# Mixed, correlated random networks

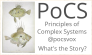
Last updated: 2022/08/28, 08:34:20 EDT

## Principles of Complex Systems, Vols. 1, 2, & 3D
## CSYS/MATH 300, 303, & 394, 2022–2023 | @pocsvox


Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont







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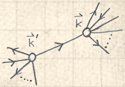

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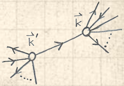

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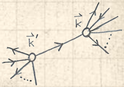
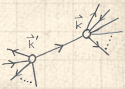


# Outline






THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TAILS OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE BOOKS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
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THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
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BIO HAZARD
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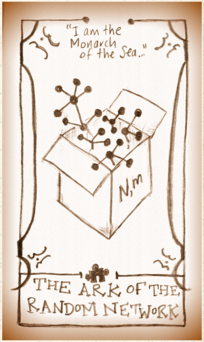
THE CHAOS OF FETTERED CONTAGION
THE END


"I am the Monarch of the Sea..."
N,m
THE ARK OF THE RANDOM NETWORK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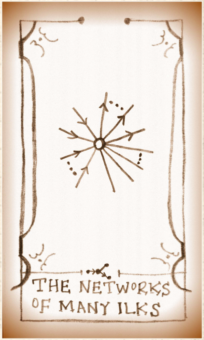
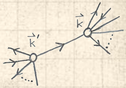

THE NETWORKS OF MANY ILKS

# Random directed networks:


- So far, we've largely studied networks with undirected, unweighted edges.







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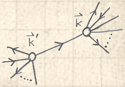
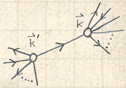


# Random directed networks:

- So far, we've largely studied networks with undirected, unweighted edges.
- Now consider directed, unweighted edges.
- Nodes have $k_{\mathrm{i}}$ and $k_{\mathrm{o}}$ incoming and outgoing edges, otherwise random.







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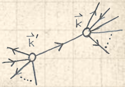

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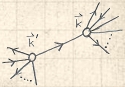

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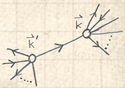
- Marginal in-degree and out-degree distributions:

$$P_{k_{\mathrm{i}}} = \sum_{k_{\mathrm{o}}=0}^{\infty} P_{k_{\mathrm{i}},k_{\mathrm{o}}} \text{ and } P_{k_{\mathrm{o}}} = \sum_{k_{\mathrm{i}}=0}^{\infty} P_{k_{\mathrm{i}},k_{\mathrm{o}}}$$



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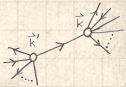
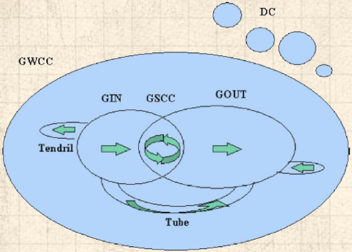
- Required balance:

$$\langle k_{\mathrm{i}} \rangle = \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{i}} P_{k_{\mathrm{i}},k_{\mathrm{o}}} = \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{o}} P_{k_{\mathrm{i}},k_{\mathrm{o}}} = \langle k_{\mathrm{o}} \rangle$$

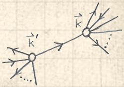


# Directed network structure:




From Boguñá and Serano. [1]

- GWCC = Giant Weakly Connected Component (directions removed);
- GIN = Giant In-Component;
- GOUT = Giant Out-Component;
- GSCC = Giant Strongly Connected Component;
- DC = Disconnected Components (finite).



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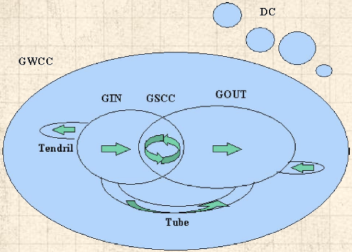


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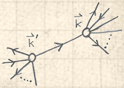
- When moving through a family of increasingly connected directed random networks, GWCC usually appears before GIN, GOUT, and GSCC which tend to appear together. [4, 1]

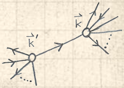


# Outline







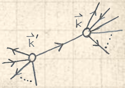


# Important observation:

- Directed and undirected random networks are separate families ...







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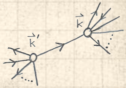
- Directed and undirected random networks are separate families ...
- ...and analyses are also disjoint.







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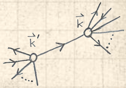
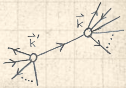
- Directed and undirected random networks are separate families ...
- ...and analyses are also disjoint.
- Need to examine a larger family of random networks with mixed directed and undirected edges.









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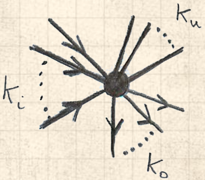


- Consider nodes with three types of edges:
  1. $k_u$ undirected edges,
  2. $k_i$ incoming directed edges,
  3. $k_o$ outgoing directed edges.

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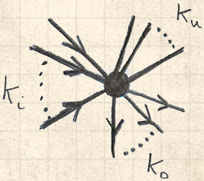
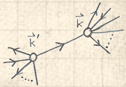



- Consider nodes with three types of edges:
  1. $k_{\mathrm{u}}$ undirected edges,
  2. $k_{\mathrm{i}}$ incoming directed edges,
  3. $k_{\mathrm{o}}$ outgoing directed edges.
- Define a node by generalized degree:

$$\vec{k} = [\, k_{\mathrm{u}} \;\; k_{\mathrm{i}} \;\; k_{\mathrm{o}} \,]^{\mathsf{T}}.$$

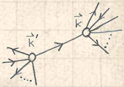


# Joint degree distribution:

$$P_{\vec{k}} \text{ where } \vec{k} = [\, k_{\mathrm{u}} \;\; k_{\mathrm{i}} \;\; k_{\mathrm{o}} \,]^{\mathsf{T}}.$$







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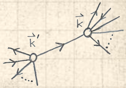
- As for directed networks, require in- and out-degree averages to match up:

$$\langle k_{\mathrm{i}} \rangle = \sum_{k_{\mathrm{u}}=0}^{\infty} \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{i}} P_{\vec{k}} = \sum_{k_{\mathrm{u}}=0}^{\infty} \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{o}} P_{\vec{k}} = \langle k_{\mathrm{o}} \rangle$$







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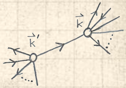
- Otherwise, no other restrictions and connections are random.



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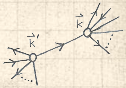
- Otherwise, no other restrictions and connections are random.
- Directed and undirected random networks are disjoint subfamilies:

$$\text{Undirected: } P_{\vec{k}} = P_{k_{\mathsf{u}}} \delta_{k_{\mathsf{i}},0} \delta_{k_{\mathsf{o}},0},$$

$$\text{Directed: } P_{\vec{k}} = \delta_{k_{\mathsf{u}},0} P_{k_{\mathsf{i}},k_{\mathsf{o}}}.$$



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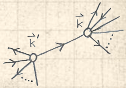
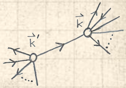


# Correlations:

- Now add correlations (two point or Markovian) ☐:







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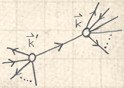

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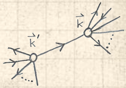
  2. $P^{(\mathrm{i})}(\vec{k} \,|\, \vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an in-directed edge relative to the destination node.



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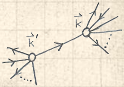

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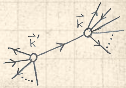

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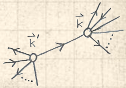

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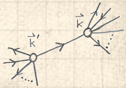

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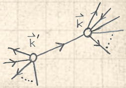

# Correlations—Undirected edge balance:





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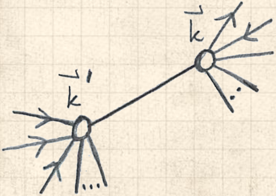


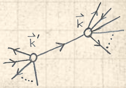

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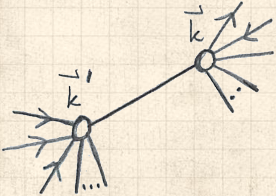


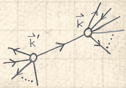

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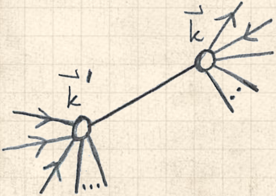






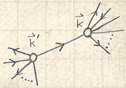

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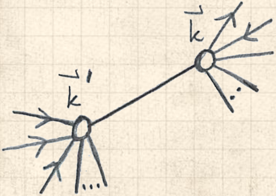

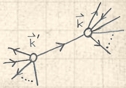

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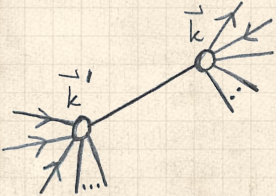


- Conditional probability connection:

$$P^{(\mathrm{u})}(\vec{k},\vec{k}') = P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')\frac{k'_{\mathrm{u}}P(\vec{k}')}{\langle k'_{\mathrm{u}}\rangle}$$

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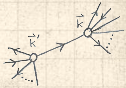

# Correlations—Directed edge balance:

- The quantities

$$\frac{k_{\text{o}} P(\vec{k})}{\langle k_{\text{o}} \rangle} \text{ and } \frac{k_{\text{i}} P(\vec{k})}{\langle k_{\text{i}} \rangle}$$

give the probabilities that in starting at a random end of a randomly selected edge, we begin at a degree $\vec{k}$ node and then find ourselves travelling:

1. along an outgoing edge, or
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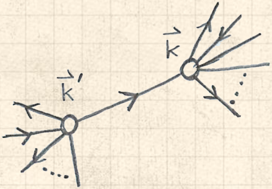






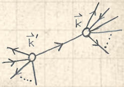

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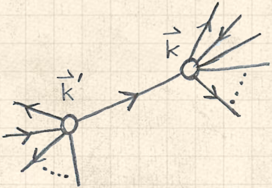


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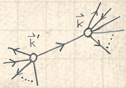
- We therefore have

$$P^{(\mathrm{dir})}(\vec{k}, \vec{k}') = P^{(\mathrm{i})}(\vec{k} \,|\, \vec{k}') \frac{k'_{\mathrm{o}} P(\vec{k}')}{\langle k'_{\mathrm{o}} \rangle} = P^{(\mathrm{o})}(\vec{k}' \,|\, \vec{k}) \frac{k_{\mathrm{i}} P(\vec{k})}{\langle k_{\mathrm{i}} \rangle}.$$



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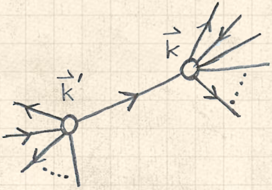


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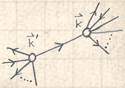


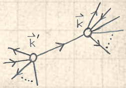

- Note that $P^{(\mathrm{dir})}(\vec{k}, \vec{k}')$ and $P^{(\mathrm{dir})}(\vec{k}', \vec{k})$ are in general not related if $\vec{k} \neq \vec{k}'$.

# Outline







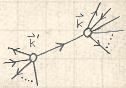


# Global spreading condition: [2]

## When are cascades possible?:







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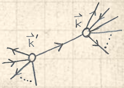
- Consider uncorrelated mixed networks first.







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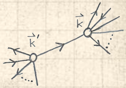
$$\mathbf{R} = \sum_{k_u=0}^{\infty} \frac{k_u P_{k_u}}{\langle k_u \rangle} \bullet (k_u - 1) \bullet B_{k_u,1} > 1.$$







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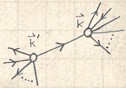

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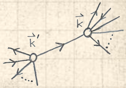
- Both are composed of (1) probability of connection to a node of a given type; (2) number of newly infected edges if successful; and (3) probability of infection.







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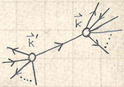
## Local growth equation:

- Define number of infected edges leading to nodes a distance $d$ away from the original seed as $f(d)$.







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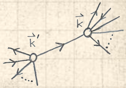
$$f(d+1) = \mathbf{R} f(d).$$



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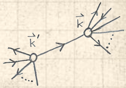

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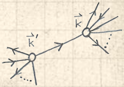

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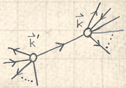
- Also allows for recovery of nodes (SIR).



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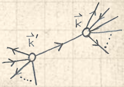
- Now have two types of edges spreading infection: directed and undirected.







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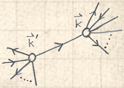

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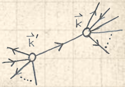


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- Gain ratio now more complicated:
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- Define $f^{(u)}(d)$ and $f^{(o)}(d)$ as the expected number of infected undirected and directed edges leading to nodes a distance $d$ from seed.

Gain ratio now has a matrix form:

$$\begin{bmatrix} f^{(\mathsf{u})}(d+1) \\ f^{(\mathsf{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathsf{u})}(d) \\ f^{(\mathsf{o})}(d) \end{bmatrix}$$

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- Two separate gain equations:

$$f^{(\mathrm{u})}(d+1) = \sum_{\vec{k}} \left[ \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{u})}(d) + \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{u}} \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{o})}(\right.$$

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$$\begin{bmatrix} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{bmatrix}$$

- Two separate gain equations:

$$f^{(\mathrm{u})}(d+1) = \sum_{\vec{k}} \left[ \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{u})}(d) + \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{u}} \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{o})}($$

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$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) & \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{u}} \\ \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet k_{\mathrm{o}} & \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1}$$

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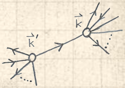
- Spreading condition: max eigenvalue of $\mathbf{R} > 1$.

# Global spreading condition:

- Useful change of notation for making results more general: write $P^{(\mathrm{u})}(\vec{k}\,|\,*) = \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle}$ and $P^{(\mathrm{i})}(\vec{k}\,|\,*) = \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle}$ where $*$ indicates the starting node's degree is irrelevant (no correlations).





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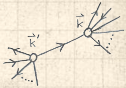
- Also write $B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$ to indicate a more general infection probability, but one that does not depend on the edge's origin.



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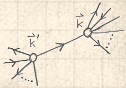
- Now have, for the example of mixed, uncorrelated random networks:

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet (k_{\mathrm{u}} - 1) & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{u}} \\ P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$$



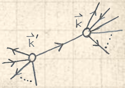
# Summary of contagion conditions for uncorrelated networks:

- I. Undirected, Uncorrelated—$f(d+1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_u} P^{(u)}(k_u \mid *) \bullet (k_u - 1) \bullet B_{k_u,*}$$







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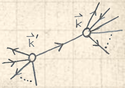

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- III. Mixed Directed and Undirected, Uncorrelated—

$$\begin{bmatrix} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{bmatrix}$$

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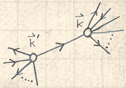
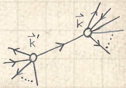


# Correlated version:

- Now have to think of transfer of infection from edges emanating from degree $\vec{k}'$ nodes to edges emanating from degree $\vec{k}$ nodes.



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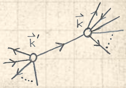

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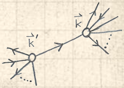

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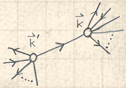

# Summary of contagion conditions for correlated networks:

- IV. Undirected, Correlated— $f_{k_u}(d+1) = \sum_{k'_u} R_{k_u k'_u} f_{k'_u}(d)$

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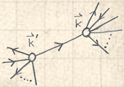

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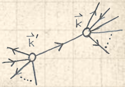

# Summary of contagion conditions for correlated networks:

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- VI. Mixed Directed and Undirected, Correlated—

$$\begin{bmatrix} f^{(\mathrm{u})}_{\vec{k}}(d+1) \\ f^{(\mathrm{o})}_{\vec{k}}(d+1) \end{bmatrix} = \sum_{k'} \mathbf{R}_{\vec{k}\vec{k}'} \begin{bmatrix} f^{(\mathrm{u})}_{\vec{k}'}(d) \\ f^{(\mathrm{o})}_{\vec{k}'}(d) \end{bmatrix}$$

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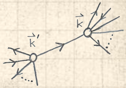
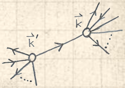


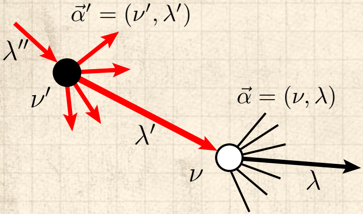
# Outline







# Full generalization:




$$f_{\vec{\alpha}}(d+1) = \sum_{\vec{\alpha}'} R_{\vec{\alpha}\vec{\alpha}'} f_{\vec{\alpha}'}(d)$$

$R_{\vec{\alpha}\vec{\alpha}'}$ is the gain ratio matrix and has the form:

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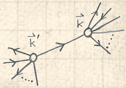

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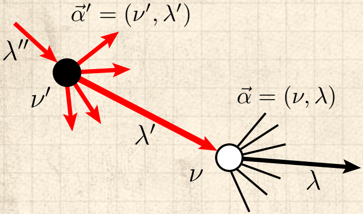


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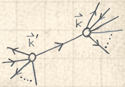

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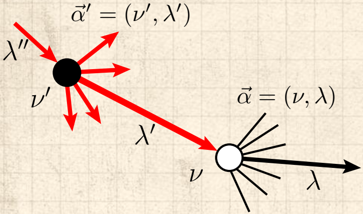


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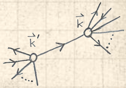

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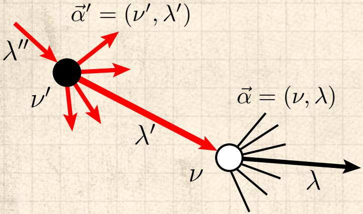


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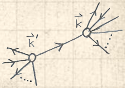

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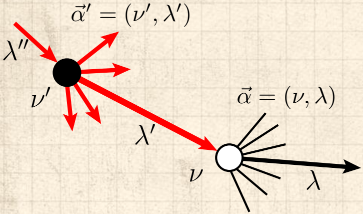


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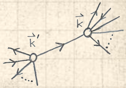
- Generalized contagion condition:

$$\max|\mu| : \mu \in \sigma(\mathbf{R}) > 1$$



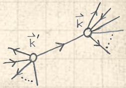



# Outline









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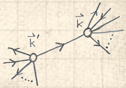
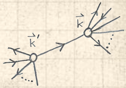
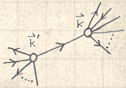


- As we saw earlier, the triggering probability for simple contagion on random networks can be determined with a straightforward physical argument.
- Two good things:

$$Q_{\text{trig}} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet B_{k1} \bullet \left[1 - \left(1 - Q_{\text{trig}}\right)^{k-1}\right],$$

$$P_{\text{trig}} = S_{\text{trig}} = \sum_{k} P_k \bullet \left[1 - (1 - Q_{\text{trig}})^{k}\right].$$



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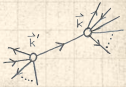

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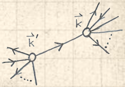
- Equivalent to result found via the eldritch route of generating functions.
- Generating functions arguably make some kinds of calculations easier (but perhaps we don't care about component sizes that much).
- On the other hand, a plainspoken physical argument helps us generalize to correlated networks more easily.



# Summary of triggering probabilities for uncorrelated networks: [3]

I. Undirected, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_{\text{u}}} P^{(\text{u})}(k'_{\text{u}} \mid \cdot) B_{k'_{\text{u}}1} \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{u}}-1}\right]$$

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- II. Directed, Uncorrelated—

$$Q_{\mathrm{trig}} = \sum_{k'_{\mathrm{i}}, k'_{\mathrm{o}}} P^{(\mathrm{u})}(k'_{\mathrm{i}}, k'_{\mathrm{o}} | \cdot) B_{k'_{\mathrm{i}}1} \left[1 - (1 - Q_{\mathrm{trig}})^{k'_{\mathrm{o}}}\right]$$

$$S_{\mathrm{trig}} = \sum_{k'_{\mathrm{i}}, k'_{\mathrm{o}}} P(k'_{\mathrm{i}}, k'_{\mathrm{o}}) \left[1 - (1 - Q_{\mathrm{trig}})^{k'_{\mathrm{o}}}\right]$$

# Summary of triggering probabilities for uncorrelated networks:

- III. Mixed Directed and Undirected, Uncorrelated—

$$Q_{\text{trig}}^{(\text{u})} = \sum_{\vec{k}'} P^{(\text{u})}(\vec{k}' \,|\, \cdot) B_{\vec{k}'1} \left[ 1 - (1 - Q_{\text{trig}}^{(\text{u})})^{k'_{\text{u}}-1} (1 - Q_{\text{trig}}^{(\text{o})})^{k'_{\text{o}}} \right]$$

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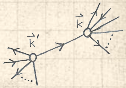

# Summary of triggering probabilities for correlated networks:

- IV. Undirected, Correlated— $Q_{\text{trig}}(k_{\text{u}}) = \sum_{k'_{\text{u}}} P^{(\text{u})}(k'_{\text{u}} \,|\, k_{\text{u}}) B_{k'_{\text{u}}1} \left[1 - (1 - Q_{\text{trig}}(k'_{\text{u}}))^{k'_{\text{u}}-1}\right]$

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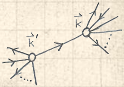

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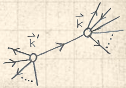
- V. Directed, Correlated— $Q_{\text{trig}}(k_{\text{i}}, k_{\text{o}}) = \sum_{k'_{\text{i}}, k'_{\text{o}}} P^{(\text{u})}(k'_{\text{i}}, k'_{\text{o}} | \, k_{\text{i}}, k_{\text{o}}) B_{k'_{\text{i}}1} \left[1 - (1 - Q_{\text{trig}}(k'_{\text{i}}, k'_{\text{o}}))^{k'_{\text{o}}}\right]$

$$S_{\text{trig}} = \sum_{k'_{\text{i}}, k'_{\text{o}}} P(k'_{\text{i}}, k'_{\text{o}}) \left[1 - (1 - Q_{\text{trig}}(k'_{\text{i}}, k'_{\text{o}}))^{k'_{\text{o}}}\right]$$



## Summary of triggering probabilities for correlated networks:

- VI. Mixed Directed and Undirected, Correlated—

$$Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}-1} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

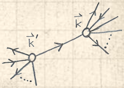
$$Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

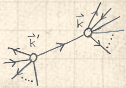
$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

## Nutshell:

- Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.







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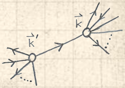

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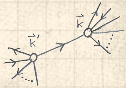
- Spreading conditions and triggering probabilities of contagion processes can be determined using a direct, physical approach.
- These conditions can be generalized to arbitrary random networks with arbitrary node and edge types.
- More generalizations: bipartite affiliation graphs and multilayer networks.



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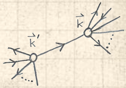

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