Assortativity and Mixing

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Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394, 2022–2023 | @pocsvox

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Outline

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- Build on work by Newman [5, 6], and Boguñá and Serano. [1].

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- Write $\mathbf{E} = [e_{\mu\nu}]$, $\vec{a} = [a_{\mu}]$, and $\vec{b} = [b_{\nu}]$.
- Requirements:

$$\sum_{\mu}\sum_{\nu} e_{\mu\nu} = 1, \ \sum_{\nu} e_{\mu\nu} = a_{\mu}, \text{ and } \sum_{\mu} e_{\mu\nu} = b_{\nu}.$$

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- Disassortative networks can be hard to build and may require constraints on the $e_{\mu\nu}$.
- Basic story: level of assortativity reflects the degree to which nodes are connected to nodes within their group.

Correlation coefficient:

- Quantify the level of assortativity with the following assortativity coefficient [6]:

$$r = \frac{\sum_{\mu} e_{\mu\mu} - \sum_{\mu} a_{\mu} b_{\mu}}{1 - \sum_{\mu} a_{\mu} b_{\mu}} = \frac{\text{Tr}\,\mathbf{E} - ||E^2||_1}{1 - ||E^2||_1}$$

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- When $e_{\mu\mu} = a_\mu b_\mu$, we have $r = 0$. ✔

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-
$$r_{\text{min}} = \frac{-||E^2||_1}{1 - ||E^2||_1}$$

where $-1 \leq r_{\text{min}} < 0$.

Watch your step

zzzhhhhwoooommmmmm

NuhnuhNuhnuhNuhnuhNuhnuhNuhnuhNuhnuh

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- This is the observed normalized deviation from randomness in the product jk.

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- Directed networks still fine but we will assume from here on that $e_{jk} = e_{kj}$.

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- Notation reconciliation for undirected networks:

$$r = \frac{\sum_{j\,k} j\,k(e_{jk} - R_j R_k)}{\sigma_R^2}$$

where, as before, R_k is the probability that a randomly chosen edge leads to a node of degree $k+1$, and

$$\sigma_R^2 = \sum_j j^2 R_j - \left[\sum_j j R_j\right]^2.$$

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- Mildly sneaky as variables need to be independent for us to be truly happy and edges are correlated...

Measurements of degree-degree correlations

	Group	Network	Type	Size n	Assortativity r	Error σ_r
	a	Physics coauthorship	undirected	52 909	0.363	0.002
	a	Biology coauthorship	undirected	1 520 251	0.127	0.0004
	b	Mathematics coauthorship	undirected	253 339	0.120	0.002
Social	c	Film actor collaborations	undirected	449 913	0.208	0.0002
	d	Company directors	undirected	7 673	0.276	0.004
	e	Student relationships	undirected	573	−0.029	0.037
	f	Email address books	directed	16 881	0.092	0.004
	g	Power grid	undirected	4 941	−0.003	0.013
Technological	h	Internet	undirected	10 697	−0.189	0.002
	i	World Wide Web	directed	269 504	−0.067	0.0002
	j	Software dependencies	directed	3 162	−0.016	0.020
	k	Protein interactions	undirected	2 115	−0.156	0.010
	l	Metabolic network	undirected	765	−0.240	0.007
Biological	m	Neural network	directed	307	−0.226	0.016
	n	Marine food web	directed	134	−0.263	0.037
	o	Freshwater food web	directed	92	−0.326	0.031

- Social networks tend to be assortative (homophily)
- Technological and biological networks tend to be disassortative

Hot lava

"I like it"

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  2. find the probability and extent of spread for simple disease models.
  3. find the probability of spreading for simple threshold models.

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- Plan: Find the generating function $F_j(x;\vec{B}_1) = \sum_{n=0}^{\infty} f_{n,j} x^n$.

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- Recursive relationship:

$$F_j(x;\vec{B}_1) = x^0 \sum_{k=0}^{\infty} \frac{e_{jk}}{R_j}(1 - B_{k+1,1}) + x \sum_{k=0}^{\infty} \frac{e_{jk}}{R_j} B_{k+1,1} \left[F_k(x;\vec{B}_1)\right]^k .$$

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- First term = **Pr** (that the first node we reach is not in the game).
- Second term involves **Pr** (we hit an active node which has k outgoing edges).
- Next: find average size of active components reached by following a link from a degree $j+1$ node = $F_j'(1;\vec{B}_1)$.

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- Rearranging and introducing a sneaky δ_{jk}:

$$\sum_{k=0}^{\infty} \left(\delta_{jk} R_k - k B_{k+1,1} e_{jk}\right) F_k'(1;\vec{B}_1) = \sum_{k=0}^{\infty} e_{jk} B_{k+1,1}.$$

Spreading on degree-correlated networks

- In matrix form, we have

$$\mathbf{A}_{\mathbf{E},\vec{B}_1}\vec{F}'(1;\vec{B}_1) = \mathbf{E}\vec{B}_1$$

where

$$\left[\mathbf{A}_{\mathbf{E},\vec{B}_1}\right]_{j+1,k+1} = \delta_{jk}R_k - kB_{k+1,1}e_{jk},$$

$$\left[\vec{F}'(1;\vec{B}_1)\right]_{k+1} = F'_k(1;\vec{B}_1),$$

$$[\mathbf{E}]_{j+1,k+1} = e_{jk}, \text{ and } \left[\vec{B}_1\right]_{k+1} = B_{k+1,1}.$$

Spreading on degree-correlated networks

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- General condition details:

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- Bonusville: We'll find a much better version of this set of conditions later...

Retrieving the cascade condition for uncorrelated networks

Outline

Spreading on degree-correlated networks

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- Generating function:

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- Generating function for vulnerable component size is more complicated.

Spreading on degree-correlated networks

- Want probability of not reaching a finite component.

$$P_{\text{trig}} = S_{\text{trig}} = 1 - H(1; \vec{B}_1)$$
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- Iterative methods should work here.

Outline

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- Need to compute $\theta_{j,t}$, the probability that an edge leading to a degree j node is infected at time t.
- Evolution of edge activity probability:

$$\theta_{j,t+1} = G_j(\vec{\theta}_t) = \phi_0 + (1-\phi_0)\times$$

$$\sum_{k=1}^{\infty} \frac{e_{j-1,k-1}}{R_{j-1}} \sum_{i=0}^{k-1} \binom{k-1}{i} \theta_{k,t}^{i} (1-\theta_{k,t})^{k-1-i} B_{ki}.$$

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- Overall active fraction's evolution:

$$\phi_{t+1} = \phi_0 + (1-\phi_0) \sum_{k=0}^{\infty} P_k \sum_{i=0}^{k} \binom{k}{i} \theta_{k,t}^{i} (1-\theta_{k,t})^{k-i} B_{ki}.$$

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$$\theta_{j,t+1} = G_j(\vec{0}) + \sum_{k=1}^{\infty} \frac{\partial G_j(\vec{0})}{\partial \theta_{k,t}} \theta_{k,t} + \frac{1}{2!} \sum_{k=1}^{\infty} \frac{\partial^2 G_j(\vec{0})}{\partial \theta_{k,t}^2} \theta_{k,t}^2 + \ldots$$

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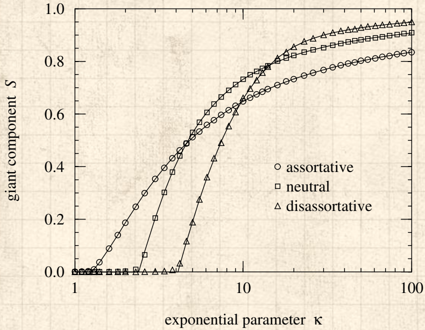
- If $G_j(\vec{0}) \neq 0$ for at least one j, always have some infection.
- If $G_j(\vec{0}) = 0 \, \forall \, j$, want largest eigenvalue $\left[\frac{\partial G_j(\vec{0})}{\partial \theta_{k,t}}\right] > 1$.
- Condition for spreading is therefore dependent on eigenvalues of this matrix:

$$\frac{\partial G_j(\vec{0})}{\partial \theta_{k,t}} = \frac{e_{j-1,k-1}}{R_{j-1}} (k-1) B_{k1}$$

Insert question from assignment 9

How the giant component changes with assortativity:



from Newman, 2002 [5]

- More assortative networks percolate for lower average degrees
- But disassortative networks end up with higher extents of spreading.

Toy guns don't pretend blow up things ...

Splsshht

Robust-yet-Fragileness of the Death Star

References I

[1] M. Boguñá and M. Ángeles Serrano.
Generalized percolation in random directed networks.
Phys. Rev. E, 72:016106, 2005. pdf

[2] P. S. Dodds and J. L. Payne.
Analysis of a threshold model of social contagion on degree-correlated networks.
Phys. Rev. E, 79:066115, 2009. pdf

[3] B. Efron and C. Stein.
The jackknife estimate of variance.
The Annals of Statistics, 9:586–596, 1981. pdf

[4] J. P. Gleeson.
Cascades on correlated and modular random networks.
Phys. Rev. E, 77:046117, 2008. pdf

References II

[5] M. Newman.
Assortative mixing in networks.
Phys. Rev. Lett., 89:208701, 2002. pdf

[6] M. E. J. Newman.
Mixing patterns in networks.
Phys. Rev. E, 67:026126, 2003. pdf