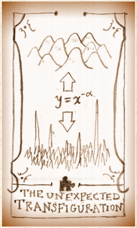
Mechanisms for Generating Power-Law Size Distributions, Part 2

Last updated: 2021/10/06, 20:26:04 EDT

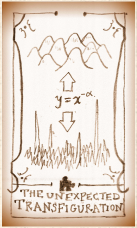
Principles of Complex Systems, Vols. 1 & 2
CSYS/MATH 300 and 303, 2021–2022 | @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont



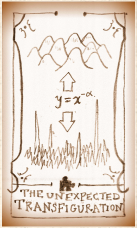
These slides are brought to you by:



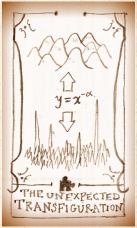
These slides are also brought to you by:

Special Guest Executive Producer

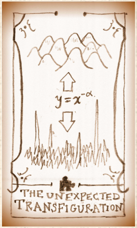
On Instagram at pratchett_the_cat



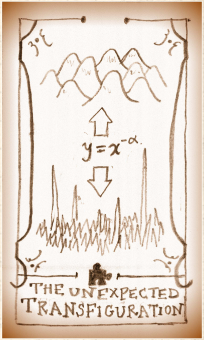
Outline



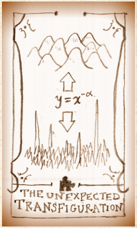
The Boggoracle Speaks:




Au
THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
p~s
THE LAWS OF BEGATING POWER
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
1 2 5 4 3 6 7 8 9
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
S I R
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
D-1 D
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END

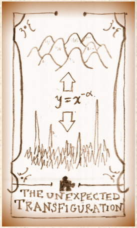

y=x^{-α}
THE UNEXPECTED TRANSFIGURATION

Outline



Variable Transformation

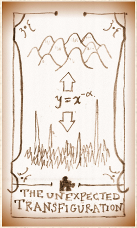
Understand power laws as arising from



Variable Transformation

Understand power laws as arising from

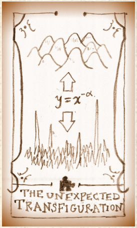
1. Elementary distributions (e.g., exponentials).



Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

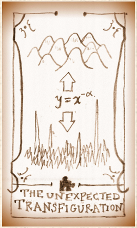


Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x

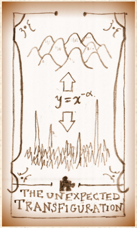


Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

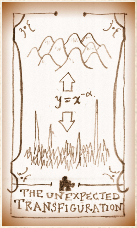


Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.
- $$P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y))\frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$$



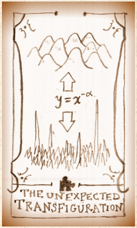
Variable Transformation

Understand power laws as arising from

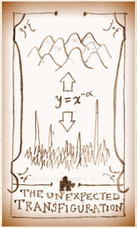
1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

- $P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y))\frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$
- Often easier to do by hand...

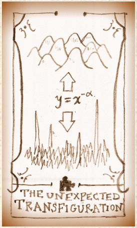


General Example



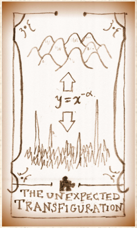
General Example

- Assume relationship between x and y is 1-1.



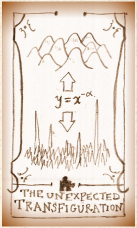
General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables:
  $y = cx^{-\alpha}$, $\alpha > 0$



General Example

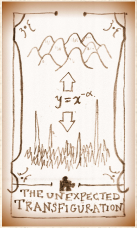
- Assume relationship between x and y is 1-1.
- Power-law relationship between variables:
  $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small



General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
-

$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$

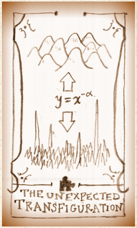


General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
-

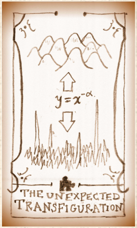
$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$

$$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$



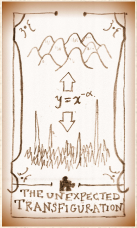
General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
- $$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$
  $$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$
  $$\text{invert: } \mathrm{d}x = \frac{-1}{c\alpha}x^{\alpha+1}\mathrm{d}y$$



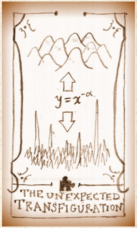
General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
- $$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$
  $$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$
  $$\text{invert: } \mathrm{d}x = \frac{-1}{c\alpha}x^{\alpha+1}\mathrm{d}y$$
  $$\mathrm{d}x = \frac{-1}{c\alpha}\left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha}\mathrm{d}y$$



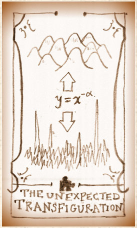
General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
- $$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$
  $$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$
  $$\text{invert: } \mathrm{d}x = \frac{-1}{c\alpha}x^{\alpha+1}\mathrm{d}y$$
  $$\mathrm{d}x = \frac{-1}{c\alpha}\left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha}\mathrm{d}y$$
  $$\mathrm{d}x = \frac{-c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y$$



Now make transformation:

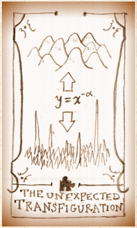
$$P_y(y)\mathsf{d}y = P_x(x)\mathsf{d}x$$



Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

$$P_y(y)\mathrm{d}y = P_x \overbrace{\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)}^{(x)} \overbrace{\frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha}\mathrm{d}y}^{\mathrm{d}x}$$



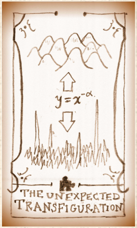
Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

$$P_y(y)\mathrm{d}y = P_x\Bigg(\overbrace{\left(\frac{y}{c}\right)^{-1/\alpha}}^{(x)}\Bigg)\overbrace{\frac{c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y}^{\mathrm{d}x}$$

- If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$



Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

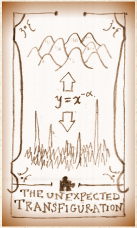
$$P_y(y)\mathrm{d}y = P_x\overbrace{\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)}^{(x)}\overbrace{\frac{c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y}^{\mathrm{d}x}$$

- If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

- If $P_x(x) \rightarrow x^\beta$ as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$

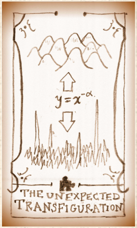


Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$



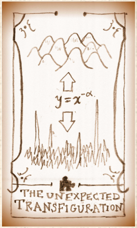
Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness (easy) ...



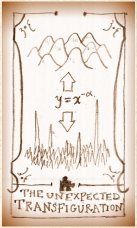
Example

Exponential distribution

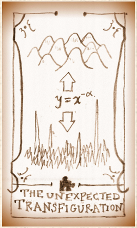
Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness (easy) ...
- More later when we cover robustness.



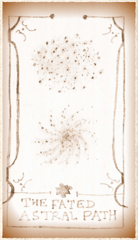
Outline



Gravity

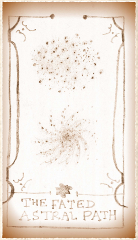
- Select a random point in the universe $\vec{x}$.

[1] Stigler's Law of Eponymy.



Gravity

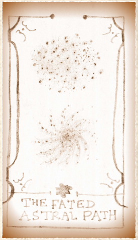
- Select a random point in the universe $\vec{x}$.
- Measure the force of gravity $F(\vec{x})$.



[1] Stigler's Law of Eponymy.

Gravity

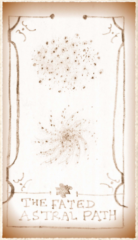
- Select a random point in the universe $\vec{x}$.
- Measure the force of gravity $F(\vec{x})$.
- Observe that $P_F(F) \sim F^{-5/2}$.



[1] Stigler's Law of Eponymy.

Gravity

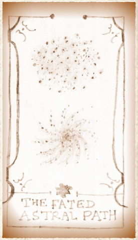
- Select a random point in the universe $\vec{x}$.
- Measure the force of gravity $F(\vec{x})$.
- Observe that $P_F(F) \sim F^{-5/2}$.
- Distribution named after Holtsmark who was thinking about electrostatics and plasma [1].



[1] Stigler's Law of Eponymy.

Gravity

- Select a random point in the universe $\vec{x}$.
- Measure the force of gravity $F(\vec{x})$.
- Observe that $P_F(F) \sim F^{-5/2}$.
- Distribution named after Holtsmark who was thinking about electrostatics and plasma [1].
- Again, the humans naming things after humans, poorly.[1]



[1] Stigler's Law of Eponymy.

Matter is concentrated in stars: [2]

- F is distributed unevenly

Matter is concentrated in stars: [2]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

Matter is concentrated in stars: [2]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)

Matter is concentrated in stars: [2]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.

Matter is concentrated in stars: [2]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

Matter is concentrated in stars: [2]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-\frac{1}{2}}$$

Matter is concentrated in stars: [2]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-\frac{1}{2}}$$

- Connect differentials: $\mathrm{d}r \propto \mathrm{d}F^{-\frac{1}{2}} \propto F^{-\frac{3}{2}}\mathrm{d}F$

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

$$\propto P_r(\mathrm{const} \times F^{-1/2})F^{-3/2}\mathrm{d}F$$

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

- $$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

- $$\propto P_r(\mathrm{const} \times F^{-1/2})F^{-3/2}\mathrm{d}F$$

- $$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

$$\propto P_r(\mathrm{const} \times F^{-1/2})F^{-3/2}\mathrm{d}F$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$

$$= F^{-1-3/2}\mathrm{d}F$$

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

$$\propto P_r(\mathrm{const} \times F^{-1/2})F^{-3/2}\mathrm{d}F$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$

$$= F^{-1-3/2}\mathrm{d}F$$

$$= F^{-5/2}\mathrm{d}F\,.$$

Gravity:

$$P_F(F) = F^{-5/2} \mathrm{d}F$$

Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

$$\gamma = 5/2$$

Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

- $$\gamma = 5/2$$
- Mean is finite.

Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

- $$\gamma = 5/2$$
- Mean is finite.
- Variance = ∞.

Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

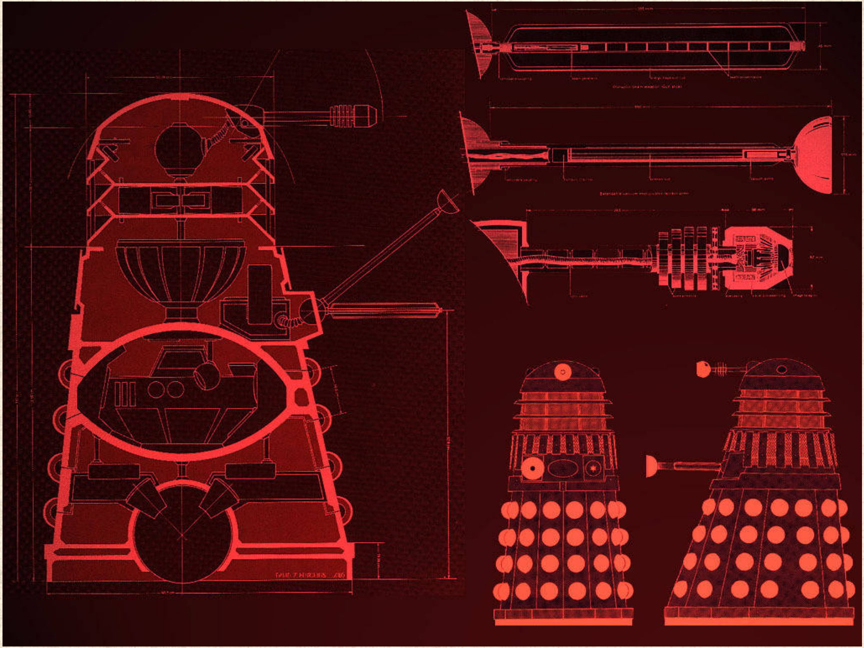
- $$\gamma = 5/2$$
- Mean is finite.
- Variance = ∞.
- A wild distribution.

Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

- $$\gamma = 5/2$$
- Mean is finite.
- Variance = ∞.
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

☐ Todo: Build Dalek army.



Outline

Extreme Caution!

- PLIPLO = Power law in, power law out

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...
- Don't do this!!! (slap, slap)

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is the stuff.

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is the stuff.
- In general: We need mechanisms!

References I

[1] J. Holtsmark.
Über die verbreiterung von spektrallinien.
Ann. Phys., 58:577–630, 1919. pdf

[2] D. Sornette.
Critical Phenomena in Natural Sciences.
Springer-Verlag, Berlin, 1st edition, 2003.