes on networks.


* Degree distribution is the elephant in the room that we are now all very aware of...




1. degree distribution P_k

 P_k is the probability that a randomly selected node has degree k .


 k = node degree = number of connections.


 **ex 1:** Erdős-Rényi random networks have Poisson degree distributions:

Insert question from assignment 7 

$$P_k = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

 **ex 2: "Scale-free" networks:** $P_k \propto k^{-\gamma} \Rightarrow$ 'hubs'.

 link cost controls skew.

 hubs may facilitate or impede contagion.



Note:

- ⊞ Erdős-Rényi random networks are a *mathematical construct*.
- ⊞ 'Scale-free' networks are **growing networks** that form according to a **plausible mechanism**.
- ⊞ Randomness is out there, just not to the degree of a completely random network.

A problem

Degree distributions

Assortativity

Clustering

Motifs

Concurrency

Branching ratios

Network distances



Interconnectedness


Nutshell


References




2. Assortativity/3. Homophily:

 Social networks: Homophily  = birds of a feather

 e.g., degree is standard property for sorting:
measure degree-degree correlations.

 **Assortative** network: ^[5] similar degree nodes
connecting to each other.
Often social: company directors, coauthors, actors.

 **Disassortative** network: high degree nodes
connecting to low degree nodes.
*Often technological or biological: Internet, WWW,
protein interactions, neural networks, food webs.*



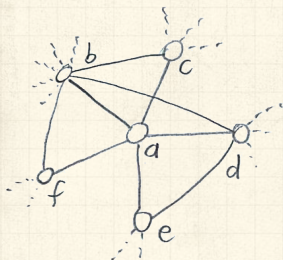
4. Clustering:



Your friends tend to know each other.



Two measures (explained on following slides):



1. Watts & Strogatz [8]

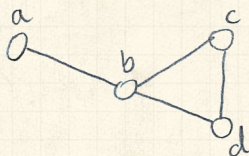
$$C_1 = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i$$

2. Newman [6]

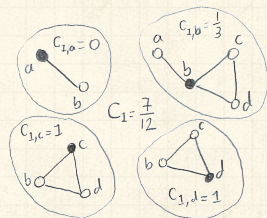
$$C_2 = \frac{3 \times \# \text{triangles}}{\# \text{triples}}$$





Example network:



Calculation of C_1 :




 C_1 is the **average fraction of pairs of neighbors who are connected**.

 Fraction of pairs of neighbors who are connected is

$$\frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2}$$

where k_i is node i 's degree, and \mathcal{N}_i is the set of i 's neighbors.

 Averaging over all nodes, we have:

$$C_1 = \frac{1}{n} \sum_{i=1}^n \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i$$

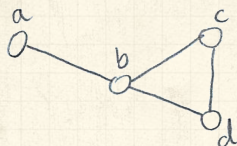
Nutshell

References

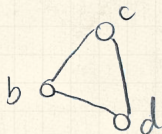


Triples and triangles

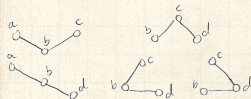
Example network:



Triangles:



Triples:



Nodes i_1 , i_2 , and i_3 form a **triple** around i_1 if i_1 is connected to i_2 and i_3 .



Nodes i_1 , i_2 , and i_3 form a **triangle** if each pair of nodes is connected



The definition $C_2 = \frac{3 \times \# \text{triangles}}{\# \text{triples}}$ measures the fraction of **closed triples**



The '3' appears because for each triangle, we have 3 closed triples.



Social Network Analysis (SNA): fraction of **transitive triples**.



Clustering:

Sneaky counting for undirected, unweighted networks:

- ☰ If the path $i-j-l$ exists then $a_{ij}a_{jl} = 1$.
- ☰ Otherwise, $a_{ij}a_{jl} = 0$.
- ☰ We want $i \neq l$ for good triples.
- ☰ In general, a path of n edges between nodes i_1 and i_n travelling through nodes i_2, i_3, \dots, i_{n-1} exists $\iff a_{i_1 i_2} a_{i_2 i_3} a_{i_3 i_4} \cdots a_{i_{n-2} i_{n-1}} a_{i_{n-1} i_n} = 1$.



$$\# \text{triples} = \frac{1}{2} \left(\sum_{i=1}^N \sum_{\ell=1}^N [A^2]_{i\ell} - \text{Tr} A^2 \right)$$



$$\# \text{triangles} = \frac{1}{6} \text{Tr} A^3$$



Properties

- For sparse networks, C_1 tends to discount highly connected nodes.
- C_2 is a useful and often preferred variant
- In general, $C_1 \neq C_2$.
- C_1 is a global average of a local ratio.
- C_2 is a ratio of two global quantities.

PoCS
@pocsvox

Properties of
Complex
Networks

Properties of
Complex
Networks



A problem
Degree distributions
Assortativity
Clustering
Motifs
Concurrency
Branching ratios
Network distances
Interconnectedness

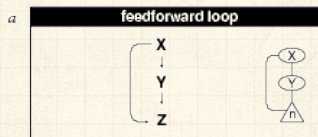
Nutshell

References



5. motifs:

-  small, recurring functional subnetworks
-  e.g., Feed Forward Loop:



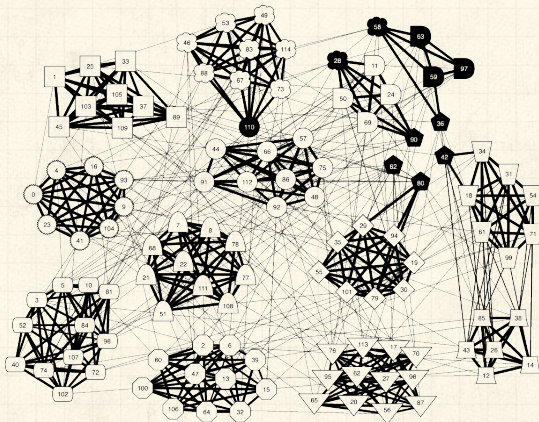
Shen-Orr, Uri Alon, *et al.* [7]

A problem
Degree distributions
Assortativity
Clustering
Motifs
Concurrency
Branching ratios
Network distances
Interconnectedness



Properties

6. modularity and structure/community detection:



Clauset *et al.*, 2006 ^[2]: NCAA football

PoCS
@pocsvox

Properties of
Complex
Networks

Properties of
Complex
Networks

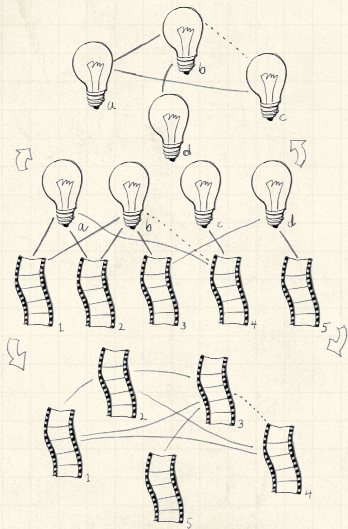
- A problem
- Degree distributions
- Assortativity
- Clustering
- Motifs**
- Concurrency
- Branching ratios
- Network distances
- Interconnectedness


Nutshell


References




Bipartite/multipartite affiliation structures:



 Many real-world networks have an underlying multi-partite structure.

-  Stories-tropes.
-  Boards and directors.
-  Films-actors-directors.
-  Classes-teachers-students.
-  Upstairs-downstairs.

 Unipartite networks may be induced or co-exist.

PoCS
@pocsvox

Properties of
Complex
Networks

Properties of
Complex
Networks

A problem
Degree distributions
Assortativity
Clustering
Motifs
Concurrency
Branching ratios
Network distances
Interconnectedness

Nutshell

References



7. concurrency:

- transmission of a contagious element only occurs during contact
- rather obvious but easily missed in a simple model
- dynamic property—static networks are not enough
- knowledge of previous contacts crucial
- beware cumulated network data
- Kretzschmar and Morris, 1996 ^[4]
- “Temporal networks” become a concrete area of study for Piranha Physicus in 2013.



8. Horton-Strahler ratios:



Metrics for branching networks:

- Method for ordering streams hierarchically
- Number: $R_n = N_\omega / N_{\omega+1}$
- Segment length: $R_l = \langle l_{\omega+1} \rangle / \langle l_\omega \rangle$
- Area/Volume: $R_a = \langle a_{\omega+1} \rangle / \langle a_\omega \rangle$



Properties of
Complex
Networks

A problem
Degree distributions
Assortativity
Clustering
Motifs
Concurrency
Branching ratios
Network distances
Interconnectedness

Nutshell

References



9. network distances:

(a) shortest path length d_{ij} :






- 🧱 Fewest number of steps between nodes i and j .
- 🧱 (Also called the chemical distance between i and j .)

(b) average path length $\langle d_{ij} \rangle$:

- 🧱 Average shortest path length in whole network.
- 🧱 Good algorithms exist for calculation.
- 🧱 Weighted links can be accommodated.





9. network distances:


-  **network diameter** d_{\max} :
Maximum shortest path length between any two nodes.
-  **closeness** $d_{cl} = [\sum_{ij} d_{ij}^{-1} / \binom{n}{2}]^{-1}$:
Average 'distance' between any two nodes.
-  Closeness handles disconnected networks
($d_{ij} = \infty$)
-  $d_{cl} = \infty$ only when all nodes are isolated.
-  Closeness perhaps compresses too much into one number





10. centrality:

 Many such measures of a node's 'importance.'

 **ex 1:** Degree centrality: k_i .

 **ex 2:** Node i 's betweenness
= fraction of shortest paths that pass through i .

 **ex 3:** Edge ℓ 's betweenness
= fraction of shortest paths that travel along ℓ .

 **ex 4:** Recursive centrality: Hubs and Authorities
(Jon Kleinberg [3])



Properties

PoCS
@pocsvox

Properties of
Complex
Networks

Interconnected networks and robustness (two for one deal):

“Catastrophic cascade of failures in interdependent networks” [1]. Buldyrev et al., Nature 2010.

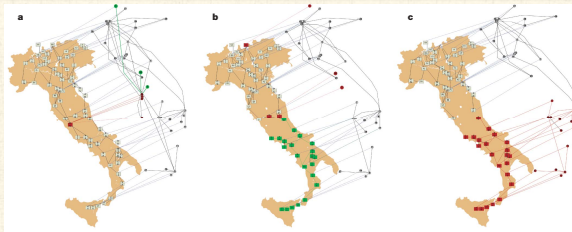


Figure 1 | Modelling a blackout in Italy. Illustration of an iterative process of a cascade of failures using real-world data from a power network (located on the map of Italy) and an Internet network (shifted above the map) that were implicated in an electrical blackout that occurred in Italy in September 2003³⁹. The networks are drawn using the real geographical locations and every Internet server is connected to the geographically nearest power station. **a.** One power station is removed (red node on map) from the power network and as a result the Internet nodes depending on it are removed from the Internet network (red nodes above the map). The nodes that will be disconnected from the giant cluster (a cluster that spans the entire network)

at the next step are marked in green. **b.** Additional nodes that were disconnected from the Internet communication network giant component are removed (red nodes above map). As a result the power stations depending on them are removed from the power network (red nodes on map). Again, the nodes that will be disconnected from the giant cluster at the next step are marked in green. **c.** Additional nodes that were disconnected from the giant component of the power network are removed (red nodes on map) as well as the nodes in the Internet network that depend on them (red nodes above map).

Properties of
Complex
Networks






A problem
Degree distributions
Assortativity
Clustering
Motifs
Concurrency
Branching ratios
Network distances
Interconnectedness

Nutshell

References



Overview Key Points:

-  The field of complex networks came into existence in the late 1990s.
-  Explosion of papers and interest since 1998/99.
-  Hardened up much thinking about complex systems.
-  Specific focus on networks that are **large-scale**, **sparse**, **natural** or **man-made**, **evolving** and **dynamic**, and (crucially) **measurable**.
-  Three main (blurred) categories:
 1. **Physical** (e.g., river networks),
 2. **Interactional** (e.g., social networks),
 3. **Abstract** (e.g., thesauri).

Properties of Complex Networks

A problem
Degree distributions
Assortativity
Clustering
Motifs
Concurrency
Branching ratios
Network distances
Interconnectedness

Nutshell

References



PoCS
@pocsvox

Properties of Complex Networks

Properties of Complex Networks

A problem
Degree distributions
Assortativity
Clustering
Motifs
Concurrency
Branching ratios
Network distances
Interconnectedness

Nutshell

References



References I

- [1] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin.

Catastrophic cascade of failures in interdependent networks.

[Nature](#), 464:1025–1028, 2010. pdf ↗

- [2] A. Clauset, C. Moore, and M. E. J. Newman.

Structural inference of hierarchies in networks,

2006. pdf ↗

- [3] J. M. Kleinberg.

Authoritative sources in a hyperlinked environment.

[Proc. 9th ACM-SIAM Symposium on Discrete Algorithms](#), 1998. pdf ↗



References II

- [4] M. Kretzschmar and M. Morris.
Measures of concurrency in networks and the spread of infectious disease.
[Math. Biosci., 133:165–95, 1996. pdf](#)
- [5] M. Newman.
Assortative mixing in networks.
[Phys. Rev. Lett., 89:208701, 2002. pdf](#)
- [6] M. E. J. Newman.
The structure and function of complex networks.
[SIAM Rev., 45\(2\):167–256, 2003. pdf](#)
- [7] S. S. Shen-Orr, R. Milo, S. Mangan, and U. Alon.
Network motifs in the transcriptional regulation network of *Escherichia coli*.
[Nature Genetics, 31:64–68, 2002. pdf](#)



- [8] D. J. Watts and S. J. Strogatz.
Collective dynamics of 'small-world' networks.
[Nature, 393:440–442, 1998.](#) pdf 