

Mechanisms for Generating Power-Law Size Distributions, Part 2

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Principles of Complex Systems, Vols. 1 & 2
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Variable
transformation
Basics
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General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables:
 $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small

$$dy = d(cx^{-\alpha})$$

$$= c(-\alpha)x^{-\alpha-1}dx$$

$$\text{invert: } dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$$

$$dx = \frac{-1}{c\alpha} \left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha} dy$$

$$dx = \frac{-c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$

Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_y(y)dy = P_x \left(\left(\frac{y}{c}\right)^{-1/\alpha} \right) \frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$

- If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

- If $P_x(x) \rightarrow x^\beta$ as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$

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Outline

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Variable Transformation

Understand power laws as arising from

- Elementary distributions (e.g., exponentials).
- Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

$$P_y(y)dy = \sum_{x|f(x)=y} P_x(x)dx = \sum_{y|f(x)=y} P_x(f^{-1}(y)) \left| \frac{dy}{f^{-1}(y)} \right|$$

- Often easier to do by hand...

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Gravity

- Select a random point in the universe \vec{x} .
- Measure the force of gravity $F(\vec{x})$.
- Observe that $P_F(F) \sim F^{-5/2}$.
- Distribution named after Holtmark who was thinking about electrostatics and plasma [1].
- Again, the humans naming things after humans, poorly.¹



¹Stigler's Law of Eponymy

Matter is concentrated in stars: [2]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)dr \propto r^2 dr$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at \vec{x} .
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

- Connect differentials: $dr \propto dF^{-1/2} \propto F^{-3/2} dF$

Transformation:

$$\text{Using } \boxed{r \propto F^{-1/2}}, \boxed{dr \propto F^{-3/2} dF}, \text{ and } \boxed{P_r(r) \propto r^2}$$

$$\begin{aligned} P_F(F)dF &= P_r(r)dr \\ &\propto P_r(\text{const} \times F^{-1/2}) F^{-3/2} dF \\ &\propto (F^{-1/2})^2 F^{-3/2} dF \\ &= F^{-1-3/2} dF \\ &= F^{-5/2} dF. \end{aligned}$$

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Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$$

- Exponentials arise from randomness (easy) ...
- More later when we cover robustness.

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Gravity:

$$P_F(F) = F^{-5/2} dF$$

$$\gamma = 5/2$$



- Mean is finite.
- Variance = ∞ .
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...



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Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is the stuff.
- In general: We need mechanisms!



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References I

- J. Holtsmark. Über die verbreiterung von spektrallinien. *Ann. Phys.*, 58:577–630, 1919. [pdf](#)
- D. Sornette. *Critical Phenomena in Natural Sciences*. Springer-Verlag, Berlin, 1st edition, 2003.



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□ Todo: Build Dalek army.

