**Principles of Complex Systems, Vols. 1, 2, & 3D**
**CSYS/MATH 300, 303, & 394**
**University of Vermont, Fall 2022**
**Assignment 18**
**Huge raving maniac with national borders and an anthem**

**Due:** Friday, February 17, by 11:59 pm
https://pdodds.w3.uvm.edu/teaching/courses/2022-2023pocsverse/assignments/18/
*Some useful reminders:*
**Deliverator:** Prof. Peter Sheridan Dodds (contact through Teams)
**Assistant Deliverator:** Dylan Casey (contact through Teams)
**Office:** The Ether
**Office hours:** See Teams calendar
**Course website:** https://pdodds.w3.uvm.edu/teaching/courses/2022-2023pocsverse
**Overleaf:** LaTeX templates and settings for all assignments are available at
https://www.overleaf.com/project/631238b0281a33de67fc1c2b.

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All parts are worth 3 points unless marked otherwise. Please show all your workingses clearly and list the names of others with whom you ~~conspired~~ collaborated.

For coding, we recommend you improve your skills with Python, R, and/or Julia. The (evil) Deliverator uses (evil) Matlab.

Graduate students are requested to use LaTeX (or related TeX variant). If you are new to LaTeX, please endeavor to submit at least $n$ questions per assignment in LaTeX, where $n$ is the assignment number.

**Assignment submission:**

Via Blackboard.

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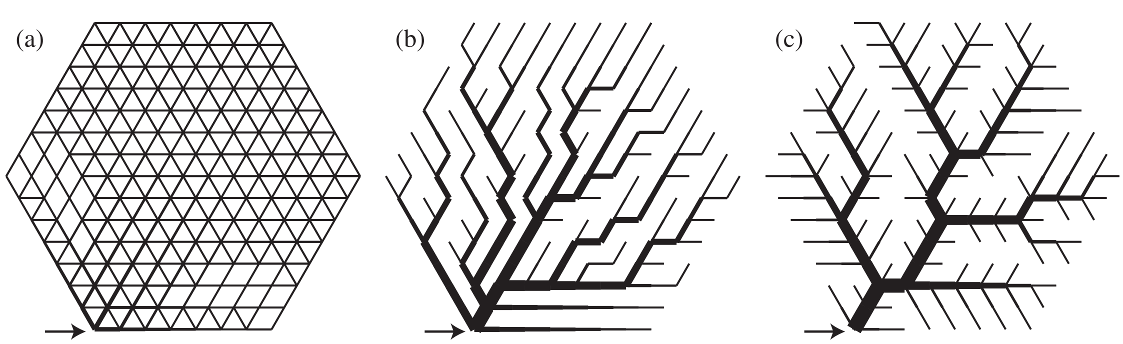
1. (3 + 3) Reproduce Bohn and Magnasco's Figs. 2a and 2b in [1]:



Steps are given below but please read through the paper to understand how they set things up.

The full team is encouraged to work together on Teams.

(a) Done (previous assignment): Construct an adjacency matrix $\mathbf{A}$ representing the hexagonal lattice used in [1]. Plot this adjacency matrix.

(b) Run a minimization procedure to construct Figs. 2a and 2b which correspond to $\gamma = 2$ and $\gamma = 1/2$. Steps:

   i. Set each link's length to 1 (the $d_{kl}$). The goal then reduces to minimizing the cost
$$C = \sum_{k,l} |I_{kl}|^{\Gamma}$$
   where $I_{kl}$ is the current on link $kl$ and $\Gamma = 2\gamma/(\gamma + 1)$.

   ii. Place a current source of nominal size $i_0$ at one node (as indicated in Fig. 2 above).

   iii. All other nodes are sinks, drawing a current of
$$i_k = -\frac{i_0}{N_{\text{nodes}-1}}.$$

   iv. Suggest setting $i_0 = 1000$ (arbitrary but useful value given the size of the network).

   v. Generate an initial set of random conductances for each link, the $\{\kappa_{kl}\}$. From the paper, these must sum to some global constraint as
$$K = \left( \sum_{k,l} \kappa_{kl}^{\gamma} \right)^{1/\gamma}.$$
   This constraint is meant to represent a limitation on the amount of material that can be used to build the network.
   Note: There seems to be no reason not to set $K = 1$. However, taking the initial value of $K$ determined by the initial set of random conductances would work.
   To our notational peril, we now have a lot of $k$ types on deck.

   vi. Solve the following to determine the potential $U$ at each node, and hence the current on each link using:
$$i_k = \sum_{l} \kappa_{kl}(U_k - U_l),$$
   and then
$$I_{kl} = \kappa_{kl}(U_l - U_k).$$

Note: the paper erroneously has $I_{kl} = R_{kl}(U_l - U_k)$ below equation 4; there are a few other instances of similar miswritings of $R_{kl}$ instead of $\kappa_{kl}$.

vii. Now, use scaling in equation (10) to compute a new set of $\{\kappa_{kl}\}$ from the $I_{kl}$. Everything boils down to

$$\kappa_{kl} \propto |I_{kl}|^{-(\Gamma-2)},$$

where the constant of proportionality is determined by again making sure $K^\gamma = \sum_{k,l} \kappa_{kl}^\gamma$.

Some help—Let's sort out the key equation:

$$i_k = \sum_l \kappa_{kl}(U_k - U_l).$$

Each time we loop around through this equation, we know the $i_k$ and the $\kappa_{kl}$ and must determine the $U_k$. In matrixology, we love $A\vec{x} = \vec{b}$ problems so let's see if we can fashion one:

$$i_k = \sum_l \kappa_{kl}(U_k - U_l)$$

$$= \sum_l \kappa_{kl} U_k - \sum_l \kappa_{kl} U_l$$

$$= U_k \sum_l \kappa_{kl} - \sum_l \mathbf{K}_{kl} U_l$$

$$= \lambda_k U_k - [\mathbf{K}\vec{U}]_k$$

where we have set $\lambda_k = \sum_l \kappa_{kl}$, the sum of the $k$th row of the matrix $\mathbf{K}$. We now construct a diagonal matrix $\Lambda$ with the $\lambda_k$ on the diagonal, and obtain:

$$\vec{i} = (\Lambda - \mathbf{K})\,\vec{U}.$$

The above is in the form $A\vec{x} = \vec{b}$ so we can solve for $\vec{U}$ using standard features of R, Matlab, Python, … (hopefully).

# References


[1] S. Bohn and M. O. Magnasco. Structure, scaling, and phase transition in the optimal transport network. *Phys. Rev. Lett.*, 98:088702, 2007. pdf