

Optimal Supply Networks I: Branching

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Principles of Complex Systems, Vols. 1 & 2
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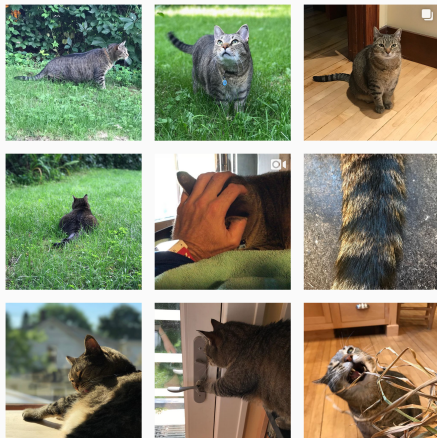
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Outline

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Optimal branching

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Murray meets Tokunaga

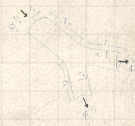
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What's the best way to distribute stuff?

 Stuff = medical services, energy, people, ...

 **Some** fundamental network problems:

1. Distribute stuff from a **single source** to **many sinks**
2. Distribute stuff from **many sources** to many sinks
3. **Redistribute** stuff between nodes that are both sources and sinks

 Supply and Collection are equivalent problems

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Single source optimal supply

Basic question for distribution/supply networks:

 How does flow behave given cost:


$$C = \sum_j I_j^\gamma Z_j$$

where

I_j = current on link j

and

Z_j = link j 's impedance?

 Example: $\gamma = 2$ for electrical networks.

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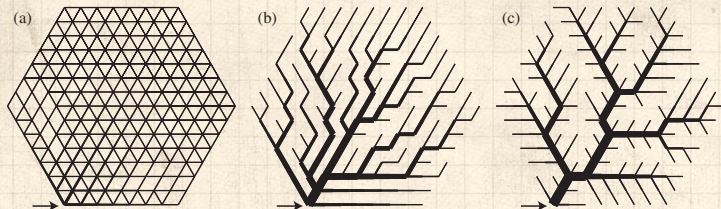
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Single source optimal supply



(a) $\gamma > 1$: Braided (bulk) flow

(b) $\gamma < 1$: Local minimum: Branching flow

(c) $\gamma < 1$: Global minimum: Branching flow

 Note: This is a single source supplying a region.

From Bohn and Magnasco ^[3]

See also Banavar *et al.* ^[1]: “Topology of the Fittest Transportation Network”; focus is on presence or absence of loops—same story

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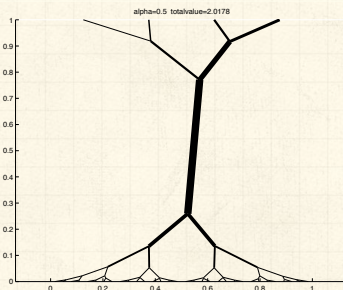
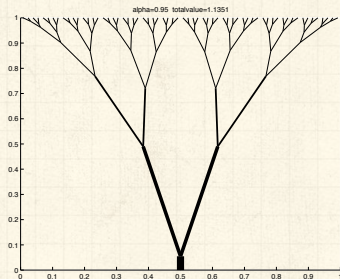


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Optimal paths related to transport (Monge)
problems ↗:



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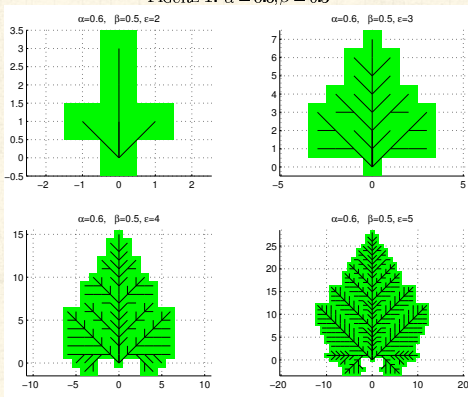
"Optimal paths related to transport
problems" ↗


Qinglan Xia,
Communications in Contemporary
Mathematics, **5**, 251–279, 2003. ^[19]




Growing networks—two parameter model: [20]

FIGURE 1. $\alpha = 0.6, \beta = 0.5$



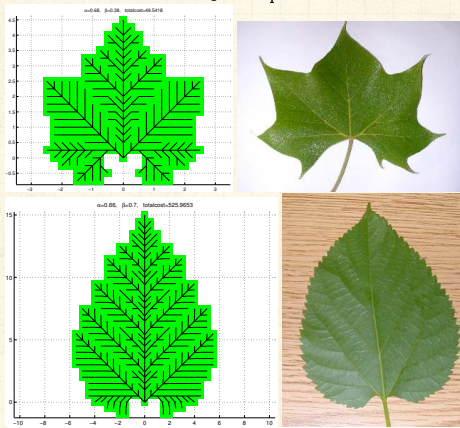
 Parameters control impedance ($0 \leq \alpha < 1$) and angles of junctions ($0 < \beta$)

 For this example: $\alpha = 0.6$ and $\beta = 0.5$



Growing networks: [20]

FIGURE 3. A maple leaf




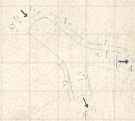
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 Top: $\alpha = 0.66$, $\beta = 0.38$; Bottom: $\alpha = 0.66$, $\beta = 0.70$





Single source optimal supply

An immensely controversial issue ...

-  The form of natural branching networks:
Random, optimal, or some
combination? [6, 18, 2, 5, 4]
-  River networks, blood networks, trees, ...

Two observations:

-  Self-similar networks appear everywhere in nature
for single source supply/single sink collection.
-  Real networks differ in **details of scaling** but
reasonably agree in **scaling relations**.

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Optimality:

-  Optimal channel networks^[13]
-  Thermodynamic analogy^[14]

versus ...

Randomness:

-  Scheidegger's directed random networks
-  Undirected random networks

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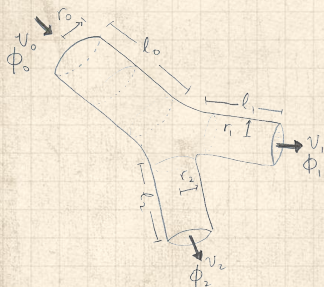
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Murray's law (1926)
connects branch radii at
forks: [11, 10, 12, 7, 16]

$$r_0^3 = r_1^3 + r_2^3$$

where r_0 = radius of main
branch, and r_1 and r_2 are
radii of sub-branches.



Holds up well for outer branchings of blood
networks.



Also found to hold for trees [12, 8] when xylem is
not a supporting structure [9].



See D'Arcy Thompson's "On Growth and Form" for
background and general inspiration [15, 16].

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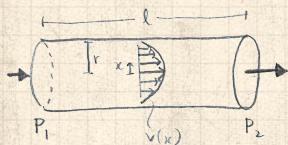




Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

where Δp = pressure difference, Φ = flux.



Fluid mechanics: Poiseuille impedance for smooth Poiseuille flow in a tube of radius r and length l :

$$Z = \frac{8\eta l}{\pi r^4}$$



η = dynamic viscosity (units: $ML^{-1}T^{-1}$).



Power required to overcome impedance:

$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z.$$



Also have rate of energy expenditure in maintaining blood given metabolic constant c :

$$P_{\text{metabolic}} = cr^2 l$$



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Aside on P_{drag}

 Work done = $F \cdot d$ = energy transferred by force F

 Power = P = rate work is done = $F \cdot v$

 Δp = Force per unit area


 Φ = Volume per unit time
= cross-sectional area \cdot velocity

 So $\Phi \Delta p$ = Force \cdot velocity





Optimization—Murray's law



Murray's law:

 Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell$$

 Observe power increases linearly with ℓ

 But r 's effect is nonlinear:

-  increasing r makes flow easier **but increases metabolic cost** (as r^2)
-  decreasing r decrease metabolic cost **but impedance goes up** (as r^{-4})

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Murray's law:

 Minimize P with respect to r :

$$\begin{aligned}\frac{\partial P}{\partial r} &= \frac{\partial}{\partial r} \left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell \right) \\ &= -4\Phi^2 \frac{8\eta\ell}{\pi r^5} + c2r\ell = 0\end{aligned}$$

 Rearrange/cancel/slap:

$$\Phi^2 = \frac{c\pi r^6}{16\eta} = k^2 r^6$$

where $k = \text{constant}$.

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Murray's law:


 So we now have:

$$\Phi = kr^3$$

 Flow rates at each branching have to add up (else our organism is in serious trouble ...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches

 All of this means we have a groovy cube-law:

$$r_0^3 = r_1^3 + r_2^3$$

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Murray meets Tokunaga:


 Φ_ω = volume rate of flow into an order ω vessel segment

 Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

 Using $\phi_\omega = kr_\omega^3$

$$r_\omega^3 = 2r_{\omega-1}^3 + \sum_{k=1}^{\omega-1} T_k r_{\omega-k}^3$$

 Find Horton ratio for vessel radius $R_r = r_\omega/r_{\omega-1} \dots$

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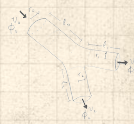


Murray meets Tokunaga:

-  Find R_r^3 satisfies same equation as R_n and R_v
(v is for volume):

$$R_r^3 = R_n = R_v$$

-  Is there more we could do here to constrain the
Horton ratios and Tokunaga constants?



Murray meets Tokunaga:

Isometry: $V_\omega \propto \ell_\omega^3$

Gives

$$R_\ell^3 = R_r^3 = R_n = R_v$$

We need one more constraint ...

West *et al.* (1997)^[18] achieve similar results following Horton's laws (but this work is disaster).

So does Turcotte *et al.* (1998)^[17] using Tokunaga (sort of).

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