

Generating Functions and Networks

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Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394, 2022-2023 | @pocsvox

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Simple examples:

Rolling dice and flipping coins:

$P_k^{(\text{d})} = \Pr(\text{throwing a } k) = 1/6 \text{ where } k = 1, 2, \dots, 6.$

$$F^{(\text{d})}(x) = \sum_{k=1}^6 p_k^{(\text{d})} x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

$p_0^{(\text{coin})} = \Pr(\text{head}) = 1/2, p_1^{(\text{coin})} = \Pr(\text{tail}) = 1/2.$

$$F^{(\text{coin})}(x) = p_0^{(\text{coin})}x^0 + p_1^{(\text{coin})}x^1 = \frac{1}{2}(1 + x).$$

A generating function for a probability distribution is called a **Probability Generating Function (p.g.f.)**.

We'll come back to these simple examples as we derive various delicious properties of generating functions.

Example

Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where geometrically, we have $c = 1 - e^{-\lambda}$

The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}.$$

Notice that $F(1) = c/(1 - e^{-\lambda}) = 1.$

For probability distributions, we must always have $F(1) = 1$ since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k = \sum_{k=0}^{\infty} P_k = 1.$$

Check die and coin p.g.f.'s.

Properties:

Average degree:

$$\begin{aligned} \langle k \rangle &= \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \Big|_{x=1} \\ &= \frac{d}{dx} F(x) \Big|_{x=1} = F'(1) \end{aligned}$$

In general, many calculations become simple, if a little abstract.

For our exponential example:

$$F'(x) = \frac{(1 - e^{-\lambda})e^{-\lambda}}{(1 - xe^{-\lambda})^2}.$$

$$\text{So: } \langle k \rangle = F'(1) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})}.$$

Check for die and coin p.g.f.'s.

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Useful pieces for probability distributions:

Normalization:

$$F(1) = 1$$

First moment:

$$\langle k \rangle = F'(1)$$

Higher moments:

$$\langle k^n \rangle = \left(x \frac{d}{dx} \right)^n F(x) \Big|_{x=1}$$

kth element of sequence (general):

$$P_k = \frac{1}{k!} \frac{d^k}{dx^k} F(x) \Big|_{x=0}$$

A beautiful, fundamental thing:

The generating function for the sum of two random variables

$$W = U + V$$

is

$$F_W(x) = F_U(x)F_V(x).$$

Conolve yourself with Convolutions: [Insert question from assignment 5](#)

Try with die and coin p.g.f.'s.

1. Add two coins (tail=0, head=1).
2. Add two dice.
3. Add a coin flip to one die roll.

Edge-degree distribution

Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

Let's re-express our condition in terms of generating functions.

We first need the g.f. for R_k .

We'll now use this notation:

$$\begin{aligned} F_P(x) &\text{ is the g.f. for } P_k. \\ F_R(x) &\text{ is the g.f. for } R_k. \end{aligned}$$

Giant component condition in terms of g.f. is:

$$\langle k \rangle_R = F'_R(1) > 1.$$

Now find how F_R is related to F_P ...

Outline

Generating Functions

- Definitions
- Basic Properties
- Giant Component Condition
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Generatingfunctionology^[1]

Idea: Given a sequence a_0, a_1, a_2, \dots , associate each element with a distinct function or other mathematical object.

Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

Definition:

The **generating function (g.f.)** for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Roughly: transforms a vector in R^∞ into a function defined on R^1 .

Related to Fourier, Laplace, Mellin, ...

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Edge-degree distribution

☞ We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to $j = k + 1$ and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{d}{dx} x^j$$

$$= \frac{1}{\langle k \rangle} \frac{d}{dx} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{d}{dx} (F_P(x) - P_0) = \frac{1}{\langle k \rangle} F'_P(x).$$

Finally, since $\langle k \rangle = F'_P(1)$,

$$F_R(x) = \frac{F'_P(x)}{F'_P(1)}$$

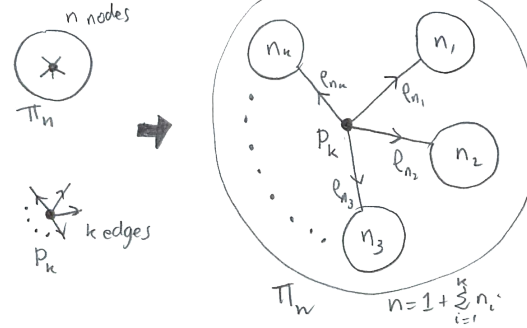
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Connecting probabilities:



☞ Markov property of random networks connects π_n , ρ_n , and P_k .

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Useful results we'll need for g.f.'s

Sneaky Result 1:

☞ Consider two random variables U and V whose values may be $0, 1, 2, \dots$

☞ Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V .

☞ SR1: If a third random variable is defined as

$$W = \sum_{i=1}^U V^{(i)} \text{ with each } V^{(i)} \stackrel{d}{=} V$$

then

$$F_W(x) = F_U(F_V(x))$$

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Edge-degree distribution

☞ Recall giant component condition is $\langle k \rangle_R = F'_R(1) > 1$.

☞ Since we have $F_R(x) = F'_P(x)/F'_P(1)$,

$$F'_R(x) = \frac{F''_P(x)}{F'_P(1)}$$

☞ Setting $x = 1$, our condition becomes

$$\frac{F''_P(1)}{F'_P(1)} > 1$$

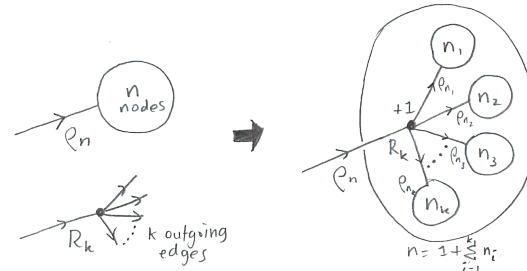
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Connecting probabilities:



☞ Markov property of random networks connects ρ_n and R_k .

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Proof of SR1:

Write probability that variable W has value k as W_k .

$$W_k = \sum_{j=0}^{\infty} U_j \times \Pr(\text{sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\{i_1, i_2, \dots, i_j\}} V_{i_1} V_{i_2} \dots V_{i_j}$$

$i_1 + i_2 + \dots + i_j = k$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} U_j \sum_{\{i_1, i_2, \dots, i_j\}} V_{i_1} V_{i_2} \dots V_{i_j} x^k$$

$i_1 + i_2 + \dots + i_j = k$

$$= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\{i_1, i_2, \dots, i_j\}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j}$$

$i_1 + i_2 + \dots + i_j = k$

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Size distributions

To figure out the **size of the largest component** (S_1), we need more resolution on component sizes.

Definitions:

☞ π_n = probability that a random node belongs to a finite component of size $n < \infty$.

☞ ρ_n = probability that a random end of a random link leads to a finite subcomponent of size $n < \infty$.

Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

$$\text{neighbors} \Leftrightarrow \text{components}$$

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G.f.'s for component size distributions:

☞

$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

The largest component:

☞ Subtle key: $F_{\pi}(1)$ is the probability that a node belongs to a **finite** component.

☞ Therefore: $S_1 = 1 - F_{\pi}(1)$.

Our mission, which we accept:

☞ Determine and connect the four generating functions

$$F_P, F_R, F_{\pi}, \text{ and } F_{\rho}.$$

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Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\{i_1, i_2, \dots, i_j\}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j}$$

$i_1 + i_2 + \dots + i_j = k$

$$\underbrace{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j}_{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j} = (F_V(x))^j$$

$$= \sum_{j=0}^{\infty} U_j (F_V(x))^j$$

$$= F_U(F_V(x))$$

☞ Alternate, groovier proof in the accompanying assignment.

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Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k ($k = 0, 1, 2, \dots$)

- SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

- Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.

$$\begin{aligned} \therefore F_V(x) &= \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k \\ &= x \sum_{j=0}^{\infty} U_j x^j = xF_U(x). \end{aligned}$$



Connecting generating functions:

- π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr(\text{sum of sizes of subcomponents at end of } k \text{ random links} = n - 1)$$



$$\text{Therefore: } \boxed{F_\pi(x) = x \underbrace{F_P(F_\rho(x))}_{\text{SR2}} \underbrace{F_R(x)}_{\text{SR1}}}$$

- Extra factor of x accounts for random node itself.



Connecting generating functions:

- We now have two functional equations connecting our generating functions:

$$F_\pi(x) = xF_P(F_\rho(x)) \text{ and } F_\rho(x) = xF_R(F_\rho(x))$$

- Taking stock: We know $F_P(x)$ and $F_R(x) = F'_P(x)/F'_P(1)$.
- We first untangle the second equation to find F_ρ
- We can do this because it **only involves** F_ρ and F_R .
- The first equation then immediately gives us F_π in terms of F_ρ and F_R .



Useful results we'll need for g.f.'s

Generalization of SR2:

- (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

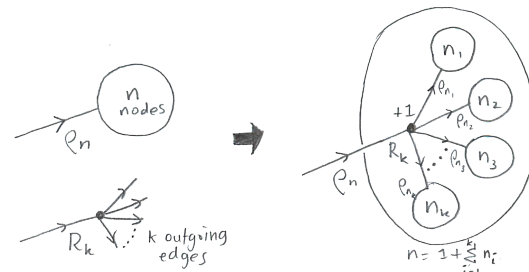
- (2) If $V = U - i$ then

$$F_V(x) = x^{-i} F_U(x)$$

$$= x^{-i} \sum_{k=0}^{\infty} U_k x^k$$



Connecting generating functions:



- Relate ρ_n to R_k and ρ_n through one step of recursion.



Component sizes

- Remembering vaguely what we are doing: Finding F_π to obtain the **fractional size of the largest component** $S_1 = 1 - F_\pi(1)$.

- Set $x = 1$ in our two equations:

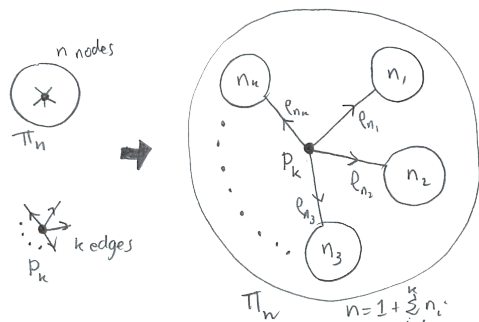
$$F_\pi(1) = F_P(F_\rho(1)) \text{ and } F_\rho(1) = F_R(F_\rho(1))$$

- Solve second equation numerically for $F_\rho(1)$.
- Plug $F_\rho(1)$ into first equation to obtain $F_\pi(1)$.



Connecting generating functions:

- Goal: figure out forms of the component generating functions, F_π and F_ρ .



- Relate π_n to P_k and ρ_n through one step of recursion.



Connecting generating functions:

- ρ_n = probability that a random link leads to a finite subcomponent of size n .

- Invoke one step of recursion:

ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n - 1$,

$$= \sum_{k=0}^{\infty} R_k \times \Pr(\text{sum of sizes of subcomponents at end of } k \text{ random links} = n - 1)$$



$$\text{Therefore: } \boxed{F_\rho(x) = x \underbrace{F_R(F_\rho(x))}_{\text{SR2}} \underbrace{F_P(x)}_{\text{SR1}}}$$

- Again, extra factor of x accounts for random node itself.



Component sizes

Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\Rightarrow F_R(x) = F'_P(x)/F'_P(1)$$

$$= \langle k \rangle e^{-\langle k \rangle (1-x)} / \langle k \rangle e^{-\langle k \rangle (1-x')} \Big|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)} = F_P(x) \quad \dots \text{aha!}$$

- RHS's of our two equations are the same.
- So $F_\pi(x) = F_\rho(x) = xF_R(F_\rho(x)) = xF_R(F_\pi(x))$
- Consistent with how our dirty (but wrong) trick worked earlier ...
- $\pi_n = \rho_n$ just as $P_k = R_k$.



Component sizes

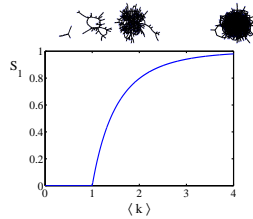
We are down to $F_\pi(x) = xF_R(F_\pi(x))$ and $F_R(x) = e^{-(k)(1-x)}$.

$$\therefore F_\pi(x) = xe^{-(k)(1-F_\pi(x))}$$

We're first after $S_1 = 1 - F_\pi(1)$ so set $x = 1$ and replace $F_\pi(1)$ by $1 - S_1$:

$$1 - S_1 = e^{-kS_1}$$

$$\text{Or: } \langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$$



Just as we found with our dirty trick ...
Again, we (usually) have to resort to numerics ...

A few simple random networks to contemplate and play around with:

Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

- $P_k = \delta_{k1}$.
- $P_k = \delta_{k2}$.
- $P_k = \delta_{k3}$.
- $P_k = \delta_{kk'}$ for some fixed $k' \geq 0$.
- $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.
- $P_k = a\delta_{k1} + (1-a)\delta_{k3}$, with $0 \leq a \leq 1$.
- $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{kk'}$ for some fixed $k' \geq 2$.
- $P_k = a\delta_{k1} + (1-a)\delta_{kk'}$ for some fixed $k' \geq 2$ with $0 \leq a \leq 1$.

A joyful example \square :

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$$

- We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}$.
- A giant component exists because: $\langle k \rangle_R = 0 \times 1/4 + 2 \times 3/4 = 3/2 > 1$.
- Generating functions for P_k and R_k :

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2$$

- Check for goodness:
 - $F_R(x) = F'_P(x)/F'_P(1)$ and $F_P(1) = F_R(1) = 1$.
 - $F'_P(1) = \langle k \rangle_P = 2$ and $F'_R(1) = \langle k \rangle_R = \frac{3}{2}$.

Things to figure out: Component size generating functions for π_n and ρ_n , and the size of the giant component.

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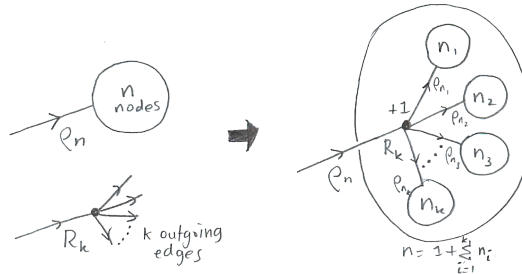
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Find $F_\rho(x)$ first:

We know: $F_\rho(x) = xF_R(F_\rho(x))$.



Sticking things in things, we have:

$$F_\rho(x) = x \left(\frac{1}{4} + \frac{3}{4} [F_\rho(x)]^2 \right)$$

Rearranging:

$$3x [F_\rho(x)]^2 - 4F_\rho(x) + x = 0$$

Please and thank you:

$$F_\rho(x) = \frac{2}{3x} \left(1 \pm \sqrt{1 - \frac{3}{4}x^2} \right)$$

- Time for a Taylor series expansion.
- The promise: non-negative powers of x with non-negative coefficients.
- First: which sign do we take?

- Because ρ_n is a probability distribution, we know $F_\rho(1) \leq 1$ and $F_\rho(x) \leq 1$ for $0 \leq x \leq 1$.
- Thinking about the limit $x \rightarrow 0$ in

$$F_\rho(x) = \frac{2}{3x} \left(1 \pm \sqrt{1 - \frac{3}{4}x^2} \right)$$

we see that the positive sign solution blows to smithereens, and the negative one is okay.

So we must have:

$$F_\rho(x) = \frac{2}{3x} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right)$$

We can now deploy the Taylor expansion:

$$(1+z)^\theta = \binom{\theta}{0}z^0 + \binom{\theta}{1}z^1 + \binom{\theta}{2}z^2 + \binom{\theta}{3}z^3 + \dots$$

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Let's define a binomial for arbitrary θ and $k = 0, 1, 2, \dots$:

$$\binom{\theta}{k} = \frac{\Gamma(\theta+1)}{\Gamma(k+1)\Gamma(\theta-k+1)}$$

For $\theta = \frac{1}{2}$, we have:

$$\begin{aligned} (1+z)^{\frac{1}{2}} &= \binom{\frac{1}{2}}{0}z^0 + \binom{\frac{1}{2}}{1}z^1 + \binom{\frac{1}{2}}{2}z^2 + \dots \\ &= \frac{\Gamma(\frac{3}{2})}{\Gamma(1)\Gamma(\frac{3}{2})}z^0 + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)\Gamma(\frac{1}{2})}z^1 + \frac{\Gamma(\frac{3}{2})}{\Gamma(3)\Gamma(-\frac{1}{2})}z^2 + \dots \\ &= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \dots \end{aligned}$$

where we've used $\Gamma(x+1) = x\Gamma(x)$ and noted that $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.

Note: $(1+z)^\theta \sim 1 + \theta z$ always.

Totally psyched, we go back to here:

$$F_\rho(x) = \frac{2}{3x} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right)$$

Setting $z = -\frac{3}{4}x^2$ and expanding, we have:

$$\begin{aligned} F_\rho(x) &= \frac{2}{3x} \left(1 - \left[1 + \frac{1}{2} \left(-\frac{3}{4}x^2 \right)^1 - \frac{1}{8} \left(-\frac{3}{4}x^2 \right)^2 + \frac{1}{16} \left(-\frac{3}{4}x^2 \right)^3 + \dots \right] \right) \end{aligned}$$

Giving:

$$\begin{aligned} F_\rho(x) &= \sum_{n=0}^{\infty} \rho_n x^n = \frac{1}{4}x + \frac{3}{64}x^3 + \frac{9}{512}x^5 + \dots + \frac{2}{3} \left(\frac{3}{4} \right)^k \frac{(-1)^{k+1} \Gamma(\frac{3}{2})}{\Gamma(k+1)\Gamma(\frac{3}{2}-k)} x^{2k-1} + \dots \end{aligned}$$

Do odd powers make sense?

We can now find $F_\pi(x)$ with:

$$\begin{aligned} F_\pi(x) &= xF_P(F_\rho(x)) \\ &= x \frac{1}{2} \left((F_\rho(x))^1 + (F_\rho(x))^3 \right) \\ &= x \frac{1}{2} \left[\frac{2}{3x} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right) + \frac{2^3}{(3x)^3} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right)^3 \right] \end{aligned}$$

- Delicious.
- In principle, we can now extract all the π_n .
- But let's just find the size of the giant component.

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First, we need $F_\rho(1)$:

$$F_\rho(x)|_{x=1} = \frac{2}{3 \cdot 1} \left(1 - \sqrt{1 - \frac{3}{4}1^2} \right) = \frac{1}{3}.$$

This is the probability that a random edge leads to a sub-component of finite size.

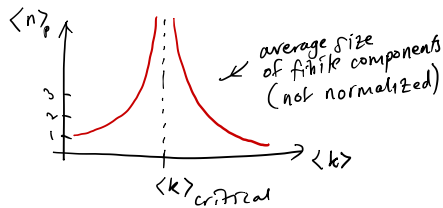
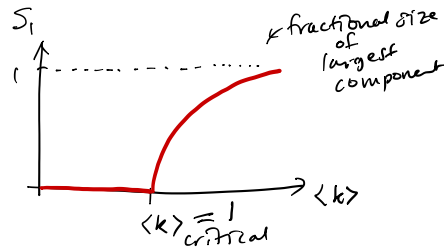
Next:

$$F_\pi(1) = 1 - F_P(F_\rho(1)) = F_P\left(\frac{1}{3}\right) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \left(\frac{1}{3}\right)^3 = \frac{5}{27}.$$

This is the probability that a random chosen node belongs to a finite component.

Finally, we have

$$S_1 = 1 - F_\pi(1) = 1 - \frac{5}{27} = \frac{22}{27}.$$



Average component size

Example: Standard random graphs.

Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.

Two differentiated equations reduce to only one:

$$F'_\pi(x) = F_P(F_\pi(x)) + xF'_\pi(x)F'_P(F_\pi(x))$$

$$\text{Rearrange: } F'_\pi(x) = \frac{F_P(F_\pi(x))}{1 - xF'_P(F_\pi(x))}$$

Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$

Replace $F_P(F_\pi(x))$ using $F_\pi(x) = xF_P(F_\pi(x))$.

Set $x = 1$ and replace $F_\pi(1)$ with $1 - S_1$.

$$\text{End result: } \langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$

Average component size

Our result for standard random networks:

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$

Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.

Look at what happens when we increase $\langle k \rangle$ to 1 from below.

We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

This blows up as $\langle k \rangle \rightarrow 1$.

Reason: we have a power law distribution of component sizes at $\langle k \rangle = 1$.

Typical critical point behavior ...

Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.

We're after:

$$\langle n \rangle = F'_\pi(1) = F_P(F_\rho(1)) + F'_\rho(1)F'_P(F_\rho(1))$$

where we first need to compute

$$F'_\rho(1) = F_R(F_\rho(1)) + F'_\rho(1)F'_R(F_\rho(1)).$$

Place stick between teeth, and recall that we have:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2.$$

Differentiation gives us:

$$F'_P(x) = \frac{1}{2} + \frac{3}{2}x^2 \text{ and } F'_R(x) = \frac{3}{2}x.$$

We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

$$\begin{aligned} F'_\pi(1) &= F_R(F_\rho(1)) + F'_\rho(1)F'_R(F_\rho(1)) \\ &= F_R\left(\frac{1}{3}\right) + F'_\rho(1)F'_R\left(\frac{1}{3}\right) \\ &= \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3^2} + F'_\rho(1) \cdot \frac{3}{2} \cdot \frac{1}{3}. \end{aligned}$$

After some reallocation of objects, we have $F'_\rho(1) = \frac{13}{2}$.

$$\begin{aligned} \text{Finally: } \langle n \rangle &= F'_\pi(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2}F'_P\left(\frac{1}{3}\right) \\ &= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3^3} + \frac{13}{2} \left(\frac{1}{2} + \frac{3}{2} \cdot \frac{1}{3^2} \right) = \frac{5}{27} + \frac{13}{3} = \frac{122}{27}. \end{aligned}$$

So, kinda small.

Average component size

Next: find **average size** of **finite** components $\langle n \rangle$.

Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.

Try to avoid finding $F_\pi(x)$...

Starting from $F_\pi(x) = xF_P(F_\rho(x))$, we differentiate:

$$F'_\pi(x) = F_P(F_\rho(x)) + xF'_\rho(x)F'_P(F_\rho(x))$$

While $F_\rho(x) = xF_R(F_\rho(x))$ gives

$$F'_\rho(x) = F_R(F_\rho(x)) + xF'_\rho(x)F'_R(F_\rho(x))$$

Now set $x = 1$ in both equations.

We solve the second equation for $F'_\rho(1)$ (we must already have $F_\rho(1)$).

Plug $F'_\rho(1)$ and $F_\rho(1)$ into first equation to find $F'_\pi(1)$.

Average component size

Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$

As $\langle k \rangle \rightarrow 0$, $S_1 = 0$, and $\langle n \rangle \rightarrow 1$.

All nodes are isolated.

As $\langle k \rangle \rightarrow \infty$, $S_1 \rightarrow 1$ and $\langle n \rangle \rightarrow 0$.

No nodes are outside of the giant component.

Extra on largest component size:

For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}/N$.

For $\langle k \rangle < 1$, $S_1 \sim (\log N)/N$.

Nutshell

Generating functions allow us to strangely calculate features of random networks.

They're a bit scary and magical.


We'll find generating functions useful for contagion.

But we'll also see that more direct, physics-bearing calculations are possible.

References I

[1] H. S. Wilf.

Generatingfunctionology.

A K Peters, Natick, MA, 3rd edition, 2006. pdf 

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