Optimal Supply Networks III: Redistribution

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Principles of Complex Systems, Vols. 1 & 2
CSYS/MATH 300 and 303, 2021–2022 | @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

The PoCSverse
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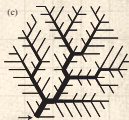Distributed
Sources
Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
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Sealie & Lambie Productions

The PoCSverse
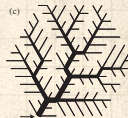Optimal Supply
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Special Guest Executive Producer

 On Instagram at pratchett_the_cat

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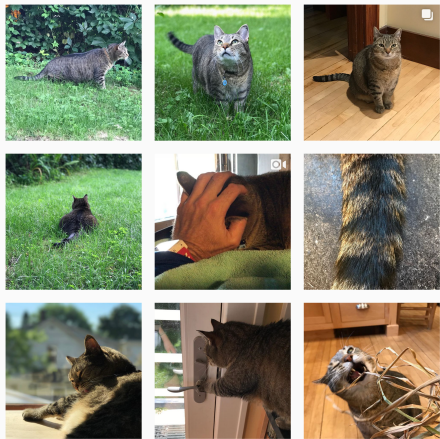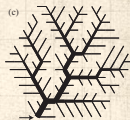References

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Many sources, many sinks

How do we distribute sources?

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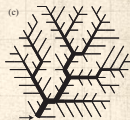A reasonable derivation
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Many sources, many sinks

How do we distribute sources?

🎲 Focus on 2-d (results generalize to higher dimensions).

Many sources, many sinks

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How do we distribute sources?

- 📦 Focus on 2-d (results generalize to higher dimensions).
- 📦 Sources = hospitals, post offices, pubs, ...

Many sources, many sinks

How do we distribute sources?

- 📦 Focus on 2-d (results generalize to higher dimensions).
- 📦 Sources = hospitals, post offices, pubs, ...
- 📦 Key problem: How do we cope with uneven population densities?

Many sources, many sinks

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How do we distribute sources?

- Focus on 2-d (results generalize to higher dimensions).
- Sources = hospitals, post offices, pubs, ...
- Key problem: How do we cope with uneven population densities?
- Obvious: if density is uniform then sources are best distributed uniformly.

Many sources, many sinks

How do we distribute sources?

- 🎲 Focus on 2-d (results generalize to higher dimensions).
- 🎲 Sources = hospitals, post offices, pubs, ...
- 🎲 Key problem: How do we cope with uneven population densities?
- 🎲 Obvious: if density is uniform then sources are best distributed uniformly.
- 🎲 Which lattice is optimal?

Many sources, many sinks

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How do we distribute sources?

- 🎲 Focus on 2-d (results generalize to higher dimensions).
- 🎲 Sources = hospitals, post offices, pubs, ...
- 🎲 Key problem: How do we cope with uneven population densities?
- 🎲 Obvious: if density is uniform then sources are best distributed uniformly.
- 🎲 Which lattice is optimal? The hexagonal lattice
- 🎲 Q2: Given population density is uneven, what do we do?

Many sources, many sinks

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How do we distribute sources?

- Focus on 2-d (results generalize to higher dimensions).
- Sources = hospitals, post offices, pubs, …
- Key problem: How do we cope with uneven population densities?
- Obvious: if density is uniform then sources are best distributed uniformly.
- Which lattice is optimal? The hexagonal lattice
- Q2: Given population density is uneven, what do we do?
- We'll follow work by Stephan (1977, 1984) [4, 5], Gastner and Newman (2006) [2], Um *et al.* (2009) [6], and work cited by them.

THE SUPPLY
MOST FAIR

Optimal source allocation

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Solidifying the basic problem

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Solidifying the basic problem

♟ Given a region with some population distribution ρ, most likely uneven.

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Solidifying the basic problem

- Given a region with some population distribution ρ, most likely uneven.
- Given resources to build and maintain N facilities.

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Solidifying the basic problem

- Given a region with some population distribution ρ, most likely uneven.
- Given resources to build and maintain N facilities.
- Q: How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?

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"Optimal design of spatial distribution networks"

Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]
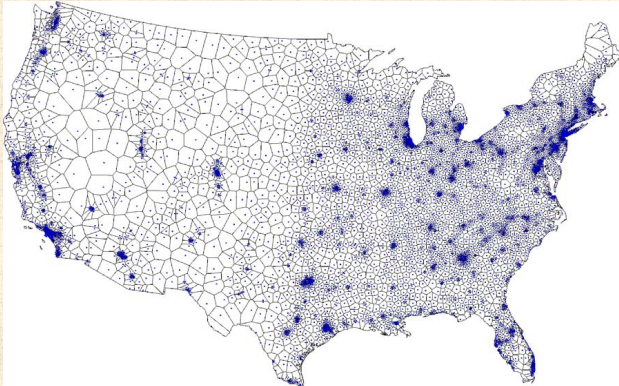
🎲 Approximately optimal location of 5000 facilities.

🎲 Based on 2000 Census data.

Optimal source allocation

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Optimal facility density ρ_{fac} vs. population density ρ_{pop}.

Optimal source allocation

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Optimal facility density ρ_{fac} vs. population density ρ_{pop}.

Fit is $\rho_{\text{fac}} \propto \rho_{\text{pop}}^{0.66}$ with $r^2 = 0.94$.

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- Optimal facility density ρ_{fac} vs. population density ρ_{pop}.
- Fit is $\rho_{\text{fac}} \propto \rho_{\text{pop}}^{0.66}$ with $r^2 = 0.94$.
- Looking good for a 2/3 power ...

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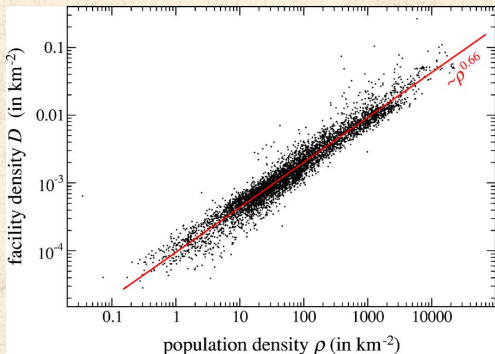   Global redistribution
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Size-density law:
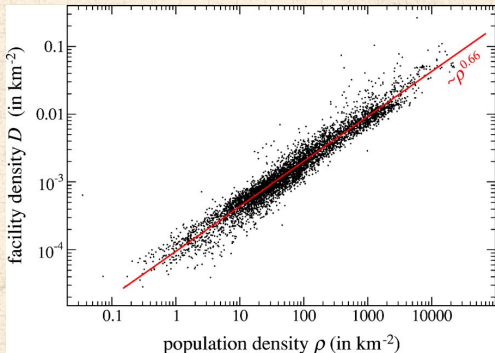
$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{2/3}$$

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Size-density law:

$$\rho_{\mathsf{fac}} \propto \rho_{\mathsf{pop}}^{2/3}$$

Why?

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Size-density law:

$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{2/3}$$

Why?

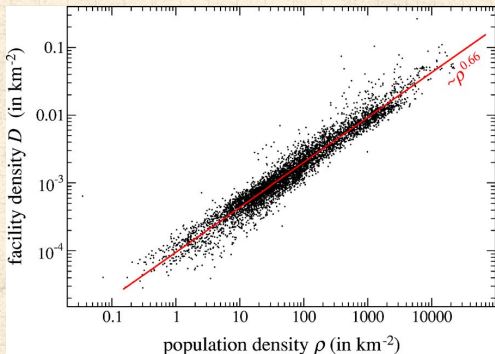Again: Different story to branching networks where there was either one source or one sink.

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Size-density law:

$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{2/3}$$

Why?

Again: Different story to branching networks where there was either one source or one sink.

Now sources & sinks are distributed throughout region.

Optimal source allocation

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"Territorial Division: The Least-Time
Constraint Behind the Formation of
Subnational Boundaries" ☑
G. Edward Stephan,
Science, **196**, 523–524, 1977. [4]

We first examine Stephan's treatment (1977) [4, 5]

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"Territorial Division: The Least-Time
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G. Edward Stephan,
Science, **196**, 523–524, 1977. [4]

🧊 We first examine Stephan's treatment (1977) [4, 5]

🧊 Zipf-like approach: invokes principle of minimal
effort.

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Constraint Behind the Formation of
Subnational Boundaries" ⬀
G. Edward Stephan,
Science, **196**, 523–524, 1977. [4]

- We first examine Stephan's treatment (1977) [4, 5]
- Zipf-like approach: invokes principle of minimal effort.
- Also known as the Homer Simpson principle.

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Consider a region of area A and population P with a single functional center that everyone needs to access every day.

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References

- Consider a region of area A and population P with a single functional center that everyone needs to access every day.
- Build up a general cost function based on time expended to access and maintain center.

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References

- Consider a region of area A and population P with a single functional center that everyone needs to access every day.

- Build up a general cost function based on time expended to access and maintain center.

- Write average travel distance to center as $\bar{d}$ and assume average speed of travel is $\bar{v}$.

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- Write average travel distance to center as $\bar{d}$ and assume average speed of travel is $\bar{v}$.

- Assume isometry: average travel distance $\bar{d}$ will be on the length scale of the region which is $\sim A^{1/2}$

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- Assume isometry: average travel distance $\bar{d}$ will be on the length scale of the region which is $\sim A^{1/2}$

- Average time expended per person in accessing facility is therefore

$$\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$$

where c is an unimportant shape factor.

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Next assume facility requires regular maintenance (person-hours per day).

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- Next assume facility requires regular maintenance (person-hours per day).
- Call this quantity τ.

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- Next assume facility requires regular maintenance (person-hours per day).

- Call this quantity τ.

- If burden of mainenance is shared then average cost per person is τ/P where P = population.

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- Next assume facility requires regular maintenance (person-hours per day).
- Call this quantity τ.
- If burden of mainenance is shared then average cost per person is τ/P where P = population.
- Replace P by $\rho_{\text{pop}}A$ where ρ_{pop} is density.

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- Important assumption: uniform density.

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- Replace P by $\rho_{\mathsf{pop}} A$ where ρ_{pop} is density.
- Important assumption: uniform density.
- Total average time cost per person:

$$T = \bar{d}/\bar{v} + \tau/(\rho_{\mathsf{pop}} A)$$

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$$T = \bar{d}/\bar{v} + \tau/(\rho_{\mathsf{pop}}A) = cA^{1/2}/\bar{v} + \tau/(\rho_{\mathsf{pop}}A).$$

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- Now Minimize with respect to A ...

Optimal source allocation

Differentiating ...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\bar{v} + \tau/(\rho_{\mathsf{pop}}A)\right)$$

Optimal source allocation

🧊 Differentiating ...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\bar{v} + \tau/(\rho_{\text{pop}}A)\right)$$

$$= \frac{c}{2\bar{v}A^{1/2}} - \frac{\tau}{\rho_{\text{pop}}A^2}$$

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♣ Differentiating …

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A} \left(cA^{1/2}/\bar{v} + \tau/(\rho_{\mathsf{pop}}A) \right)$$

$$= \frac{c}{2\bar{v}A^{1/2}} - \frac{\tau}{\rho_{\mathsf{pop}}A^2} = 0$$

♣ Rearrange:

$$A = \left(\frac{2\bar{v}\tau}{c\rho_{\mathsf{pop}}} \right)^{2/3}$$

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♟ Differentiating …

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\bar{v} + \tau/(\rho_{\mathsf{pop}}A)\right)$$

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$$A = \left(\frac{2\bar{v}\tau}{c\rho_{\mathsf{pop}}}\right)^{2/3} \propto \rho_{\mathsf{pop}}^{-2/3}$$

Optimal source allocation

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🔷 Rearrange:

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🔷 # facilities per unit area ρ_{fac}:

$$\rho_{\mathsf{fac}} \propto A^{-1} \propto \rho_{\mathsf{pop}}^{2/3}$$

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- Differentiating ...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\bar{v} + \tau/(\rho_{\mathsf{pop}}A)\right)$$

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- Rearrange:

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- \# facilities per unit area ρ_{fac}:

$$\rho_{\mathsf{fac}} \propto A^{-1} \propto \rho_{\mathsf{pop}}^{2/3}$$

- Groovy ...

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An issue:

🎲 Maintenance (τ) is assumed to be independent of population and area (P and A)

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An issue:

- Maintenance (τ) is assumed to be independent of population and area (P and A)

- Stephan's online book
  "The Division of Territory in Society" is here🔗.
- (It used to be here🔗.)
- The Readme🔗 is well worth reading (1995).

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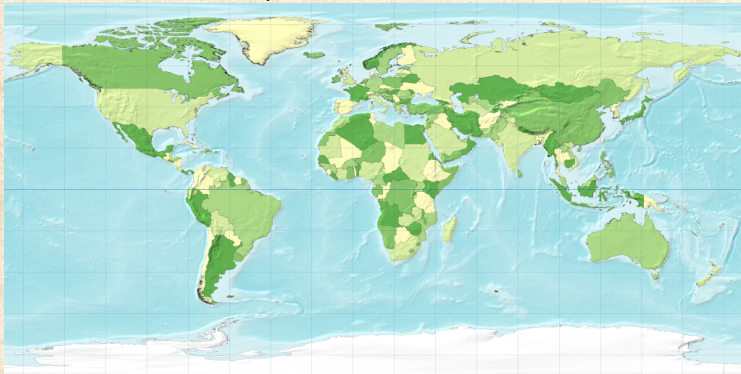18 of 48

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Standard world map:

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Cartogram of countries 'rescaled' by population:

Cartograms

Diffusion-based cartograms:

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Diffusion-based cartograms:

🧊 Idea of cartograms is to distort areas to more accurately represent some local density ρ_{pop} (e.g. population).

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- Idea of cartograms is to distort areas to more accurately represent some local density ρ_{pop} (e.g. population).

- Many methods put forward—typically involve some kind of physical analogy to spreading or repulsion.

Cartograms

Diffusion-based cartograms:

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♻ Idea of cartograms is to distort areas to more accurately represent some local density ρ_{pop} (e.g. population).

♻ Many methods put forward—typically involve some kind of physical analogy to spreading or repulsion.

♻ Algorithm due to Gastner and Newman (2004)[1] is based on standard diffusion:

$$\nabla^2 \rho_{\mathsf{pop}} - \frac{\partial \rho_{\mathsf{pop}}}{\partial t} = 0.$$

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Diffusion-based cartograms:

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- Idea of cartograms is to distort areas to more accurately represent some local density ρ_{pop} (e.g. population).

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- Algorithm due to Gastner and Newman (2004)[1] is based on standard diffusion:

$$\nabla^2 \rho_{\text{pop}} - \frac{\partial \rho_{\text{pop}}}{\partial t} = 0.$$

- Allow density to diffuse and trace the movement of individual elements and boundaries.

Cartograms

Diffusion-based cartograms:

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- ❧ Idea of cartograms is to distort areas to more accurately represent some local density ρ_{pop} (e.g. population).

- ❧ Many methods put forward—typically involve some kind of physical analogy to spreading or repulsion.

- ❧ Algorithm due to Gastner and Newman (2004) [1] is based on standard diffusion:

$$\nabla^2 \rho_{\mathsf{pop}} - \frac{\partial \rho_{\mathsf{pop}}}{\partial t} = 0.$$

- ❧ Allow density to diffuse and trace the movement of individual elements and boundaries.

- ❧ Diffusion is constrained by boundary condition of surrounding area having density $\bar{\rho}_{\mathsf{pop}}$.

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Child mortality:
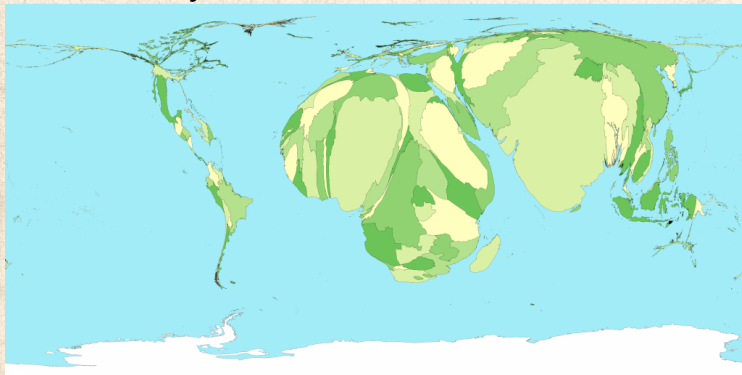
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Energy consumption:
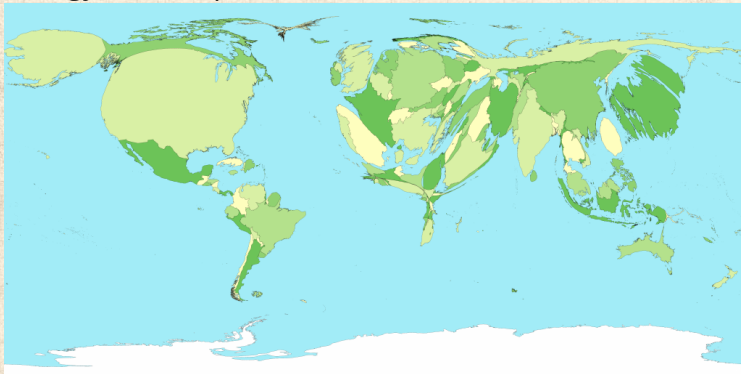
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Gross domestic product:
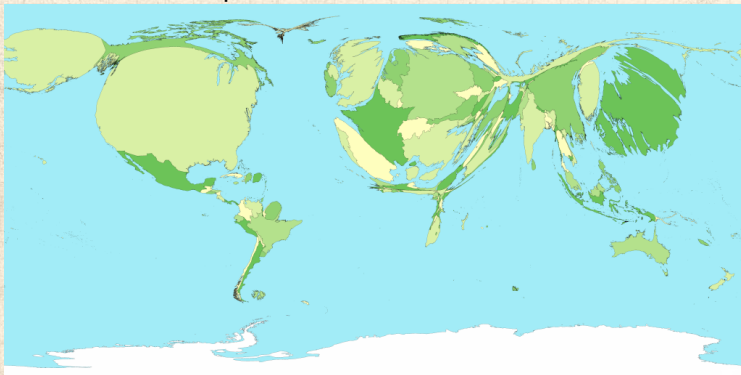
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Greenhouse gas emissions:
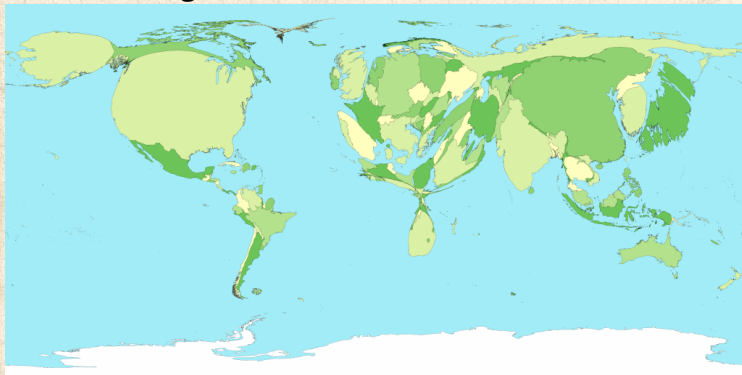
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Spending on healthcare:
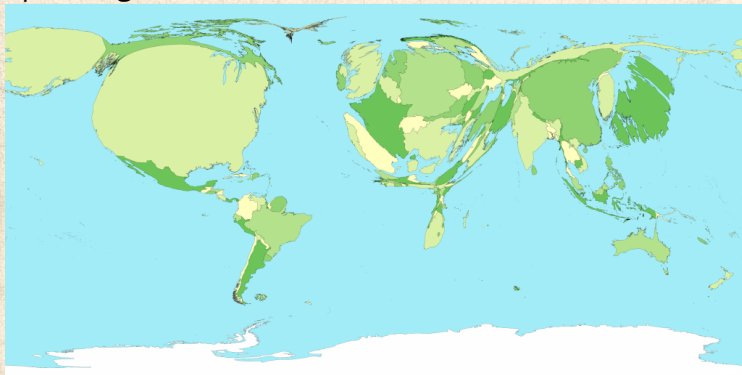
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People living with HIV:
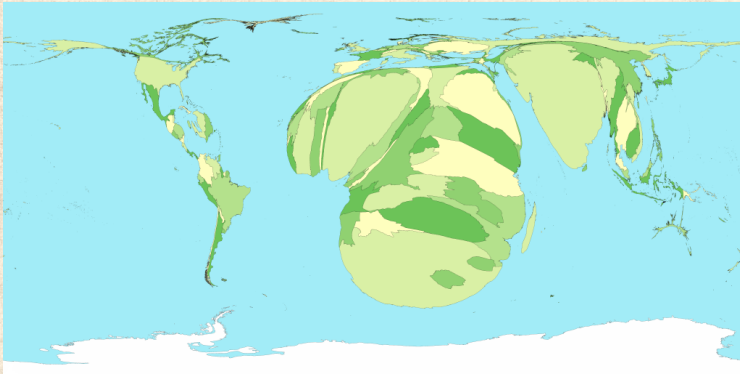
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- The preceding sampling of Gastner & Newman's cartograms lives here ⬀.
- A larger collection can be found at worldmapper.org ⬀.

W🌎RLDMAPPER *The world as you've never seen it before*

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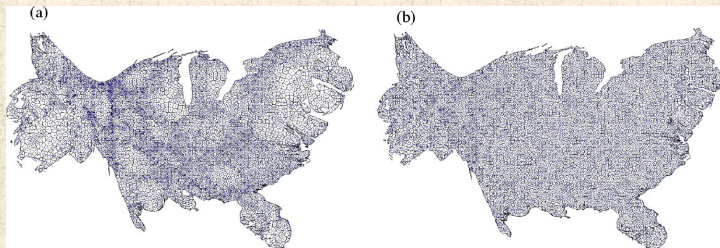A reasonable derivation
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"Optimal design of spatial distribution networks" 🗗
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]

(a)   (b)

Left: population density-equalized cartogram.

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"Optimal design of spatial distribution networks" ⬀
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]

(a)   (b)

Left: population density-equalized cartogram.

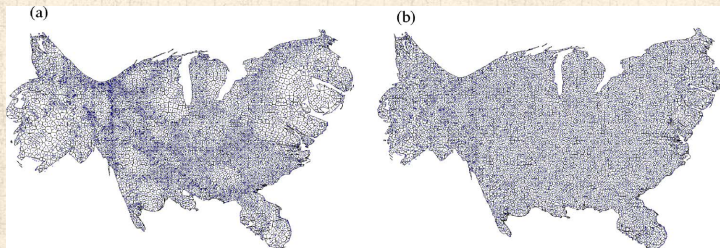**Right:** (population density)$^{2/3}$-equalized cartogram.

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"Optimal design of spatial distribution networks" ⤷
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]

(a)   (b)

- **Left:** population density-equalized cartogram.
- **Right:** (population density)$^{2/3}$-equalized cartogram.
- Facility density is uniform for $\rho_{pop}^{2/3}$ cartogram.

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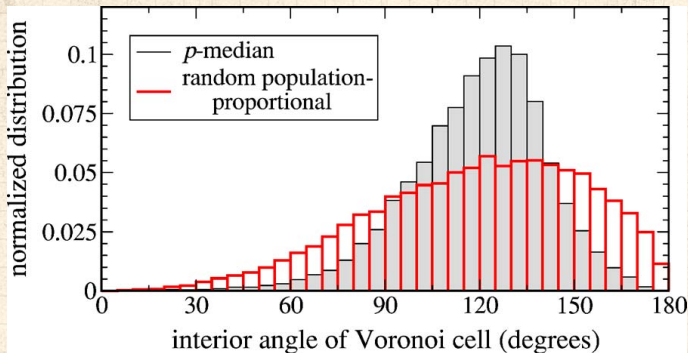Networks III
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From Gastner and Newman (2006)[2]

Cartogram's Voronoi cells are somewhat hexagonal.

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Deriving the optimal source distribution:

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Deriving the optimal source distribution:

♣ **Basic idea:** Minimize the average distance from a random individual to the nearest facility. [2]

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Deriving the optimal source distribution:

- **Basic idea:** Minimize the average distance from a random individual to the nearest facility. [2]

- Assume given a fixed population density ρ_{pop} defined on a spatial region Ω.

Size-density law

Deriving the optimal source distribution:

- ♻ **Basic idea:** Minimize the average distance from a random individual to the nearest facility. [2]

- ♻ Assume given a fixed population density ρ_{pop} defined on a spatial region Ω.

- ♻ Formally, we want to find the locations of n sources $\{\vec{x}_1, \dots, \vec{x}_n\}$ that minimizes the cost function

$$F(\{\vec{x}_1, \dots, \vec{x}_n\}) = \int_{\Omega} \rho_{\mathsf{pop}}(\vec{x}) \min_i \|\vec{x} - \vec{x}_i\| \mathsf{d}\vec{x}.$$

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Deriving the optimal source distribution:

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⛏ Basic idea: Minimize the average distance from a random individual to the nearest facility. [2]

⛏ Assume given a fixed population density ρ_{pop} defined on a spatial region Ω.

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$$ F(\{\vec{x}_1, \dots, \vec{x}_n\}) = \int_\Omega \rho_{\mathsf{pop}}(\vec{x}) \min_i ||\vec{x} - \vec{x}_i|| \mathsf{d}\vec{x} . $$

⛏ Also known as the p-median problem.

Size-density law

Deriving the optimal source distribution:

- **Basic idea:** Minimize the average distance from a random individual to the nearest facility. [2]

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- Also known as the p-median problem.

- Not easy ...in fact this one is an NP-hard problem. [2]

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- ⚙ Also known as the p-median problem.

- ⚙ Not easy ...in fact this one is an NP-hard problem. [2]

- ⚙ Approximate solution originally due to Gusein-Zade [3].

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Approximations:

- For a given set of source placements $\{\vec{x}_1, \ldots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells ⤤, one per source.

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Approximations:

🧊 For a given set of source placements $\{\vec{x}_1, \dots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells ↗, one per source.

🧊 Define $A(\vec{x})$ as the area of the Voronoi cell containing $\vec{x}$.

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- Define $A(\vec{x})$ as the area of the Voronoi cell containing $\vec{x}$.

- As per Stephan's calculation, estimate typical distance from $\vec{x}$ to the nearest source (say i) as

$$c_i A(\vec{x})^{1/2}$$

where c_i is a shape factor for the ith Voronoi cell.

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- For a given set of source placements $\{\vec{x}_1, \ldots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells ⬚, one per source.

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where c_i is a shape factor for the ith Voronoi cell.

- Approximate c_i as a constant c.

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Carrying on:

🧩 The cost function is now

$$F = c \int_{\Omega} \rho_{\mathsf{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathsf{d}\vec{x} \,.$$

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Carrying on:

🎲 The cost function is now

$$F = c \int_{\Omega} \rho_{\mathsf{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathsf{d}\vec{x} \,.$$

🎲 We also have that the constraint that Voronoi cells divide up the overall area of Ω: $\sum_{i=1}^{n} A(\vec{x}_i) = A_{\Omega}$.

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🎲 We also have that the constraint that Voronoi cells divide up the overall area of Ω: $\sum_{i=1}^{n} A(\vec{x}_i) = A_{\Omega}$.

🎲 Sneakily turn this into an integral constraint:

$$\int_{\Omega} \frac{\mathrm{d}\vec{x}}{A(\vec{x})} = n.$$

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Carrying on:

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🔷 Within each cell, $A(\vec{x})$ is constant.

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Carrying on:

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🎲 We also have that the constraint that Voronoi cells divide up the overall area of Ω: $\sum_{i=1}^{n} A(\vec{x}_i) = A_{\Omega}$.

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$$\int_{\Omega} \frac{\mathsf{d}\vec{x}}{A(\vec{x})} = n.$$

🎲 Within each cell, $A(\vec{x})$ is constant.

🎲 So ...integral over each of the n cells equals 1.

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Now a Lagrange multiplier story:

By varying $\{\vec{x}_1, \ldots, \vec{x}_n\}$, minimize

$$G(A) = c \int_\Omega \rho_{\mathsf{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathsf{d}\vec{x} - \lambda \left(n - \int_\Omega \left[A(\vec{x}) \right]^{-1} \mathsf{d}\vec{x} \right)$$

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- I Can Haz Calculus of Variations ☐↗?

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🎲 I Can Haz Calculus of Variations ☐?

🎲 Compute $\delta G / \delta A$, the functional derivative ☐ of the functional $G(A)$.

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🎲 I Can Haz Calculus of Variations ℂ?

🎲 Compute $\delta G/\delta A$, the functional derivative ℂ of the functional $G(A)$.

🎲 This gives

$$\int_{\Omega} \left[\frac{c}{2} \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{-1/2} - \lambda [A(\vec{x})]^{-2} \right] \mathrm{d}\vec{x} = 0.$$

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$$G(A) = c \int_\Omega \rho_{\mathsf{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathsf{d}\vec{x} - \lambda \left(n - \int_\Omega [A(\vec{x})]^{-1} \mathsf{d}\vec{x} \right)$$

🔲 I Can Haz Calculus of Variations 🗗?

🔲 Compute $\delta G/\delta A$, the functional derivative 🗗 of the functional $G(A)$.

🔲 This gives

$$\int_\Omega \left[\frac{c}{2} \rho_{\mathsf{pop}}(\vec{x}) A(\vec{x})^{-1/2} - \lambda [A(\vec{x})]^{-2} \right] \mathsf{d}\vec{x} = 0.$$

🔲 Setting the integrand to be zilch, we have:

$$\rho_{\mathsf{pop}}(\vec{x}) = 2\lambda c^{-1} A(\vec{x})^{-3/2}.$$

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Now a Lagrange multiplier story:

🎲 Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho_{\text{pop}}^{-2/3}.$$

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Now a Lagrange multiplier story:

▣ Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho_{\text{pop}}^{-2/3}.$$

▣ Finally, we indentify $1/A(\vec{x})$ as $\rho_{\text{fac}}(\vec{x})$, an approximation of the local source density.

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Now a Lagrange multiplier story:

⬡ Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho_{\mathsf{pop}}^{-2/3}.$$

⬡ Finally, we indentify $1/A(\vec{x})$ as $\rho_{\mathsf{fac}}(\vec{x})$, an approximation of the local source density.

⬡ Substituting $\rho_{\mathsf{fac}} = 1/A$, we have

$$\rho_{\mathsf{fac}}(\vec{x}) = \left(\frac{c}{2\lambda} \rho_{\mathsf{pop}} \right)^{2/3}.$$

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Now a Lagrange multiplier story:

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- Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho_{\mathsf{pop}}^{-2/3}.$$

- Finally, we indentify $1/A(\vec{x})$ as $\rho_{\mathsf{fac}}(\vec{x})$, an approximation of the local source density.

- Substituting $\rho_{\mathsf{fac}} = 1/A$, we have

$$\rho_{\mathsf{fac}}(\vec{x}) = \left(\frac{c}{2\lambda}\rho_{\mathsf{pop}}\right)^{2/3}.$$

- Normalizing (or solving for λ):

$$\rho_{\mathsf{fac}}(\vec{x}) = n\frac{[\rho_{\mathsf{pop}}(\vec{x})]^{2/3}}{\int_{\Omega}[\rho_{\mathsf{pop}}(\vec{x})]^{2/3}\mathsf{d}\vec{x}} \propto [\rho_{\mathsf{pop}}(\vec{x})]^{2/3}.$$

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One more thing:

How do we supply these facilities?

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One more thing:

- How do we supply these facilities?
- How do we best redistribute mail? People?

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One more thing:

- 📦 How do we supply these facilities?
- 📦 How do we best redistribute mail? People?
- 📦 How do we get beer to the pubs?

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One more thing:

- 🪧 How do we supply these facilities?
- 🪧 How do we best redistribute mail? People?
- 🪧 How do we get beer to the pubs?
- 🪧 Gastner and Newman model: cost is a function of basic maintenance and travel time:

$$C_{\mathsf{maint}} + \gamma C_{\mathsf{travel}}.$$

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One more thing:

- 🎲 How do we supply these facilities?
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$$C_{\text{maint}} + \gamma C_{\text{travel}}.$$

- 🎲 Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance ℓ_{ij} and number of legs to journey:

$$(1 - \delta)\ell_{ij} + \delta(\#\text{hops}).$$

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- How do we supply these facilities?
- How do we best redistribute mail? People?
- How do we get beer to the pubs?
- Gastner and Newman model: cost is a function of basic maintenance and travel time:

$$C_{\mathsf{maint}} + \gamma C_{\mathsf{travel}}.$$

- Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance ℓ_{ij} and number of legs to journey:

$$(1 - \delta)\ell_{ij} + \delta(\#\mathsf{hops}).$$

- When $\delta = 1$, only number of hops matters.

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From Gastner and Newman (2006) [2]

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THE SUN

THE NETWORKS COMPLEX

THE ARK OF THE RANDOM NETWORK

THE NETWORKS FREE OF SCALE

THE SMALLNESS OF WORLDS

THE THEORY OF 5TH DEGREE

THE LANDSCAPE OF PACING PATHS

THE NETWORK OF BLOOD

THE TERMS OF REALITY

THE MEMBERS OF STREAMS

THE LAWS OF BRANCHING

THE UNKNOWN MECHANISM

THE HEAD OF THE BRANCH

THE LAW OF OPTIMAL FORKS

THE CHURCH OF QUARTERPLANT

THE TRUTHICIDE

THE FAMILY OF PANCAKE OWLS

THE FAMILY OF SPHERICAL OWLS

THE IMPERISHABLE PACHYDERM

THE ILLSCALING & ALLOMETRIC BODIES

THE SUPPLY MASTERPIX

THE SUPPLY
TO THE MOST
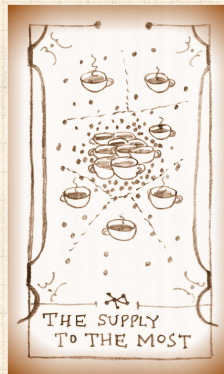
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Beyond minimizing distances:

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Beyond minimizing distances:

- "Scaling laws between population and facility densities" by Um *et al.*, Proc. Natl. Acad. Sci., 2009. [6]

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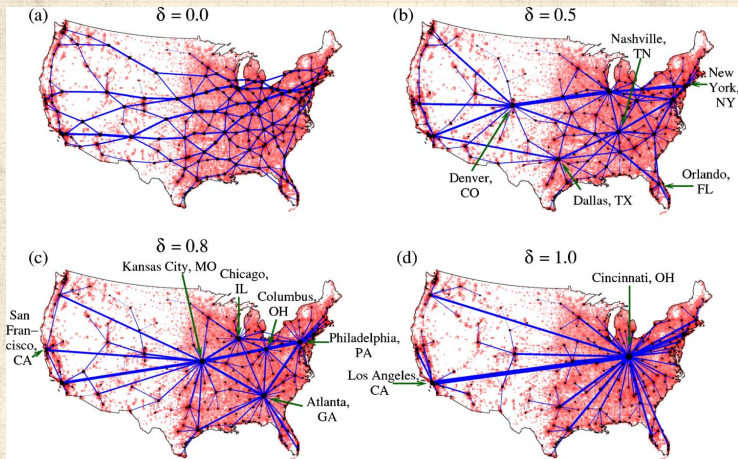Global redistribution
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Beyond minimizing distances:

🧊 "Scaling laws between population and facility densities" by Um *et al.*, Proc. Natl. Acad. Sci., 2009. [6]

🧊 Um *et al.* find empirically and argue theoretically that the connection between facility and population density

$$\rho_{\mathsf{fac}} \propto \rho_{\mathsf{pop}}^{\alpha}$$

does not universally hold with $\alpha = 2/3$.

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- ♣ Two idealized limiting classes:
    1. For-profit, commercial facilities: $\alpha = 1$;

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- Two idealized limiting classes:
  1. For-profit, commercial facilities: $\alpha = 1$;
  2. Pro-social, public facilities: $\alpha = 2/3$.

- Um *et al.* investigate facility locations in the United States and South Korea.

Public versus private facilities: evidence

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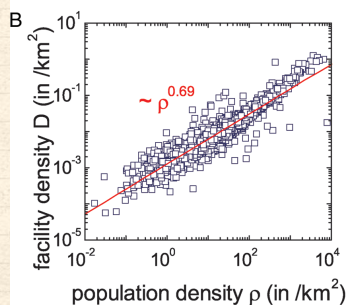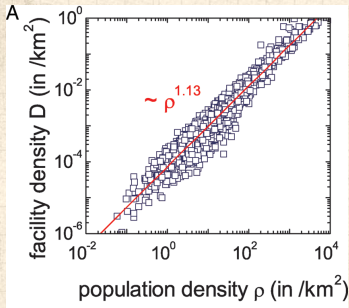
- **Left plot:** ambulatory hospitals in the U.S.
- **Right plot:** public schools in the U.S.

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- **Left plot:** ambulatory hospitals in the U.S.
- **Right plot:** public schools in the U.S.
- Note: break in scaling for public schools. Transition from $\alpha \simeq 2/3$ to $\alpha = 1$ around $\rho_{\mathsf{pop}} \simeq 100$.

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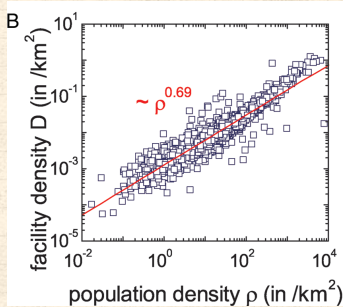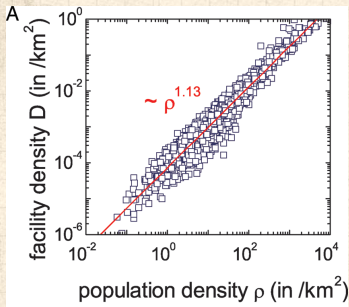Networks III
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US facility	α (SE)	R^2
Ambulatory hospital	1.13(1)	0.93
Beauty care	1.08(1)	0.86
Laundry	1.05(1)	0.90
Automotive repair	0.99(1)	0.92
Private school	0.95(1)	0.82
Restaurant	0.93(1)	0.89
Accommodation	0.89(1)	0.70
Bank	0.88(1)	0.89
Gas station	0.86(1)	0.94
Death care	0.79(1)	0.80
* Fire station	0.78(3)	0.93
* Police station	0.71(6)	0.75
Public school	0.69(1)	0.87

SK facility	α (SE)	R^2
Bank	1.18(2)	0.96
Parking place	1.13(2)	0.91
* Primary clinic	1.09(2)	1.00
* Hospital	0.96(5)	0.97
* University/college	0.93(9)	0.89
Market place	0.87(2)	0.90
* Secondary school	0.77(3)	0.98
* Primary school	0.77(3)	0.97
Social welfare org.	0.75(2)	0.84
* Police station	0.71(5)	0.94
Government office	0.70(1)	0.93
* Fire station	0.60(4)	0.93
* Public health center	0.09(5)	0.19

Rough transition
between public
and private at
$\alpha \simeq 0.8$.

Note: * indicates
analysis is at
state/province
level; otherwise
county level.

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A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.

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So what's going on?

🎲 Social institutions seek to minimize distance of travel.

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So what's going on?

- Social institutions seek to minimize distance of travel.

- Commercial institutions seek to maximize the number of visitors.

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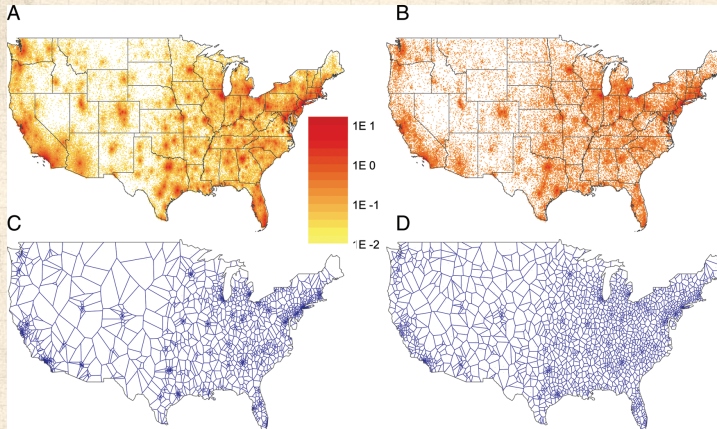A reasonable derivation
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🧊 Defns: For the ith facility and its Voronoi cell V_i, define

   📦 n_i = population of the ith cell;
   📦 $\langle r_i \rangle$ = the average travel distance to the ith facility.
   📦 A_i = area of ith cell (s_i in Um *et al.* [6])

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- 🐾 Objective function to maximize for a facility (highly constructed):

$$v_i = n_i \langle r_i \rangle^{\beta} \text{ with } 0 \leq \beta \leq 1.$$

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- Objective function to maximize for a facility (highly constructed):

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- Limits:
  - $\beta = 0$: purely commercial.
  - $\beta = 1$: purely social.

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Either proceeding as per the Gastner-Newman-Gusein-Zade calculation or, as Um *et al.* do, observing that the cost for each cell should be the same, we have:

$$\rho_{\mathsf{fac}}(\vec{x}) = n \frac{[\rho_{\mathsf{pop}}(\vec{x})]^{2/(\beta+2)}}{\int_\Omega [\rho_{\mathsf{pop}}(\vec{x})]^{2/(\beta+2)} \mathsf{d}\vec{x}} \propto [\rho_{\mathsf{pop}}(\vec{x})]^{2/(\beta+2)}.$$

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⬡ For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.

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For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.

For $\beta = 1$, $\alpha = 2/3$: social scaling is sublinear.

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