Random walks and diffusion on networks

Last updated: 2022/08/29, 00:04:32 EDT

Principles of Complex Systems, Vols. 1, 2, & 3D CSYS/MATH 300, 303, & 394, 2022-2023 | @pocsvox

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Outline

Random walks on networks

Random walks on networks—basics:

- lmagine a single random walker moving around on a network.
- At t = 0, start walker at node j and take time to be discrete.
- 🚯 Q: What's the long term probability distribution for where the walker will be?
- Befine $p_i(t)$ as the probability that at time step t, our walker is at node *i*.
- We want to characterize the evolution of $\vec{p}(t)$.
- Sirst task: connect $\vec{p}(t+1)$ to $\vec{p}(t)$.
- Let's call our walker Barry.
- 🚯 Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Norse still: Barry is texting.

Where is Barry?

- lacktrian consider simple undirected, ergodic (strongly connected) networks. As usual, represent network by adjacency matrix
 - A where
 - $a_{ij} = 1$ if *i* has an edge leading to *j*, $a_{ij} = 0$ otherwise.
 - Barry is at node j at time t with probability $p_i(t)$.
 - ln the next time step, he randomly lurches toward one of j's neighbors.
 - \bigotimes Barry arrives at node *i* from node *j* with probability $\frac{1}{k_i}$ if an edge connects *j* to *i*.
 - Equation-wise:

$$p_i(t+1) = \sum_{j=1}^n \frac{1}{k_j} a_{ji} p_j(t).$$

where k_i is j's degree. Note: $k_i = \sum_{i=1}^n a_{ij}$.

- Inebriation and diffusion:
- Random walks on networks

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Where is Barry?

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Diffusion

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- Excellent observation: The same equation applies for stuff moving around a network, such that at each time step all material at node i is sent to its neighbors.
- $x_i(t)$ = amount of stuff at node *i* at time *t*.

$$x_i(t+1) = \sum_{j=1}^n \frac{1}{k_j} a_{ji} x_j(t)$$

- Random walking is equivalent to diffusion C.
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Where is Barry?

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By inspection, we see that

$$\infty) = \frac{1}{\sum_{i=1}^{n} k_i} \vec{k}$$

- satisfies $\vec{p}(\infty) = A^{\mathsf{T}} K^{-1} \vec{p}(\infty)$ with eigenvalue 1.
- \bigotimes We will find Barry at node *i* with probability proportional to its degree k_i .
- 🗞 Beautiful implication: probability of finding Barry travelling along any edge is uniform.
- Diffusion in real space smooths things out.
- On networks, uniformity occurs on edges.
- So in fact, diffusion in real space is about the edges too but we just don't see that.

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Random walks or networks

- Solution $A^{\mathsf{T}}K^{-1}$ is similar to a real symmetric matrix if $A = A^{\mathsf{T}}$.
- Solution Consider the transformation $M = K^{-1/2}$:

 $K^{-1/2} A^{\mathsf{T}} K^{-1} K^{1/2} = K^{-1/2} A^{\mathsf{T}} K^{-1/2}.$

Since $A^{\mathsf{T}} = A$, we have

Other pieces:

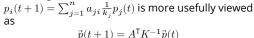
 $(K^{-1/2}AK^{-1/2})^{\mathsf{T}} = K^{-1/2}AK^{-1/2}.$

- \mathbb{R} Upshot: $A^{\mathsf{T}}K^{-1} = AK^{-1}$ has real eigenvalues and a complete set of orthogonal eigenvectors.
- Can also show that maximum eigenvalue magnitude is indeed 1.

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Linear algebra-based excitement:

where $[K_{ij}] = [\delta_{ij}k_i]$ has node degrees on the main diagonal and zeros everywhere else.

- So... we need to find the dominant eigenvalue of $A^{\mathsf{T}}K^{-1}$.
- Expect this eigenvalue will be 1 (doesn't make) sense for total probability to change).
- The corresponding eigenvector will be the limiting probability distribution (or invariant measure).
- Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.



