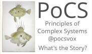
Generating Functions and Networks

Last updated: 2022/08/28, 08:34:20 EDT

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394, 2022–2023| @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Santa Fe Institute | University of Vermont



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

These slides are brought to you by:

These slides are also brought to you by:

Special Guest Executive Producer

On Instagram at pratchett_the_cat

Generating Functions
- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- A few examples
- Average Component Size

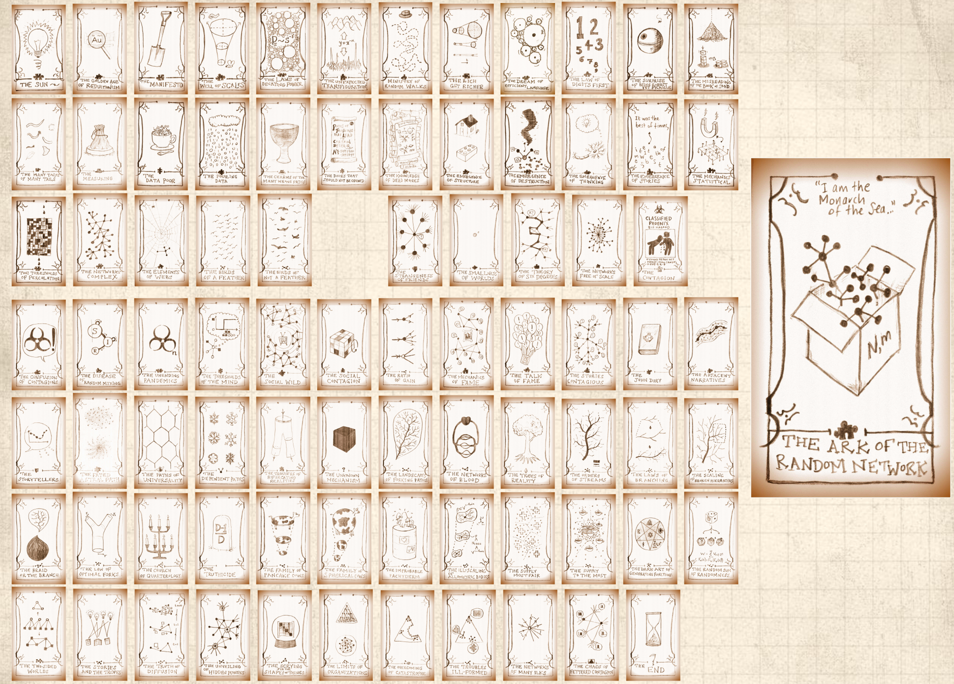
References

Outline

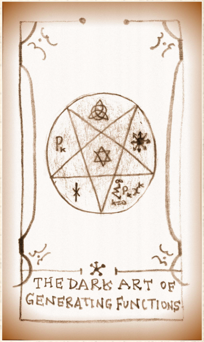
Generating Functions

- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- A few examples
- Average Component Size

References


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE ORDERS SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END
"I am the Monarch of the Sea.."
N,m
THE ARK OF THE RANDOM NETWORK


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE BOOKS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED PHOENIX
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


THE DARK ART OF GENERATING FUNCTIONS

Outline

Generating Functions

- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- A few examples
- Average Component Size

References

Generatingfunctionology [1]

- **Idea:** Given a sequence $a_0, a_1, a_2, \ldots,$ associate each element with a distinct function or other mathematical object.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Generatingfunctionology [1]

- Idea: Given a sequence $a_0, a_1, a_2, \ldots,$ associate each element with a distinct function or other mathematical object.
- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Generatingfunctionology [1]

- Idea: Given a sequence $a_0, a_1, a_2, \ldots$, associate each element with a distinct function or other mathematical object.
- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

Definition:

- The generating function (g.f.) for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Generatingfunctionology [1]

- Idea: Given a sequence $a_0, a_1, a_2, \ldots,$ associate each element with a distinct function or other mathematical object.
- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

Definition:

- The generating function (g.f.) for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- Roughly: transforms a vector in R^∞ into a function defined on R^1.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Generatingfunctionology [1]

- Idea: Given a sequence $a_0, a_1, a_2, \ldots,$ associate each element with a distinct function or other mathematical object.
- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

Definition:

- The generating function (g.f.) for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- Roughly: transforms a vector in R^∞ into a function defined on R^1.
- Related to Fourier, Laplace, Mellin, ...

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Simple examples:

Rolling dice and flipping coins:

- $p_k^{(\text{⚂})} = \mathbf{Pr}(\text{throwing a } k) = 1/6$ where $k = 1, 2, \ldots, 6$.

$$F^{(\text{⚂})}(x) = \sum_{k=1}^{6} p_k^{(\text{⚂})} x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Simple examples:

Rolling dice and flipping coins:

- $p_k^{(\boxdot)} = \textbf{Pr}(\text{throwing a } k) = 1/6$ where $k = 1, 2, \ldots, 6$.

$$F^{(\boxdot)}(x) = \sum_{k=1}^{6} p_k^{(\boxdot)} x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

- $p_0^{(\text{coin})} = \textbf{Pr}(\text{head}) = 1/2$, $p_1^{(\text{coin})} = \textbf{Pr}(\text{tail}) = 1/2$.

$$F^{(\text{coin})}(x) = p_0^{(\text{coin})} x^0 + p_1^{(\text{coin})} x^1 = \frac{1}{2}(1 + x).$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Simple examples:

Rolling dice and flipping coins:

- $p_k^{(⚂)} = \mathbf{Pr}(\text{throwing a } k) = 1/6$ where $k = 1, 2, \ldots, 6$.

$$F^{(⚂)}(x) = \sum_{k=1}^{6} p_k^{(⚂)} x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

- $p_0^{(\text{coin})} = \mathbf{Pr}(\text{head}) = 1/2$, $p_1^{(\text{coin})} = \mathbf{Pr}(\text{tail}) = 1/2$.

$$F^{(\text{coin})}(x) = p_0^{(\text{coin})} x^0 + p_1^{(\text{coin})} x^1 = \frac{1}{2}(1 + x).$$

- A generating function for a probability distribution is called a Probability Generating Function (p.g.f.).

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Simple examples:

Rolling dice and flipping coins:

- $p_k^{(\text{⚂})} = \textbf{Pr}(\text{throwing a } k) = 1/6$ where $k = 1, 2, \ldots, 6$.

$$F^{(\text{⚂})}(x) = \sum_{k=1}^{6} p_k^{(\text{⚂})} x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

- $p_0^{(\text{coin})} = \textbf{Pr}(\text{head}) = 1/2$, $p_1^{(\text{coin})} = \textbf{Pr}(\text{tail}) = 1/2$.

$$F^{(\text{coin})}(x) = p_0^{(\text{coin})} x^0 + p_1^{(\text{coin})} x^1 = \frac{1}{2}(1 + x).$$

- A generating function for a probability distribution is called a Probability Generating Function (p.g.f.).
- We'll come back to these simple examples as we derive various delicious properties of generating functions.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Example

- Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where geometricsumfully, we have $c = 1 - e^{-\lambda}$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Example

- Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where geometricsumfully, we have $c = 1 - e^{-\lambda}$

- The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Example

- Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where geometricsumfully, we have $c = 1 - e^{-\lambda}$

- The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Example

- Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where geometricsumfully, we have $c = 1 - e^{-\lambda}$

- The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}.$$

Example

- Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where geometricsumfully, we have $c = 1 - e^{-\lambda}$

- The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}.$$

- Notice that $F(1) = c/(1 - e^{-\lambda}) = 1$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Example

- Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where geometricsumfully, we have $c = 1 - e^{-\lambda}$

- The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}.$$

- Notice that $F(1) = c/(1 - e^{-\lambda}) = 1$.
- For probability distributions, we must always have $F(1) = 1$ since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Example

- Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where geometricsumfully, we have $c = 1 - e^{-\lambda}$

- The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}.$$

- Notice that $F(1) = c/(1 - e^{-\lambda}) = 1.$
- For probability distributions, we must always have $F(1) = 1$ since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k = \sum_{k=0}^{\infty} P_k$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Example

- Take a degree distribution with exponential decay:

$$P_k = c e^{-\lambda k}$$

where geometricsumfully, we have $c = 1 - e^{-\lambda}$

- The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} c e^{-\lambda k} x^k = \frac{c}{1 - x e^{-\lambda}}.$$

- Notice that $F(1) = c/(1 - e^{-\lambda}) = 1$.
- For probability distributions, we must always have $F(1) = 1$ since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k = \sum_{k=0}^{\infty} P_k = 1.$$

- Check die and coin p.g.f.'s.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Outline

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References

Properties:

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Properties:

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \Bigg|_{x=1}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Properties:

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} F(x) \bigg|_{x=1}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References

Properties:

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} F(x) \bigg|_{x=1} = F'(1)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Properties:

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \Big|_{x=1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} F(x) \Big|_{x=1} = F'(1)$$

- In general, many calculations become simple, if a little abstract.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Properties:

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} F(x) \bigg|_{x=1} = F'(1)$$

- In general, many calculations become simple, if a little abstract.
- For our exponential example:

$$F'(x) = \frac{(1 - e^{-\lambda})e^{-\lambda}}{(1 - xe^{-\lambda})^2}.$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Properties:

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} F(x) \bigg|_{x=1} = F'(1)$$

- In general, many calculations become simple, if a little abstract.
- For our exponential example:

$$F'(x) = \frac{(1 - e^{-\lambda}) e^{-\lambda}}{(1 - x e^{-\lambda})^2}.$$

-

So: $\langle k \rangle = F'(1) = \dfrac{e^{-\lambda}}{(1 - e^{-\lambda})}.$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Properties:

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} F(x) \bigg|_{x=1} = F'(1)$$

- In general, many calculations become simple, if a little abstract.
- For our exponential example:

$$F'(x) = \frac{(1 - e^{-\lambda})e^{-\lambda}}{(1 - xe^{-\lambda})^2}.$$

- So: $\langle k \rangle = F'(1) = \dfrac{e^{-\lambda}}{(1 - e^{-\lambda})}.$
- Check for die and coin p.g.f.'s.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful pieces for probability distributions:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful pieces for probability distributions:

- Normalization:

$$F(1) = 1$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References

Useful pieces for probability distributions:

- Normalization:

$$F(1) = 1$$

- First moment:

$$\langle k \rangle = F'(1)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful pieces for probability distributions:

- Normalization:

$$F(1) = 1$$

- First moment:

$$\langle k \rangle = F'(1)$$

- Higher moments:

$$\langle k^n \rangle = \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^n F(x) \bigg|_{x=1}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful pieces for probability distributions:

- Normalization:

$$F(1) = 1$$

- First moment:

$$\langle k \rangle = F'(1)$$

- Higher moments:

$$\langle k^n \rangle = \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^n F(x) \bigg|_{x=1}$$

- kth element of sequence (general):

$$P_k = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) \bigg|_{x=0}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A beautiful, fundamental thing:

- The generating function for the sum of two random variables

$$W = U + V$$

is

$$F_W(x) = F_U(x)F_V(x).$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A beautiful, fundamental thing:

- The generating function for the sum of two random variables

$$W = U + V$$

is

$$F_W(x) = F_U(x) F_V(x).$$

- Convolve yourself with Convolutions: Insert question from assignment 5 .

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A beautiful, fundamental thing:

- The generating function for the sum of two random variables

$$W = U + V$$

is

$$F_W(x) = F_U(x) F_V(x).$$

- Convolve yourself with Convolutions: Insert question from assignment 5 .
- Try with die and coin p.g.f.'s.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A beautiful, fundamental thing:

- The generating function for the sum of two random variables

$$W = U + V$$

is

$$F_W(x) = F_U(x)F_V(x).$$

- Convolve yourself with Convolutions: Insert question from assignment 5 .
- Try with die and coin p.g.f.'s.
    1. Add two coins (tail=0, head=1).

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A beautiful, fundamental thing:

- The generating function for the sum of two random variables

$$W = U + V$$

is

$$F_W(x) = F_U(x) F_V(x).$$

- Convolve yourself with Convolutions: Insert question from assignment 5 .
- Try with die and coin p.g.f.'s.
  1. Add two coins (tail=0, head=1).
  2. Add two dice.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A beautiful, fundamental thing:

- The generating function for the sum of two random variables

$$W = U + V$$

is

$$F_W(x) = F_U(x) F_V(x).$$

- Convolve yourself with Convolutions: Insert question from assignment 5 .
- Try with die and coin p.g.f.'s.
  1. Add two coins (tail=0, head=1).
  2. Add two dice.
  3. Add a coin flip to one die roll.

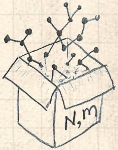
Outline

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References

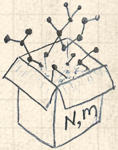


Edge-degree distribution

- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



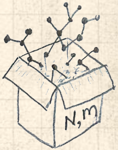
Edge-degree distribution

- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

- Let's re-express our condition in terms of generating functions.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



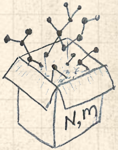
Edge-degree distribution

- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

- Let's re-express our condition in terms of generating functions.
- We first need the g.f. for R_k.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



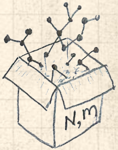
Edge-degree distribution

- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

- Let's re-express our condition in terms of generating functions.
- We first need the g.f. for R_k.
- We'll now use this notation:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Edge-degree distribution

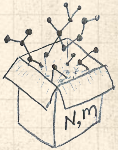
- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

- Let's re-express our condition in terms of generating functions.
- We first need the g.f. for R_k.
- We'll now use this notation:

  $F_P(x)$ is the g.f. for P_k.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Edge-degree distribution

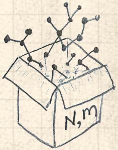
- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

- Let's re-express our condition in terms of generating functions.
- We first need the g.f. for R_k.
- We'll now use this notation:

  $F_P(x)$ is the g.f. for P_k.
  $F_R(x)$ is the g.f. for R_k.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Edge-degree distribution

- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

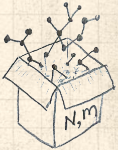
- Let's re-express our condition in terms of generating functions.
- We first need the g.f. for R_k.
- We'll now use this notation:

  $F_P(x)$ is the g.f. for P_k.
  $F_R(x)$ is the g.f. for R_k.

- Giant component condition in terms of g.f. is:

$$\langle k \rangle_R = F_R'(1) > 1.$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Edge-degree distribution

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

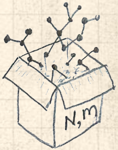
- Let's re-express our condition in terms of generating functions.
- We first need the g.f. for R_k.
- We'll now use this notation:

  $F_P(x)$ is the g.f. for P_k.
  $F_R(x)$ is the g.f. for R_k.

- Giant component condition in terms of g.f. is:

$$\langle k \rangle_R = F_R'(1) > 1.$$

- Now find how F_R is related to F_P ...

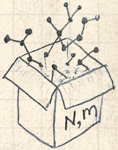


Edge-degree distribution

- We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

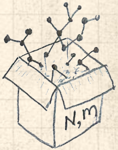


Edge-degree distribution

- We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



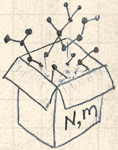
Edge-degree distribution

- We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to $j = k+1$ and pull out $\frac{1}{\langle k \rangle}$:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Edge-degree distribution

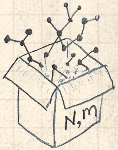
- We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k .$$

Shift index to $j = k + 1$ and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Edge-degree distribution

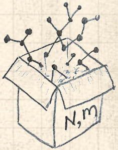
- We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to $j = k+1$ and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d}x} x^j$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Edge-degree distribution

- We have

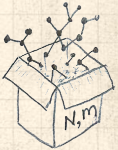
$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to $j = k + 1$ and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d}x} x^j$$

$$= \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{\infty} P_j x^j$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Edge-degree distribution

- We have

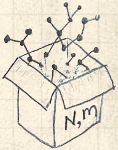
$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to $j = k + 1$ and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d}x} x^j$$

$$= \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \left(F_P(x) - P_0 \right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References

Edge-degree distribution

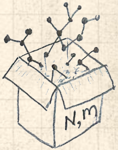
- We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to $j = k + 1$ and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d}x} x^j$$

$$= \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \left(F_P(x) - P_0 \right) = \frac{1}{\langle k \rangle} F_P'(x).$$



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Edge-degree distribution

- We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

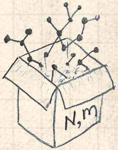
Shift index to $j = k + 1$ and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d}x} x^j$$

$$= \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \left(F_P(x) - P_0 \right) = \frac{1}{\langle k \rangle} F_P'(x).$$

Finally, since $\langle k \rangle = F_P'(1)$,

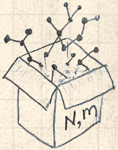
$$\boxed{F_R(x) = \frac{F_P'(x)}{F_P'(1)}}$$



Edge-degree distribution

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

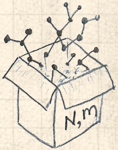
- Recall giant component condition is $\langle k \rangle_R = F'_R(1) > 1$.



Edge-degree distribution

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Recall giant component condition is $\langle k \rangle_R = F_R'(1) > 1$.
- Since we have $F_R(x) = F_P'(x)/F_P'(1)$,

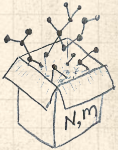


Edge-degree distribution

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Recall giant component condition is $\langle k \rangle_R = F'_R(1) > 1$.
- Since we have $F_R(x) = F'_P(x)/F'_P(1)$,

$$F'_R(x) = \frac{F''_P(x)}{F'_P(1)}.$$



Edge-degree distribution

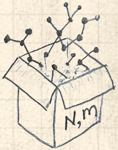
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Recall giant component condition is $\langle k \rangle_R = F'_R(1) > 1$.
- Since we have $F_R(x) = F'_P(x)/F'_P(1)$,

$$F'_R(x) = \frac{F''_P(x)}{F'_P(1)}.$$

- Setting $x = 1$, our condition becomes

$$\boxed{\frac{F''_P(1)}{F'_P(1)} > 1}$$



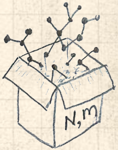
Outline

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

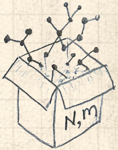
References



Size distributions

To figure out the size of the largest component (S_1), we need more resolution on component sizes.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



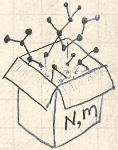
Size distributions

To figure out the size of the largest component (S_1), we need more resolution on component sizes.

Definitions:

- π_n = probability that a random node belongs to a finite component of size $n < \infty$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



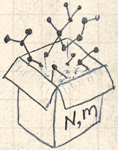
Size distributions

To figure out the size of the largest component (S_1), we need more resolution on component sizes.

Definitions:

- π_n = probability that a random node belongs to a finite component of size $n < \infty$.
- ρ_n = probability that a random end of a random link leads to a finite subcomponent of size $n < \infty$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Size distributions

To figure out the size of the largest component (S_1), we need more resolution on component sizes.

Definitions:

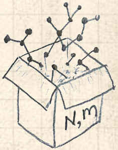
- π_n = probability that a random node belongs to a finite component of size $n < \infty$.
- ρ_n = probability that a random end of a random link leads to a finite subcomponent of size $n < \infty$.

Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

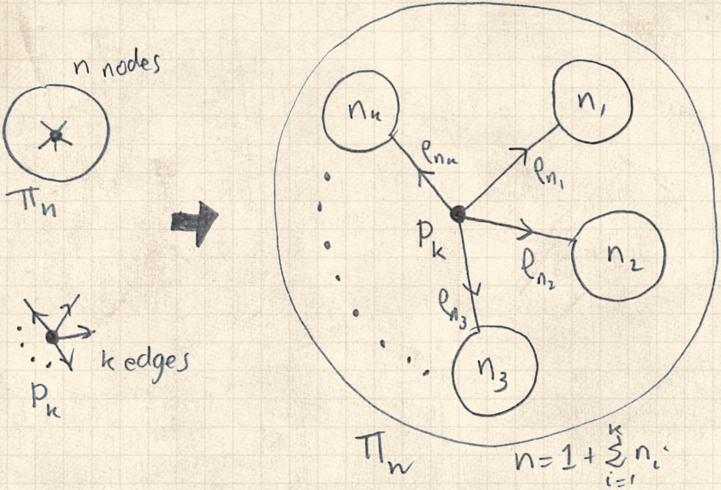
neighbors $\Leftrightarrow$ components

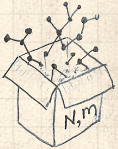
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Connecting probabilities:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

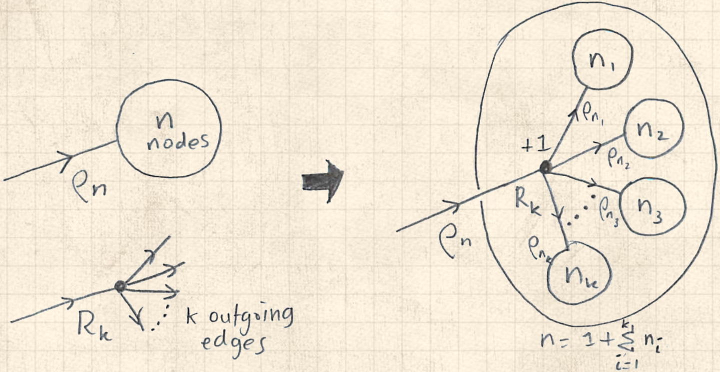


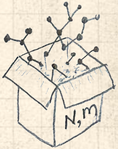


- Markov property of random networks connects π_n, ρ_n, and P_k.

Connecting probabilities:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

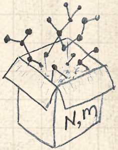




- Markov property of random networks connects ρ_n and R_k.

G.f.'s for component size distributions:

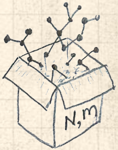
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



G.f.'s for component size distributions:

$$F_\pi(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



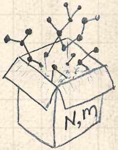
G.f.'s for component size distributions:

$$F_\pi(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

The largest component:

- Subtle key: $F_\pi(1)$ is the probability that a node belongs to a finite component.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



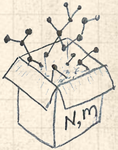
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

G.f.'s for component size distributions:

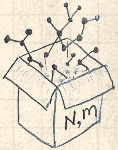
- $$F_\pi(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

The largest component:

- Subtle key: $F_\pi(1)$ is the probability that a node belongs to a finite component.
- Therefore: $S_1 = 1 - F_\pi(1)$.



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



G.f.'s for component size distributions:

- $$F_\pi(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

The largest component:

- Subtle key: $F_\pi(1)$ is the probability that a node belongs to a finite component.
- Therefore: $S_1 = 1 - F_\pi(1)$.

Our mission, which we accept:

- Determine and connect the four generating functions

$$F_P, F_R, F_\pi, \text{ and } F_\rho.$$

Outline

Generating Functions
- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- A few examples
- Average Component Size

References

Useful results we'll need for g.f.'s

Sneaky Result 1:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 1:

- Consider two random variables U and V whose values may be $0, 1, 2, \ldots$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 1:

- Consider two random variables U and V whose values may be $0, 1, 2, \ldots$
- Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 1:

- Consider two random variables U and V whose values may be $0, 1, 2, \ldots$
- Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V.
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^{U} V^{(i)} \text{ with each } V^{(i)} \stackrel{d}{=} V$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 1:

- Consider two random variables U and V whose values may be $0, 1, 2, \ldots$
- Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V.
- SR1: If a third random variable is defined as

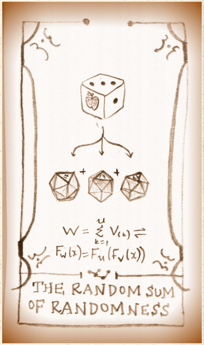
$$W = \sum_{i=1}^{U} V^{(i)} \text{ with each } V^{(i)} \stackrel{d}{=} V$$

then

$$\boxed{F_W(x) = F_U\left(F_V(x)\right)}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARKS OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING SHAPES FOR THE THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


$W = \sum_{k=1}^{U} V_{(k)} \rightleftharpoons$
$F_W(x) = F_U(F_V(x))$
THE RANDOM SUM OF RANDOMNESS

Proof of SR1:

Write probability that variable W has value k as W_k.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Proof of SR1:

Write probability that variable W has value k as W_k.

$$W_k = \sum_{j=0}^{\infty} U_j \times \text{Pr(sum of } j \text{ draws of variable } V = k)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Proof of SR1:

Write probability that variable W has value k as W_k.

$$W_k = \sum_{j=0}^{\infty} U_j \times \text{Pr(sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} V_{i_2} \cdots V_{i_j}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Proof of SR1:

Write probability that variable W has value k as W_k.

$$W_k = \sum_{j=0}^{\infty} U_j \times \text{Pr(sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} V_{i_2} \cdots V_{i_j}$$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Proof of SR1:

Write probability that variable W has value k as W_k.

$$W_k = \sum_{j=0}^{\infty} U_j \times \text{Pr(sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} V_{i_2} \cdots V_{i_j}$$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} V_{i_2} \cdots V_{i_j} x^k$$

Proof of SR1:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Write probability that variable W has value k as W_k.

$$W_k = \sum_{j=0}^{\infty} U_j \times \text{Pr(sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} V_{i_2} \cdots V_{i_j}$$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} V_{i_2} \cdots V_{i_j} x^k$$

$$= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty}$$

Proof of SR1:

Write probability that variable W has value k as W_k.

$$W_k = \sum_{j=0}^{\infty} U_j \times \text{Pr(sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} V_{i_2} \cdots V_{i_j}$$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} V_{i_2} \cdots V_{i_j} x^k$$

$$= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \underbrace{\sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1+i_2+\ldots+i_j=k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \underbrace{\underbrace{\sum_{\substack{\{i_1,i_2,\dots,i_j\}| \\ i_1+i_2+\dots+i_j=k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j}}_{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j = (F_V(x))^j}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \underbrace{\sum_{\substack{\{i_1,i_2,\ldots,i_j\}| \\ i_1+i_2+\ldots+i_j=k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j}$$

$$\underbrace{}_{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j = \left(F_V(x)\right)^j}$$

$$= \sum_{j=0}^{\infty} U_j \left(F_V(x)\right)^j$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \underbrace{\underbrace{\sum_{\substack{\{i_1,i_2,\ldots,i_j\}| \\ i_1+i_2+\ldots+i_j=k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j}}_{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j = (F_V(x))^j}$$

$$= \sum_{j=0}^{\infty} U_j \left(F_V(x)\right)^j$$

$$= F_U\left(F_V(x)\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \underbrace{\underbrace{\sum_{\substack{\{i_1,i_2,\dots,i_j\}| \\ i_1+i_2+\dots+i_j=k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j}}_{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j = (F_V(x))^j}$$

$$= \sum_{j=0}^{\infty} U_j \left(F_V(x)\right)^j$$

$$= F_U\left(F_V(x)\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \underbrace{\underbrace{\sum_{\substack{\{i_1, i_2, \dots, i_j\} | \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \cdots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j}}_{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j = (F_V(x))^j}$$

$$= \sum_{j=0}^{\infty} U_j \, (F_V(x))^j$$

$$= F_U\left(F_V(x)\right)$$

- Alternate, groovier proof in the accompanying assignment.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References

Useful results we'll need for g.f.'s

Sneaky Result 2:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \ldots)$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \ldots)$
- SR2: If a second random variable is defined as

$$V = U + 1$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \dots)$
- SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \dots)$
- SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

- Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \dots)$
- SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = x F_U(x)}$$

- Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.
-

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \dots)$
- SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

- Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.
-

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \dots)$
- SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

- Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.
-

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

$$= x \sum_{j=0}^{\infty} U_j x^j$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \dots)$
- SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

- Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.
-

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

$$= x \sum_{j=0}^{\infty} U_j x^j = xF_U(x).$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \dots)$
- SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

- Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.
-

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

$$= x \sum_{j=0}^{\infty} U_j x^j = xF_U(x).$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Generalization of SR2:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Generalization of SR2:

- (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Generalization of SR2:

- (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

- (2) If $V = U - i$ then

$$F_V(x) = x^{-i} F_U(x)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Useful results we'll need for g.f.'s

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Generalization of SR2:

- (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

- (2) If $V = U - i$ then

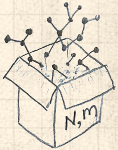
$$F_V(x) = x^{-i} F_U(x)$$

$$= x^{-i} \sum_{k=0}^{\infty} U_k x^k$$

Outline

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

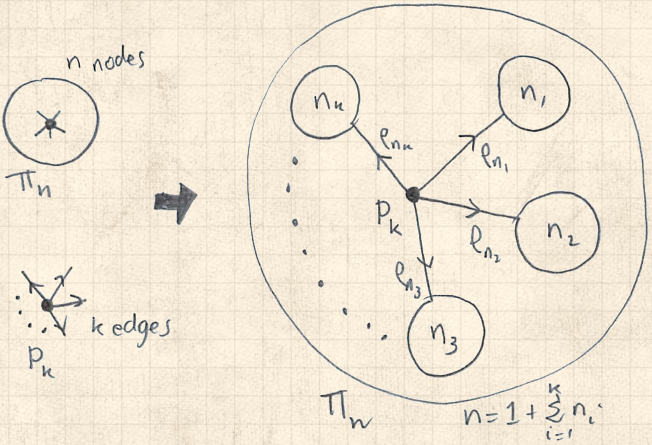
References



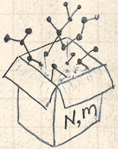
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Connecting generating functions:

- Goal: figure out forms of the component generating functions, F_π and F_ρ.



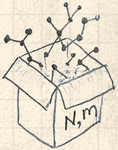
- Relate π_n to P_k and ρ_n through one step of recursion.



Connecting generating functions:

- π_n = probability that a random node belongs to a finite component of size n

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

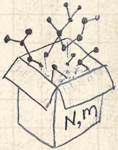


Connecting generating functions:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \mathrm{Pr}\left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array} \right)$$



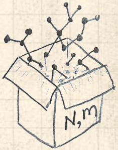
Connecting generating functions:

- π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \text{Pr}\left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array} \right)$$

- Therefore: $F_{\pi}(x) =$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



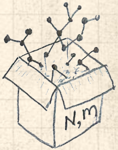
Connecting generating functions:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \text{Pr}\left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array} \right)$$

- Therefore: $$\boxed{F_\pi(x) = \underbrace{F_P\left(F_\rho(x)\right)}_{\text{SR1}}}$$



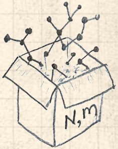
Connecting generating functions:

- π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \text{Pr}\left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array} \right)$$

- Therefore:

$$\boxed{F_\pi(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_P\left(F_\rho(x)\right)}_{\text{SR1}}}$$



Connecting generating functions:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

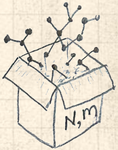
- π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \text{Pr}\left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array} \right)$$

- Therefore:

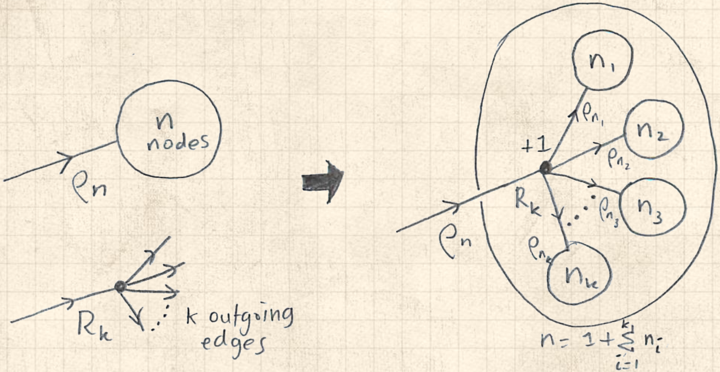
$$\boxed{F_\pi(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_P\left(F_\rho(x)\right)}_{\text{SR1}}}$$

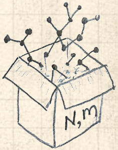
- Extra factor of x accounts for random node itself.



Connecting generating functions:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



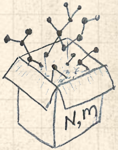


- Relate ρ_n to R_k and ρ_n through one step of recursion.

Connecting generating functions:

- ρ_n = probability that a random link leads to a finite subcomponent of size n.

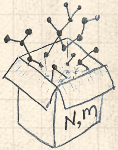
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Connecting generating functions:

- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- Invoke one step of recursion:
  ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n-1$,

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

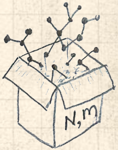


Connecting generating functions:

- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- Invoke one step of recursion:
  ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n-1$,

$$= \sum_{k=0}^{\infty} R_k \times \text{Pr}\left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array} \right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



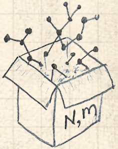
Connecting generating functions:

- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- Invoke one step of recursion:
  ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n-1$,

$$= \sum_{k=0}^{\infty} R_k \times \Pr\left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array} \right)$$

- Therefore: $F_\rho(x) =$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Connecting generating functions:

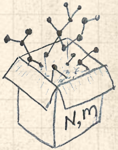
- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- Invoke one step of recursion:
  ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n-1$,

$$= \sum_{k=0}^{\infty} R_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array} \right)$$

-

Therefore: $$F_\rho(x) = \underbrace{F_R\left(F_\rho(x)\right)}_{\text{SR1}}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



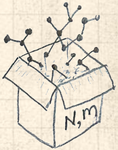
Connecting generating functions:

- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- Invoke one step of recursion:
  ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n-1$,

$$= \sum_{k=0}^{\infty} R_k \times \text{Pr}\left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array} \right)$$

- Therefore: $$F_\rho(x) = \underbrace{x}_{\text{SR2}} \; \underbrace{F_R\left(F_\rho(x)\right)}_{\text{SR1}}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Connecting generating functions:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

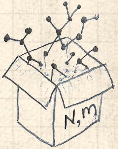
- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- Invoke one step of recursion:
  ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n-1$,

$$= \sum_{k=0}^{\infty} R_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array} \right)$$

- Therefore:

$$F_\rho(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_R\left(F_\rho(x)\right)}_{\text{SR1}}$$

- Again, extra factor of x accounts for random node itself.

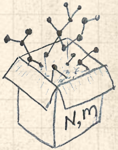


Connecting generating functions:

- We now have two functional equations connecting our generating functions:

$$F_\pi(x) = xF_P\left(F_\rho(x)\right) \text{ and } F_\rho(x) = xF_R\left(F_\rho(x)\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



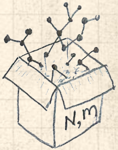
Connecting generating functions:

- We now have two functional equations connecting our generating functions:

$$F_\pi(x) = xF_P\left(F_\rho(x)\right) \text{ and } F_\rho(x) = xF_R\left(F_\rho(x)\right)$$

- Taking stock: We know $F_P(x)$ and $F_R(x) = F'_P(x)/F'_P(1)$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



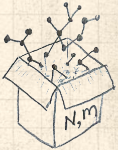
Connecting generating functions:

- We now have two functional equations connecting our generating functions:

$$F_\pi(x) = xF_P\left(F_\rho(x)\right) \text{ and } F_\rho(x) = xF_R\left(F_\rho(x)\right)$$

- Taking stock: We know $F_P(x)$ and $F_R(x) = F_P'(x)/F_P'(1)$.
- We first untangle the second equation to find F_ρ

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



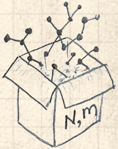
Connecting generating functions:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- We now have two functional equations connecting our generating functions:

$$F_\pi(x) = xF_P\left(F_\rho(x)\right) \text{ and } F_\rho(x) = xF_R\left(F_\rho(x)\right)$$

- Taking stock: We know $F_P(x)$ and $F_R(x) = F'_P(x)/F'_P(1)$.
- We first untangle the second equation to find F_ρ
- We can do this because it only involves F_ρ and F_R.



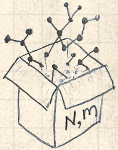
Connecting generating functions:

- We now have two functional equations connecting our generating functions:

$$F_\pi(x) = xF_P\left(F_\rho(x)\right) \text{ and } F_\rho(x) = xF_R\left(F_\rho(x)\right)$$

- Taking stock: We know $F_P(x)$ and $F_R(x) = F_P'(x)/F_P'(1)$.
- We first untangle the second equation to find F_ρ
- We can do this because it only involves F_ρ and F_R.
- The first equation then immediately gives us F_π in terms of F_ρ and F_R.

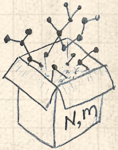
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Component sizes

- Remembering vaguely what we are doing:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

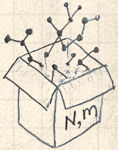


Component sizes

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Remembering vaguely what we are doing:

  Finding F_π to obtain the fractional size of the largest component $S_1 = 1 - F_\pi(1)$.



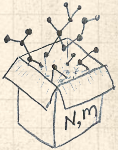
Component sizes

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Remembering vaguely what we are doing:

  Finding F_{π} to obtain the fractional size of the largest component $S_1 = 1 - F_{\pi}(1)$.

- Set $x = 1$ in our two equations:



Component sizes

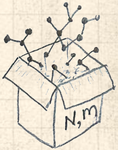
- Remembering vaguely what we are doing:

  Finding F_π to obtain the fractional size of the largest component $S_1 = 1 - F_\pi(1)$.

- Set $x = 1$ in our two equations:

$$F_\pi(1) = F_P\left(F_\rho(1)\right) \text{ and } F_\rho(1) = F_R\left(F_\rho(1)\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Component sizes

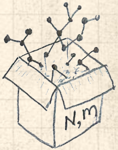
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Remembering vaguely what we are doing:

  Finding F_π to obtain the fractional size of the largest component $S_1 = 1 - F_\pi(1)$.
- Set $x = 1$ in our two equations:

$$F_\pi(1) = F_P\left(F_\rho(1)\right) \text{ and } F_\rho(1) = F_R\left(F_\rho(1)\right)$$

- Solve second equation numerically for $F_\rho(1)$.



Component sizes

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

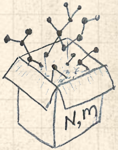
- Remembering vaguely what we are doing:

  Finding F_π to obtain the fractional size of the largest component $S_1 = 1 - F_\pi(1)$.

- Set $x = 1$ in our two equations:

$$F_\pi(1) = F_P\left(F_\rho(1)\right) \text{ and } F_\rho(1) = F_R\left(F_\rho(1)\right)$$

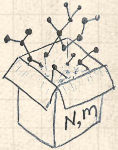
- Solve second equation numerically for $F_\rho(1)$.
- Plug $F_\rho(1)$ into first equation to obtain $F_\pi(1)$.



Component sizes

Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$



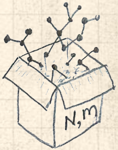
Component sizes

Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\Rightarrow F_R(x) = F_P'(x)/F_P'(1)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Component sizes

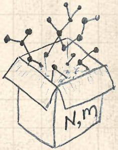
Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\Rightarrow F_R(x) = F'_P(x)/F'_P(1)$$

$$= \langle k \rangle e^{-\langle k \rangle (1-x)} / \langle k \rangle e^{-\langle k \rangle (1-x')}|_{x'=1}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Component sizes

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

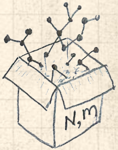
Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\Rightarrow F_R(x) = F_P'(x)/F_P'(1)$$

$$= \langle k \rangle e^{-\langle k \rangle (1-x)} / \langle k \rangle e^{-\langle k \rangle (1-x')}|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)}$$



Component sizes

Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

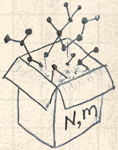
$$\Rightarrow F_R(x) = F_P'(x)/F_P'(1)$$

$$= \langle k \rangle e^{-\langle k \rangle (1-x)} / \langle k \rangle e^{-\langle k \rangle (1-x')}|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)} = F_P(x)$$

...aha!

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Component sizes

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Example: Standard random graphs.

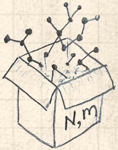
- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\Rightarrow F_R(x) = F'_P(x)/F'_P(1)$$

$$= \langle k \rangle e^{-\langle k \rangle (1-x)} / \langle k \rangle e^{-\langle k \rangle (1-x')}|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)} = F_P(x)$$ ...aha!

- RHS's of our two equations are the same.



Component sizes

Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

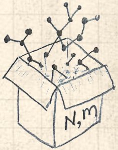
$$\Rightarrow F_R(x) = F_P'(x)/F_P'(1)$$

$$= \langle k \rangle e^{-\langle k \rangle (1-x)} / \langle k \rangle e^{-\langle k \rangle (1-x')}|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)} = F_P(x)$$ ...aha!

- RHS's of our two equations are the same.
- So $F_\pi(x) = F_\rho(x) = xF_R(F_\rho(x)) = xF_R(F_\pi(x))$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References

Component sizes

Example: Standard random graphs.

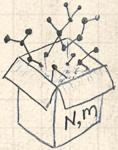
- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\Rightarrow F_R(x) = F_P'(x)/F_P'(1)$$

$$= \langle k \rangle e^{-\langle k \rangle (1-x)} / \langle k \rangle e^{-\langle k \rangle (1-x')}|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)} = F_P(x)$$ ...aha!

- RHS's of our two equations are the same.
- So $F_\pi(x) = F_\rho(x) = xF_R(F_\rho(x)) = xF_R(F_\pi(x))$
- Consistent with how our dirty (but wrong) trick worked earlier ...



Component sizes

Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

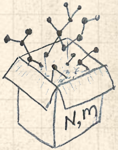
$$\Rightarrow F_R(x) = F'_P(x)/F'_P(1)$$

$$= \langle k \rangle e^{-\langle k \rangle (1-x)} / \langle k \rangle e^{-\langle k \rangle (1-x')}|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)} = F_P(x)$$ ...aha!

- RHS's of our two equations are the same.
- So $F_\pi(x) = F_\rho(x) = xF_R(F_\rho(x)) = xF_R(F_\pi(x))$
- Consistent with how our dirty (but wrong) trick worked earlier ...
- $\pi_n = \rho_n$ just as $P_k = R_k$.

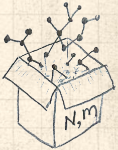
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Component sizes

- We are down to $F_\pi(x) = xF_R(F_\pi(x))$ and $F_R(x) = e^{-\langle k \rangle (1-x)}$.

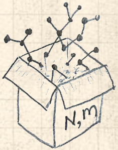
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Component sizes

- We are down to $F_\pi(x) = xF_R(F_\pi(x))$ and $F_R(x) = e^{-\langle k\rangle(1-x)}$.
- $$\therefore F_\pi(x) = xe^{-\langle k\rangle(1-F_\pi(x))}$$

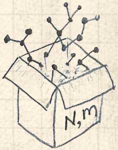
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Component sizes

- We are down to $F_\pi(x) = xF_R(F_\pi(x))$ and $F_R(x) = e^{-\langle k \rangle (1-x)}$.
- $$\therefore F_\pi(x) = xe^{-\langle k \rangle (1-F_\pi(x))}$$
- We're first after $S_1 = 1 - F_\pi(1)$ so set $x = 1$ and replace $F_\pi(1)$ by $1 - S_1$:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

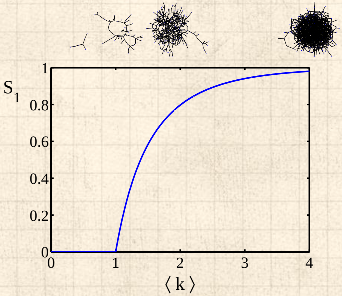


Component sizes

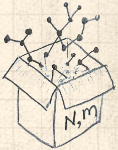
- We are down to $F_\pi(x) = xF_R(F_\pi(x))$ and $F_R(x) = e^{-\langle k \rangle (1-x)}$.
- $$\therefore F_\pi(x) = x e^{-\langle k \rangle (1 - F_\pi(x))}$$
- We're first after $S_1 = 1 - F_\pi(1)$ so set $x = 1$ and replace $F_\pi(1)$ by $1 - S_1$:

$$1 - S_1 = e^{-\langle k \rangle S_1}$$

Or: $\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

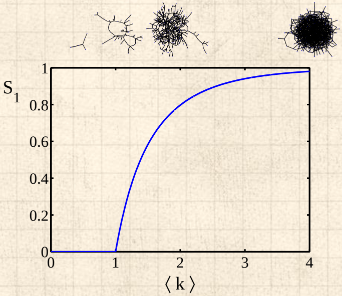


Component sizes

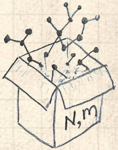
- We are down to $F_\pi(x) = xF_R(F_\pi(x))$ and $F_R(x) = e^{-\langle k \rangle (1-x)}$.
- $$\therefore F_\pi(x) = xe^{-\langle k \rangle (1-F_\pi(x))}$$
- We're first after $S_1 = 1 - F_\pi(1)$ so set $x = 1$ and replace $F_\pi(1)$ by $1 - S_1$:

$$1 - S_1 = e^{-\langle k \rangle S_1}$$

Or: $\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$



- Just as we found with our dirty trick ...



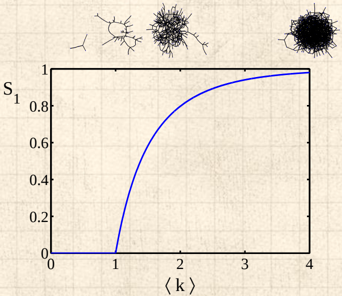
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

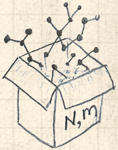
Component sizes

- We are down to $F_\pi(x) = xF_R(F_\pi(x))$ and $F_R(x) = e^{-\langle k \rangle (1-x)}$.
- $$\therefore F_\pi(x) = xe^{-\langle k \rangle (1 - F_\pi(x))}$$
- We're first after $S_1 = 1 - F_\pi(1)$ so set $x = 1$ and replace $F_\pi(1)$ by $1 - S_1$:

$$1 - S_1 = e^{-\langle k \rangle S_1}$$

Or: $$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$$



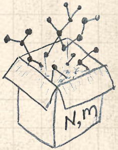


- Just as we found with our dirty trick ...
- Again, we (usually) have to resort to numerics ...

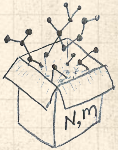
Outline

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

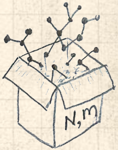


A few simple random networks to contemplate and play around with:

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A few simple random networks to contemplate and play around with:

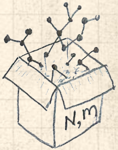
- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A few simple random networks to contemplate and play around with:

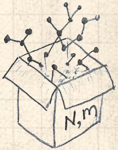
- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
- $P_k = \delta_{k1}$.



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A few simple random networks to contemplate and play around with:

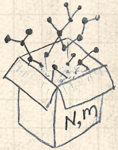
- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
- $P_k = \delta_{k1}$.
- $P_k = \delta_{k2}$.



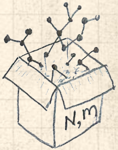
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A few simple random networks to contemplate and play around with:

- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
- $P_k = \delta_{k1}$.
- $P_k = \delta_{k2}$.
- $P_k = \delta_{k3}$.



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



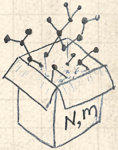
A few simple random networks to contemplate and play around with:

- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
- $P_k = \delta_{k1}$.
- $P_k = \delta_{k2}$.
- $P_k = \delta_{k3}$.
- $P_k = \delta_{kk'}$ for some fixed $k' \geq 0$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A few simple random networks to contemplate and play around with:

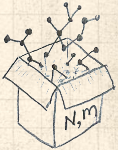
- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
- $P_k = \delta_{k1}$.
- $P_k = \delta_{k2}$.
- $P_k = \delta_{k3}$.
- $P_k = \delta_{kk'}$ for some fixed $k' \geq 0$.
- $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A few simple random networks to contemplate and play around with:

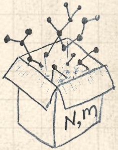
- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
- $P_k = \delta_{k1}$.
- $P_k = \delta_{k2}$.
- $P_k = \delta_{k3}$.
- $P_k = \delta_{kk'}$ for some fixed $k' \geq 0$.
- $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.
- $P_k = a\delta_{k1} + (1-a)\delta_{k3}$, with $0 \leq a \leq 1$.



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

A few simple random networks to contemplate and play around with:

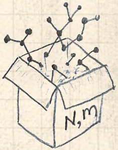
- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
- $P_k = \delta_{k1}$.
- $P_k = \delta_{k2}$.
- $P_k = \delta_{k3}$.
- $P_k = \delta_{kk'}$ for some fixed $k' \geq 0$.
- $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.
- $P_k = a\delta_{k1} + (1-a)\delta_{k3}$, with $0 \leq a \leq 1$.
- $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{kk'}$ for some fixed $k' \geq 2$.



A few simple random networks to contemplate and play around with:

- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
- $P_k = \delta_{k1}$.
- $P_k = \delta_{k2}$.
- $P_k = \delta_{k3}$.
- $P_k = \delta_{kk'}$ for some fixed $k' \geq 0$.
- $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.
- $P_k = a\delta_{k1} + (1-a)\delta_{k3}$, with $0 \leq a \leq 1$.
- $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{kk'}$ for some fixed $k' \geq 2$.
- $P_k = a\delta_{k1} + (1-a)\delta_{kk'}$ for some fixed $k' \geq 2$ with $0 \leq a \leq 1$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

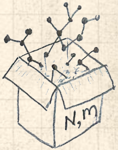


A joyful example:

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

- We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}.$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



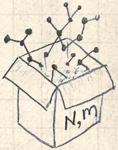
A joyful example:

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

- We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}.$
- A giant component exists because: $\langle k \rangle_R = 0 \times 1/4 + 2 \times 3/4 = 3/2 > 1.$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References



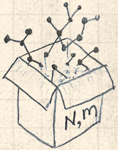
A joyful example :

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

- We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}.$
- A giant component exists because: $\langle k \rangle_R = 0 \times 1/4 + 2 \times 3/4 = 3/2 > 1.$
- Generating functions for P_k and R_k:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

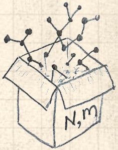
A joyful example ☐:

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

- We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}.$
- A giant component exists because: $\langle k \rangle_R = 0 \times 1/4 + 2 \times 3/4 = 3/2 > 1.$
- Generating functions for P_k and R_k:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2$$

- Check for goodness:



A joyful example:

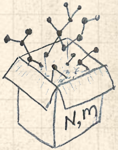
$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

- We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}$.
- A giant component exists because: $\langle k \rangle_R = 0 \times 1/4 + 2 \times 3/4 = 3/2 > 1$.
- Generating functions for P_k and R_k:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2$$

- Check for goodness:
  - $F_R(x) = F'_P(x)/F'_P(1)$ and $F_P(1) = F_R(1) = 1$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



A joyful example:

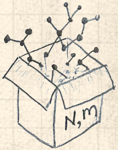
$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

- We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}.$
- A giant component exists because: $\langle k \rangle_R = 0 \times 1/4 + 2 \times 3/4 = 3/2 > 1.$
- Generating functions for P_k and R_k:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2$$

- Check for goodness:
  - $F_R(x) = F_P'(x)/F_P'(1)$ and $F_P(1) = F_R(1) = 1.$
  - $F_P'(1) = \langle k \rangle_P = 2$ and $F_R'(1) = \langle k \rangle_R = \frac{3}{2}.$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



A joyful example:

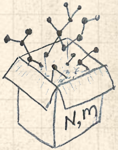
$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

- We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}$.
- A giant component exists because: $\langle k \rangle_R = 0 \times 1/4 + 2 \times 3/4 = 3/2 > 1$.
- Generating functions for P_k and R_k:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2$$

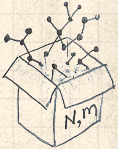
- Check for goodness:
  - $F_R(x) = F_P'(x)/F_P'(1)$ and $F_P(1) = F_R(1) = 1$.
  - $F_P'(1) = \langle k \rangle_P = 2$ and $F_R'(1) = \langle k \rangle_R = \frac{3}{2}$.
- Things to figure out: Component size generating functions for π_n and ρ_n, and the size of the giant component.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

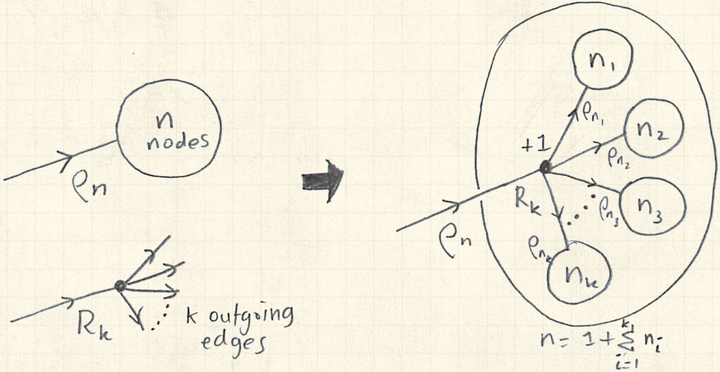
References



Find $F_\rho(x)$ first:

- We know:

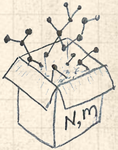
$$F_\rho(x) = x F_R\left(F_\rho(x)\right).$$



Sticking things in things, we have:

$$F_\rho(x) = x\left(\frac{1}{4} + \frac{3}{4}\left[F_\rho(x)\right]^2\right).$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



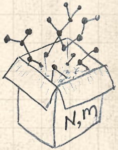
- Sticking things in things, we have:

$$F_\rho(x) = x\left(\frac{1}{4} + \frac{3}{4}\left[F_\rho(x)\right]^2\right).$$

- Rearranging:

$$3x\left[F_\rho(x)\right]^2 - 4F_\rho(x) + x = 0.$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Sticking things in things, we have:

$$F_\rho(x) = x\left(\frac{1}{4} + \frac{3}{4}\left[F_\rho(x)\right]^2\right).$$

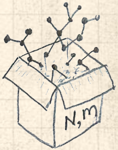
Rearranging:

$$3x\left[F_\rho(x)\right]^2 - 4F_\rho(x) + x = 0.$$

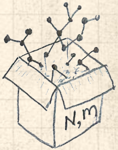
Please and thank you:

$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



- Sticking things in things, we have:

$$F_\rho(x) = x\left(\frac{1}{4} + \frac{3}{4}\left[F_\rho(x)\right]^2\right).$$

- Rearranging:

$$3x\left[F_\rho(x)\right]^2 - 4F_\rho(x) + x = 0.$$

- Please and thank you:

$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right)$$

- Time for a Taylor series expansion.

- Sticking things in things, we have:

$$F_\rho(x) = x\left(\frac{1}{4} + \frac{3}{4}\left[F_\rho(x)\right]^2\right).$$

- Rearranging:

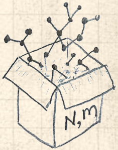
$$3x\left[F_\rho(x)\right]^2 - 4F_\rho(x) + x = 0.$$

- Please and thank you:

$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right)$$

- Time for a Taylor series expansion.
- The promise: non-negative powers of x with non-negative coefficients.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



- Sticking things in things, we have:

$$F_\rho(x) = x\left(\frac{1}{4} + \frac{3}{4}\left[F_\rho(x)\right]^2\right).$$

- Rearranging:

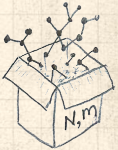
$$3x\left[F_\rho(x)\right]^2 - 4F_\rho(x) + x = 0.$$

- Please and thank you:

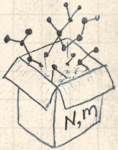
$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right)$$

- Time for a Taylor series expansion.
- The promise: non-negative powers of x with non-negative coefficients.
- First: which sign do we take?

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Because ρ_n is a probability distribution, we know $F_\rho(1) \leq 1$ and $F_\rho(x) \leq 1$ for $0 \leq x \leq 1$.

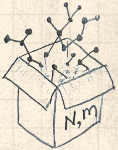


- Because ρ_n is a probability distribution, we know $F_\rho(1) \le 1$ and $F_\rho(x) \le 1$ for $0 \le x \le 1$.
- Thinking about the limit $x \to 0$ in

$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right),$$

we see that the positive sign solution blows to smithereens, and the negative one is okay.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

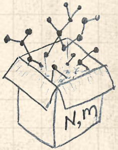
- Because ρ_n is a probability distribution, we know $F_\rho(1) \leq 1$ and $F_\rho(x) \leq 1$ for $0 \leq x \leq 1$.
- Thinking about the limit $x \to 0$ in

$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right),$$

we see that the positive sign solution blows to smithereens, and the negative one is okay.

- So we must have:

$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right),$$



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References

- Because ρ_n is a probability distribution, we know $F_\rho(1) \le 1$ and $F_\rho(x) \le 1$ for $0 \le x \le 1$.
- Thinking about the limit $x \to 0$ in

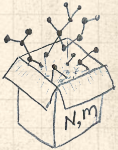
$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right),$$

  we see that the positive sign solution blows to smithereens, and the negative one is okay.
- So we must have:

$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right),$$

- We can now deploy the Taylor expansion:

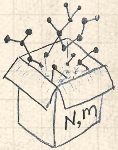
$$(1+z)^\theta = \binom{\theta}{0}z^0 + \binom{\theta}{1}z^1 + \binom{\theta}{2}z^2 + \binom{\theta}{3}z^3 + \ldots$$



Let's define a binomial for arbitrary θ and $k = 0, 1, 2, \ldots$:

$$\binom{\theta}{k} = \frac{\Gamma(\theta + 1)}{\Gamma(k + 1)\Gamma(\theta - k + 1)}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



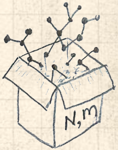
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Let's define a binomial for arbitrary θ and $k = 0, 1, 2, \ldots$:

$$\binom{\theta}{k} = \frac{\Gamma(\theta+1)}{\Gamma(k+1)\Gamma(\theta-k+1)}$$

- For $\theta = \frac{1}{2}$, we have:

$$(1+z)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} z^0 + \binom{\frac{1}{2}}{1} z^1 + \binom{\frac{1}{2}}{2} z^2 + \ldots$$



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

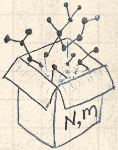
- Let's define a binomial for arbitrary θ and $k = 0, 1, 2, \ldots$:

$$\binom{\theta}{k} = \frac{\Gamma(\theta+1)}{\Gamma(k+1)\Gamma(\theta-k+1)}$$

- For $\theta = \frac{1}{2}$, we have:

$$(1+z)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} z^0 + \binom{\frac{1}{2}}{1} z^1 + \binom{\frac{1}{2}}{2} z^2 + \ldots$$

$$= \frac{\Gamma(\frac{3}{2})}{\Gamma(1)\Gamma(\frac{3}{2})} z^0 + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)\Gamma(\frac{1}{2})} z^1 + \frac{\Gamma(\frac{3}{2})}{\Gamma(3)\Gamma(-\frac{1}{2})} z^2 + \ldots$$



- Let's define a binomial for arbitrary θ and $k = 0, 1, 2, \ldots$:

$$\binom{\theta}{k} = \frac{\Gamma(\theta+1)}{\Gamma(k+1)\Gamma(\theta-k+1)}$$

- For $\theta = \frac{1}{2}$, we have:

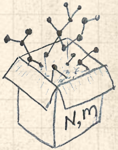
$$(1+z)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} z^0 + \binom{\frac{1}{2}}{1} z^1 + \binom{\frac{1}{2}}{2} z^2 + \ldots$$

$$= \frac{\Gamma(\frac{3}{2})}{\Gamma(1)\Gamma(\frac{3}{2})} z^0 + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)\Gamma(\frac{1}{2})} z^1 + \frac{\Gamma(\frac{3}{2})}{\Gamma(3)\Gamma(-\frac{1}{2})} z^2 + \ldots$$

$$= 1 + \frac{1}{2} z - \frac{1}{8} z^2 + \frac{1}{16} z^3 - \ldots$$

where we've used $\Gamma(x+1) = x\Gamma(x)$ and noted that $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Let's define a binomial for arbitrary θ and $k = 0, 1, 2, \ldots$:

$$\binom{\theta}{k} = \frac{\Gamma(\theta+1)}{\Gamma(k+1)\Gamma(\theta-k+1)}$$

- For $\theta = \frac{1}{2}$, we have:

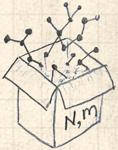
$$(1+z)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} z^0 + \binom{\frac{1}{2}}{1} z^1 + \binom{\frac{1}{2}}{2} z^2 + \ldots$$

$$= \frac{\Gamma(\frac{3}{2})}{\Gamma(1)\Gamma(\frac{3}{2})} z^0 + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)\Gamma(\frac{1}{2})} z^1 + \frac{\Gamma(\frac{3}{2})}{\Gamma(3)\Gamma(-\frac{1}{2})} z^2 + \ldots$$

$$= 1 + \frac{1}{2} z - \frac{1}{8} z^2 + \frac{1}{16} z^3 - \ldots$$

where we've used $\Gamma(x+1) = x\Gamma(x)$ and noted that $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.

- Note: $(1+z)^{\theta} \sim 1 + \theta z$ always.



Totally psyched, we go back to here:

$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right).$$

- Totally psyched, we go back to here:

$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right).$$

- Setting $z = -\frac{3}{4}x^2$ and expanding, we have:

$$F_\rho(x) =$$

$$\frac{2}{3x}\left(1 - \left[1 + \frac{1}{2}\left(-\frac{3}{4}x^2\right)^1 - \frac{1}{8}\left(-\frac{3}{4}x^2\right)^2 + \frac{1}{16}\left(-\frac{3}{4}x^2\right)^3\right] + \ldots\right)$$

- Totally psyched, we go back to here:

$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right).$$

- Setting $z = -\frac{3}{4}x^2$ and expanding, we have:

$$F_\rho(x) =$$

$$\frac{2}{3x}\left(1 - \left[1 + \frac{1}{2}\left(-\frac{3}{4}x^2\right)^1 - \frac{1}{8}\left(-\frac{3}{4}x^2\right)^2 + \frac{1}{16}\left(-\frac{3}{4}x^2\right)^3\right] + \ldots\right)$$

- Giving:

$$F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n =$$

$$\frac{1}{4}x + \frac{3}{64}x^3 + \frac{9}{512}x^5 + \ldots + \frac{2}{3}\left(\frac{3}{4}\right)^k \frac{(-1)^{k+1}\Gamma(\frac{3}{2})}{\Gamma(k+1)\Gamma(\frac{3}{2}-k)} x^{2k-1} + \ldots$$

- Totally psyched, we go back to here:

$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right).$$

- Setting $z = -\frac{3}{4}x^2$ and expanding, we have:

$$F_\rho(x) =$$

$$\frac{2}{3x}\left(1 - \left[1 + \frac{1}{2}\left(-\frac{3}{4}x^2\right)^1 - \frac{1}{8}\left(-\frac{3}{4}x^2\right)^2 + \frac{1}{16}\left(-\frac{3}{4}x^2\right)^3\right] + \ldots\right)$$

- Giving:

$$F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n =$$

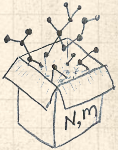
$$\frac{1}{4}x + \frac{3}{64}x^3 + \frac{9}{512}x^5 + \ldots + \frac{2}{3}\left(\frac{3}{4}\right)^k \frac{(-1)^{k+1}\Gamma(\frac{3}{2})}{\Gamma(k+1)\Gamma(\frac{3}{2}-k)}x^{2k-1} + \ldots$$

- Do odd powers make sense?

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- We can now find $F_{\pi}(x)$ with:

$$F_{\pi}(x) = x F_{P}\left(F_{\rho}(x)\right)$$

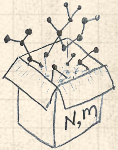


We can now find $F_\pi(x)$ with:

$$F_\pi(x) = x F_P\left(F_\rho(x)\right)$$

$$= x\frac{1}{2}\left(\left(F_\rho(x)\right)^1 + \left(F_\rho(x)\right)^3\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



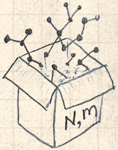
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- We can now find $F_\pi(x)$ with:

$$F_\pi(x) = x F_P\left(F_\rho(x)\right)$$

$$= x\frac{1}{2}\left(\left(F_\rho(x)\right)^1 + \left(F_\rho(x)\right)^3\right)$$

$$= x\frac{1}{2}\left[\frac{2}{3x}\left(1-\sqrt{1-\frac{3}{4}x^2}\right) + \frac{2^3}{(3x)^3}\left(1-\sqrt{1-\frac{3}{4}x^2}\right)^3\right].$$



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

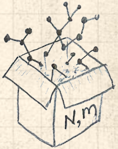
- We can now find $F_\pi(x)$ with:

$$F_\pi(x) = x F_P\left(F_\rho(x)\right)$$

$$= x\frac{1}{2}\left(\left(F_\rho(x)\right)^1 + \left(F_\rho(x)\right)^3\right)$$

$$= x\frac{1}{2}\left[\frac{2}{3x}\left(1-\sqrt{1-\frac{3}{4}x^2}\right) + \frac{2^3}{(3x)^3}\left(1-\sqrt{1-\frac{3}{4}x^2}\right)^3\right].$$

- Delicious.



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

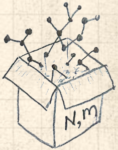
- We can now find $F_\pi(x)$ with:

$$F_\pi(x) = x F_P\left(F_\rho(x)\right)$$

$$= x\frac{1}{2}\left(\left(F_\rho(x)\right)^1 + \left(F_\rho(x)\right)^3\right)$$

$$= x\frac{1}{2}\left[\frac{2}{3x}\left(1-\sqrt{1-\frac{3}{4}x^2}\right) + \frac{2^3}{(3x)^3}\left(1-\sqrt{1-\frac{3}{4}x^2}\right)^3\right].$$

- Delicious.
- In principle, we can now extract all the π_n.



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

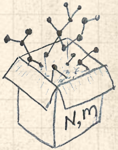
- We can now find $F_\pi(x)$ with:

$$F_\pi(x) = x F_P\left(F_\rho(x)\right)$$

$$= x\frac{1}{2}\left(\left(F_\rho(x)\right)^1 + \left(F_\rho(x)\right)^3\right)$$

$$= x\frac{1}{2}\left[\frac{2}{3x}\left(1-\sqrt{1-\frac{3}{4}x^2}\right) + \frac{2^3}{(3x)^3}\left(1-\sqrt{1-\frac{3}{4}x^2}\right)^3\right].$$

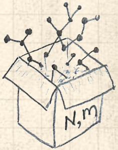
- Delicious.
- In principle, we can now extract all the π_n.
- But let's just find the size of the giant component.



First, we need $F_\rho(1)$:

$$F_\rho(x)\big|_{x=1} = \frac{2}{3\cdot 1}\left(1-\sqrt{1-\frac{3}{4}1^2}\right) = \frac{1}{3}.$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

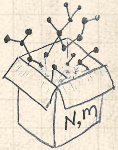


- First, we need $F_\rho(1)$:

$$F_\rho(x)\big|_{x=1} = \frac{2}{3 \cdot 1}\left(1 - \sqrt{1 - \frac{3}{4}1^2}\right) = \frac{1}{3}.$$

- This is the probability that a random edge leads to a sub-component of finite size.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



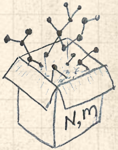
- First, we need $F_{\rho}(1)$:

$$F_{\rho}(x)\big|_{x=1} = \frac{2}{3 \cdot 1}\left(1 - \sqrt{1 - \frac{3}{4}1^2}\right) = \frac{1}{3}.$$

- This is the probability that a random edge leads to a sub-component of finite size.
- Next:

$$F_{\pi}(1) = 1 \cdot F_{P}\left(F_{\rho}(1)\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



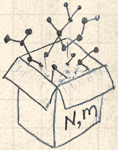
- First, we need $F_\rho(1)$:

$$F_\rho(x)\Big|_{x=1} = \frac{2}{3\cdot 1}\left(1-\sqrt{1-\frac{3}{4}1^2}\right) = \frac{1}{3}.$$

- This is the probability that a random edge leads to a sub-component of finite size.
- Next:

$$F_\pi(1) = 1\cdot F_P\left(F_\rho(1)\right) = F_P\left(\frac{1}{3}\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



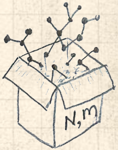
- First, we need $F_\rho(1)$:

$$F_\rho(x)\big|_{x=1} = \frac{2}{3\cdot 1}\left(1-\sqrt{1-\frac{3}{4}1^2}\right) = \frac{1}{3}.$$

- This is the probability that a random edge leads to a sub-component of finite size.
- Next:

$$F_\pi(1) = 1\cdot F_P\left(F_\rho(1)\right) = F_P\left(\frac{1}{3}\right) = \frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\left(\frac{1}{3}\right)^3$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



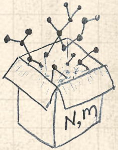
- First, we need $F_\rho(1)$:

$$F_\rho(x)\big|_{x=1} = \frac{2}{3\cdot 1}\left(1 - \sqrt{1 - \frac{3}{4}1^2}\right) = \frac{1}{3}.$$

- This is the probability that a random edge leads to a sub-component of finite size.
- Next:

$$F_\pi(1) = 1 \cdot F_P\left(F_\rho(1)\right) = F_P\left(\frac{1}{3}\right) = \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\left(\frac{1}{3}\right)^3 = \frac{5}{27}.$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

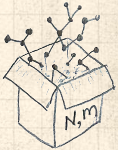
- First, we need $F_\rho(1)$:

$$F_\rho(x)\big|_{x=1} = \frac{2}{3\cdot 1}\left(1-\sqrt{1-\frac{3}{4}1^2}\right) = \frac{1}{3}.$$

- This is the probability that a random edge leads to a sub-component of finite size.
- Next:

$$F_\pi(1) = 1\cdot F_P\left(F_\rho(1)\right) = F_P\left(\frac{1}{3}\right) = \frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\left(\frac{1}{3}\right)^3 = \frac{5}{27}.$$

- This is the probability that a random chosen node belongs to a finite component.



- First, we need $F_\rho(1)$:

$$F_\rho(x)\Big|_{x=1} = \frac{2}{3 \cdot 1}\left(1 - \sqrt{1 - \frac{3}{4}1^2}\right) = \frac{1}{3}.$$

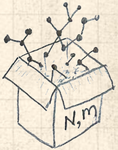
- This is the probability that a random edge leads to a sub-component of finite size.
- Next:

$$F_\pi(1) = 1 \cdot F_P\left(F_\rho(1)\right) = F_P\left(\frac{1}{3}\right) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2}\left(\frac{1}{3}\right)^3 = \frac{5}{27}.$$

- This is the probability that a random chosen node belongs to a finite component.
- Finally, we have

$$S_1 = 1 - F_\pi(1) = 1 - \frac{5}{27} = \frac{22}{27}.$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



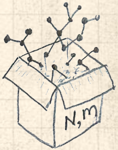
Outline

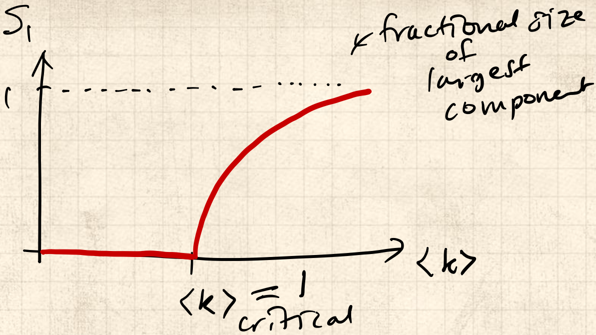
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

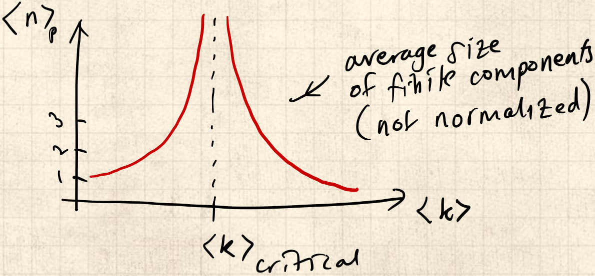
References

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size

References





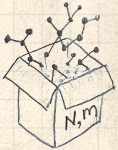


Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Average component size

- Next: find average size of finite components $\langle n \rangle$.

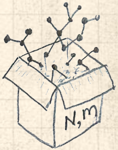
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.

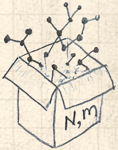
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F_\pi'(1)$.
- Try to avoid finding $F_\pi(x)$ ...

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

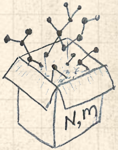


Average component size

- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.
- Try to avoid finding $F_\pi(x)$ ...
- Starting from $F_\pi(x) = xF_P\left(F_\rho(x)\right)$, we differentiate:

$$F'_\pi(x) = F_P\left(F_\rho(x)\right) + xF'_\rho(x)F'_P\left(F_\rho(x)\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

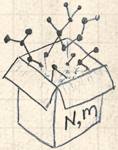
- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.
- Try to avoid finding $F_\pi(x)$ ...
- Starting from $F_\pi(x) = xF_P\left(F_\rho(x)\right)$, we differentiate:

$$F'_\pi(x) = F_P\left(F_\rho(x)\right) + xF'_\rho(x)F'_P\left(F_\rho(x)\right)$$

- While $F_\rho(x) = xF_R\left(F_\rho(x)\right)$ gives

$$F'_\rho(x) = F_R\left(F_\rho(x)\right) + xF'_\rho(x)F'_R\left(F_\rho(x)\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.
- Try to avoid finding $F_{\pi}(x)$ ...
- Starting from $F_{\pi}(x) = xF_P\left(F_{\rho}(x)\right)$, we differentiate:

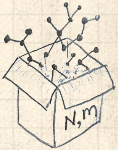
$$F'_{\pi}(x) = F_P\left(F_{\rho}(x)\right) + xF'_{\rho}(x)F'_P\left(F_{\rho}(x)\right)$$

- While $F_{\rho}(x) = xF_R\left(F_{\rho}(x)\right)$ gives

$$F'_{\rho}(x) = F_R\left(F_{\rho}(x)\right) + xF'_{\rho}(x)F'_R\left(F_{\rho}(x)\right)$$

- Now set $x = 1$ in both equations.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.
- Try to avoid finding $F_\pi(x)$ ...
- Starting from $F_\pi(x) = xF_P\left(F_\rho(x)\right)$, we differentiate:

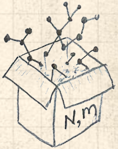
$$F'_\pi(x) = F_P\left(F_\rho(x)\right) + xF'_\rho(x)F'_P\left(F_\rho(x)\right)$$

- While $F_\rho(x) = xF_R\left(F_\rho(x)\right)$ gives

$$F'_\rho(x) = F_R\left(F_\rho(x)\right) + xF'_\rho(x)F'_R\left(F_\rho(x)\right)$$

- Now set $x = 1$ in both equations.
- We solve the second equation for $F'_\rho(1)$ (we must already have $F_\rho(1)$).

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.
- Try to avoid finding $F_\pi(x)$ ...
- Starting from $F_\pi(x) = xF_P\left(F_\rho(x)\right)$, we differentiate:

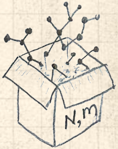
$$F'_\pi(x) = F_P\left(F_\rho(x)\right) + xF'_\rho(x)F'_P\left(F_\rho(x)\right)$$

- While $F_\rho(x) = xF_R\left(F_\rho(x)\right)$ gives

$$F'_\rho(x) = F_R\left(F_\rho(x)\right) + xF'_\rho(x)F'_R\left(F_\rho(x)\right)$$

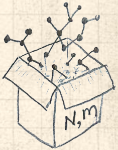
- Now set $x = 1$ in both equations.
- We solve the second equation for $F'_\rho(1)$ (we must already have $F_\rho(1)$).
- Plug $F'_\rho(1)$ and $F_\rho(1)$ into first equation to find $F'_\pi(1)$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

Example: Standard random graphs.

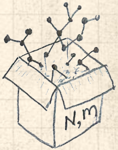


Average component size

Example: Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



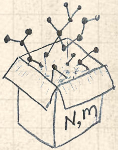
Average component size

Example: Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- Two differentiated equations reduce to only one:

$$F'_\pi(x) = F_P\left(F_\pi(x)\right) + xF'_\pi(x)F'_P\left(F_\pi(x)\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

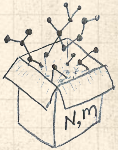
Example: Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- Two differentiated equations reduce to only one:

$$F'_\pi(x) = F_P\left(F_\pi(x)\right) + xF'_\pi(x)F'_P\left(F_\pi(x)\right)$$

Rearrange: $$F'_\pi(x) = \frac{F_P\left(F_\pi(x)\right)}{1 - xF'_P\left(F_\pi(x)\right)}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

Example: Standard random graphs.

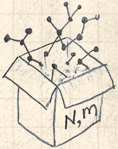
- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- Two differentiated equations reduce to only one:

$$F'_\pi(x) = F_P\left(F_\pi(x)\right) + xF'_\pi(x)F'_P\left(F_\pi(x)\right)$$

Rearrange: $$F'_\pi(x) = \frac{F_P\left(F_\pi(x)\right)}{1 - xF'_P\left(F_\pi(x)\right)}$$

- Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Average component size

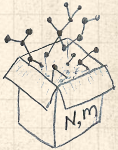
Example: Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- Two differentiated equations reduce to only one:

$$F'_\pi(x) = F_P\left(F_\pi(x)\right) + xF'_\pi(x)F'_P\left(F_\pi(x)\right)$$

Rearrange: $$F'_\pi(x) = \frac{F_P\left(F_\pi(x)\right)}{1 - xF'_P\left(F_\pi(x)\right)}$$

- Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$
- Replace $F_P(F_\pi(x))$ using $F_\pi(x) = xF_P(F_\pi(x))$.



Average component size

Example: Standard random graphs.

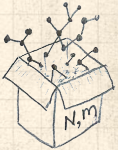
- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- Two differentiated equations reduce to only one:

$$F'_\pi(x) = F_P\left(F_\pi(x)\right) + xF'_\pi(x)F'_P\left(F_\pi(x)\right)$$

Rearrange: $$F'_\pi(x) = \frac{F_P\left(F_\pi(x)\right)}{1 - xF'_P\left(F_\pi(x)\right)}$$

- Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$
- Replace $F_P(F_\pi(x))$ using $F_\pi(x) = xF_P(F_\pi(x))$.
- Set $x = 1$ and replace $F_\pi(1)$ with $1 - S_1$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

Example: Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- Two differentiated equations reduce to only one:

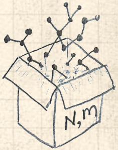
$$F'_\pi(x) = F_P\left(F_\pi(x)\right) + xF'_\pi(x)F'_P\left(F_\pi(x)\right)$$

Rearrange: $$F'_\pi(x) = \frac{F_P\left(F_\pi(x)\right)}{1 - xF'_P\left(F_\pi(x)\right)}$$

- Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$
- Replace $F_P(F_\pi(x))$ using $F_\pi(x) = xF_P(F_\pi(x))$.
- Set $x = 1$ and replace $F_\pi(1)$ with $1 - S_1$.

End result: $$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

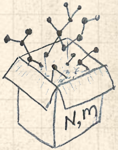


Average component size

- Our result for standard random networks:

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



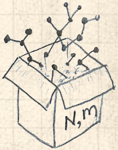
Average component size

- Our result for standard random networks:

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



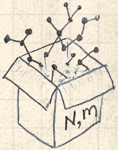
Average component size

- Our result for standard random networks:

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



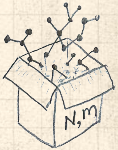
Average component size

- Our result for standard random networks:

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- We have $S_1 = 0$ for all $\langle k \rangle < 1$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

- Our result for standard random networks:

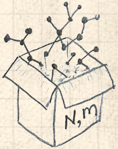
$$\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- This blows up as $\langle k \rangle \to 1$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

- Our result for standard random networks:

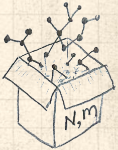
$$\langle n \rangle = F_\pi'(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- This blows up as $\langle k \rangle \to 1$.
- Reason: we have a power law distribution of component sizes at $\langle k \rangle = 1$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Average component size

- Our result for standard random networks:

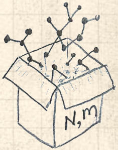
$$\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- This blows up as $\langle k \rangle \to 1$.
- Reason: we have a power law distribution of component sizes at $\langle k \rangle = 1$.
- Typical critical point behavior ...

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

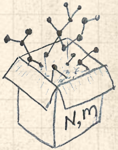


Average component size

- Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



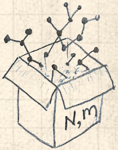
Average component size

- Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



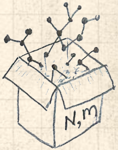
Average component size

- Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.
- All nodes are isolated.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



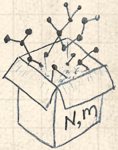
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

Average component size

- Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.
- All nodes are isolated.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.



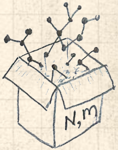
Average component size

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F_\pi'(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.
- All nodes are isolated.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.
- No nodes are outside of the giant component.



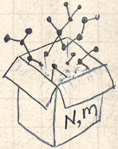
Average component size

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- As $\langle k \rangle \rightarrow 0$, $S_1 = 0$, and $\langle n \rangle \rightarrow 1$.
- All nodes are isolated.
- As $\langle k \rangle \rightarrow \infty$, $S_1 \rightarrow 1$ and $\langle n \rangle \rightarrow 0$.
- No nodes are outside of the giant component.



Extra on largest component size:

- For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}/N$.

Average component size

- Limits of $\langle k \rangle = 0$ and ∞ make sense for

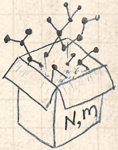
$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.
- All nodes are isolated.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.
- No nodes are outside of the giant component.

Extra on largest component size:

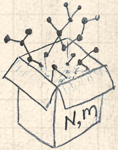
- For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}/N$.
- For $\langle k \rangle < 1$, $S_1 \sim (\log N)/N$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

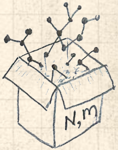
- Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.



- Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.
- We're after:

$$\langle n \rangle = F'_\pi(1) = F_P\left(F_\rho(1)\right) + F'_\rho(1) F'_P\left(F_\rho(1)\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



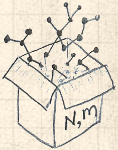
- Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.
- We're after:

$$\langle n \rangle = F'_{\pi}(1) = F_P\left(F_\rho(1)\right) + F'_\rho(1) F'_P\left(F_\rho(1)\right)$$

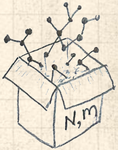
where we first need to compute

$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right).$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



- Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.
- We're after:

$$\langle n \rangle = F'_{\pi}(1) = F_P\left(F_\rho(1)\right) + F'_\rho(1) F'_P\left(F_\rho(1)\right)$$

where we first need to compute

$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right).$$

- Place stick between teeth, and recall that we have:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2.$$

- Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.
- We're after:

$$\langle n \rangle = F'_\pi(1) = F_P\left(F_\rho(1)\right) + F'_\rho(1) F'_P\left(F_\rho(1)\right)$$

where we first need to compute

$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right).$$

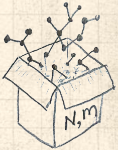
- Place stick between teeth, and recall that we have:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2.$$

- Differentiation gives us:

$$F'_P(x) = \frac{1}{2} + \frac{3}{2}x^2 \text{ and } F'_R(x) = \frac{3}{2}x.$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right)$$

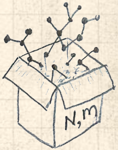
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right)$$

$$= F_R\left(\frac{1}{3}\right) + F'_\rho(1) F'_R\left(\frac{1}{3}\right)$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



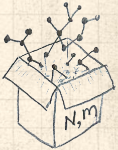
We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right)$$

$$= F_R\left(\frac{1}{3}\right) + F'_\rho(1) F'_R\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} + \frac{\cancel{3}}{4}\frac{1}{3\cancel{2}} + F'_\rho(1)\frac{\cancel{3}}{2}\frac{1}{\cancel{3}}.$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



- We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

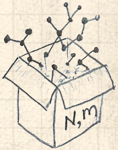
$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1)F'_R\left(F_\rho(1)\right)$$

$$= F_R\left(\frac{1}{3}\right) + F'_\rho(1)F'_R\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} + \frac{\cancel{3}}{4}\frac{1}{3^{\cancel{2}}} + F'_\rho(1)\frac{\cancel{3}}{2}\frac{1}{\cancel{3}}.$$

- After some reallocation of objects, we have $F'_\rho(1) = \frac{13}{2}$.

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



- We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

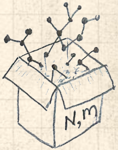
$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right)$$

$$= F_R\left(\frac{1}{3}\right) + F'_\rho(1) F'_R\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} + \frac{\cancel{3}}{4}\frac{1}{3^{\cancel{2}}} + F'_\rho(1)\frac{\cancel{3}}{2}\frac{1}{\cancel{3}}.$$

- After some reallocation of objects, we have $F'_\rho(1) = \frac{13}{2}$.
- Finally: $\langle n \rangle = F'_\pi(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2} F'_P\left(\frac{1}{3}\right)$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

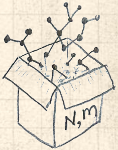
$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right)$$

$$= F_R\left(\frac{1}{3}\right) + F'_\rho(1) F'_R\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} + \frac{\cancel{3}}{4}\frac{1}{3^{\cancel{2}}} + F'_\rho(1)\frac{\cancel{3}}{2}\frac{1}{\cancel{3}}.$$

- After some reallocation of objects, we have $F'_\rho(1) = \frac{13}{2}$.
- Finally: $\langle n \rangle = F'_\pi(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2} F'_P\left(\frac{1}{3}\right)$

$$= \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3^3} + \frac{13}{2}\left(\frac{1}{2} + \frac{\cancel{3}}{2}\frac{1}{3^{\cancel{2}}}\right)$$



- We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right)$$

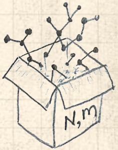
$$= F_R\left(\frac{1}{3}\right) + F'_\rho(1) F'_R\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} + \frac{\cancel{3}}{4}\frac{1}{3^{\cancel{2}}} + F'_\rho(1)\frac{\cancel{3}}{2}\frac{1}{\cancel{3}}.$$

- After some reallocation of objects, we have $F'_\rho(1) = \frac{13}{2}$.
- Finally: $\langle n \rangle = F'_\pi(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2} F'_P\left(\frac{1}{3}\right)$

$$= \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3^3} + \frac{13}{2}\left(\frac{1}{2} + \frac{\cancel{3}}{2}\frac{1}{3^{\cancel{2}}}\right) = \frac{5}{27} + \frac{13}{3}$$

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right)$$

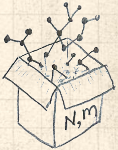
$$= F_R\left(\frac{1}{3}\right) + F'_\rho(1) F'_R\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} + \frac{\cancel{3}}{4}\frac{1}{3^{\cancel{2}}} + F'_\rho(1)\frac{\cancel{3}}{2}\frac{1}{\cancel{3}}.$$

- After some reallocation of objects, we have $F'_\rho(1) = \frac{13}{2}$.

- Finally: $\langle n \rangle = F'_\pi(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2} F'_P\left(\frac{1}{3}\right)$

$$= \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3^3} + \frac{13}{2}\left(\frac{1}{2} + \frac{\cancel{3}}{2}\frac{1}{3^{\cancel{2}}}\right) = \frac{5}{27} + \frac{13}{3} = \frac{122}{27}.$$



Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

$$F'_\rho(1) = F_R\left(F_\rho(1)\right) + F'_\rho(1) F'_R\left(F_\rho(1)\right)$$

$$= F_R\left(\frac{1}{3}\right) + F'_\rho(1) F'_R\left(\frac{1}{3}\right)$$

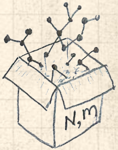
$$= \frac{1}{4} + \frac{\cancel{3}}{4}\frac{1}{3^{\cancel{2}}} + F'_\rho(1)\frac{\cancel{3}}{2}\frac{1}{\cancel{3}}.$$

- After some reallocation of objects, we have $F'_\rho(1) = \frac{13}{2}$.
-

Finally: $\langle n \rangle = F'_\pi(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2} F'_P\left(\frac{1}{3}\right)$

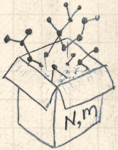
$$= \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3^3} + \frac{13}{2}\left(\frac{1}{2} + \frac{\cancel{3}}{2}\frac{1}{3^{\cancel{2}}}\right) = \frac{5}{27} + \frac{13}{3} = \frac{122}{27}.$$

- So, kinda small.



Nutshell

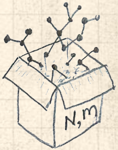
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



- Generating functions allow us to strangely calculate features of random networks.

Nutshell

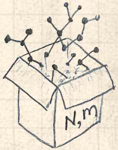
- Generating functions allow us to strangely calculate features of random networks.
- They're a bit scary and magical.



Nutshell

- Generating functions allow us to strangely calculate features of random networks.
- They're a bit scary and magical.
- We'll find generating functions useful for contagion.

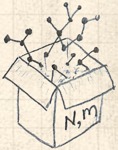
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



Nutshell

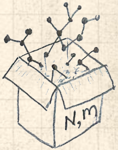
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

- Generating functions allow us to strangely calculate features of random networks.
- They're a bit scary and magical.
- We'll find generating functions useful for contagion.
- But we'll also see that more direct, physics-bearing calculations are possible.



Neural reboot (NR):

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References

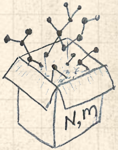


Elevation:

https://www.youtube.com/watch?v=bGBoZbT7cR8?rel=0

References I

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
A few examples
Average Component Size
References



[1] H. S. Wilf.
Generatingfunctionology.
A K Peters, Natick, MA, 3rd edition, 2006. pdf