# Mechanisms for Generating Power-Law Size Distributions, Part 1

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Principles of Complex Systems, Vols. 1 & 2
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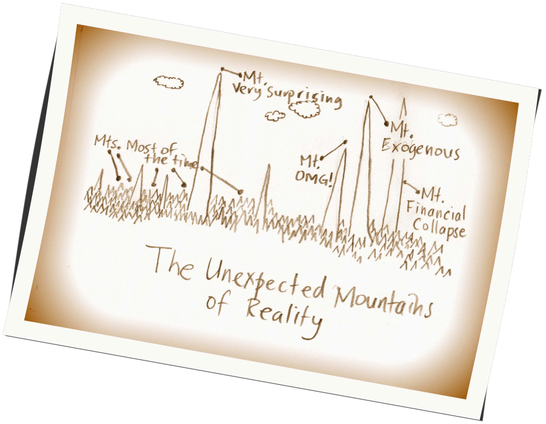
## Outline










## Mechanisms:

### A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks.

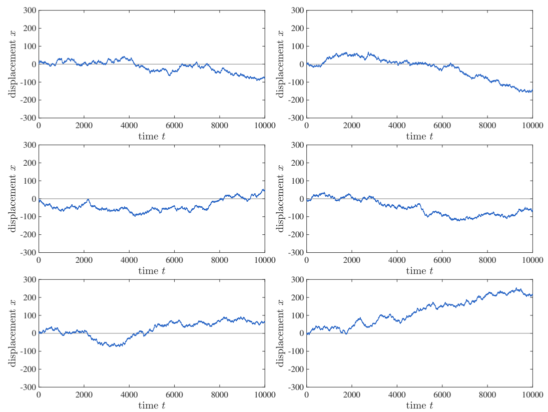
### The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a zombie texter) starts at origin $x = 0$.
- Step at time $t$ is $\epsilon_t$:

$$\epsilon_t = \begin{cases} +1 & \text{with probability 1/2} \\ -1 & \text{with probability 1/2} \end{cases}$$



## A few random random walks:






## Random walks:

### Displacement after $t$ steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

### Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our zombie texter to be back at their starting place.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our texting undead friend lurching back to $x$=0 must diminish, right?



### Variances sum:*

$$\text{Var}(x_t) = \text{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \text{Var}(\epsilon_i) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

### So typical displacement from the origin scales as:

$$\boxed{\sigma = t^{1/2}}$$

- A non-trivial scaling law arises out of additive aggregation or accumulation.



## Random walk basics:

### Counting random walks:

- Each specific random walk of length $t$ appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after $t$ time steps.
- Random walk must displace by $+(j - i)$ after $t$ steps.
- Insert question from assignment 5

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

## How does $P(x_t)$ behave for large $t$?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- $x_{2n}$ is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\textbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large $n$, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\textbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

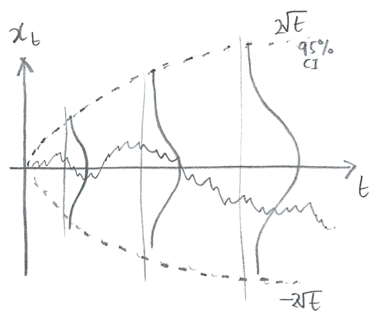
Insert question from assignment 5

- The whole is different from the parts. #nutritious
- See also: Stable Distributions





## Universality is also not left-handed:



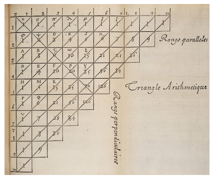

- This is Diffusion: the most essential kind of spreading (more later).
- View as Random Additive Growth Mechanism.





## So many things are connected:

### Pascal's Triangle



- Could have been the Pyramid of Pingala[1] or the Triangle of Khayyam, Jia Xian, Tartaglia, ...
- Binomials tend towards the Normal.
- Counting encoded in algebraic forms (and much more).
- $(h+t)^n = \sum_{k=0}^{n} \binom{n}{k} h^k t^{n-k}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- $(h+t)^3 = hhh + hht + hth + thh + htt + tht + tth + ttt$

[1] Stigler's Law of Eponymy showing excellent form again.















### Random walks are even weirder than you might think...

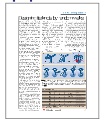
- $\xi_{r,t}$ = the probability that by time step $t$, a random walk has crossed the origin $r$ times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = $\infty$

See Feller, Intro to Probability Theory, Volume I [5]





## Applied knot theory:



"Designing tie knots by random walks"
Fink and Mao,
Nature, **398**, 31–32, 1999. [6]

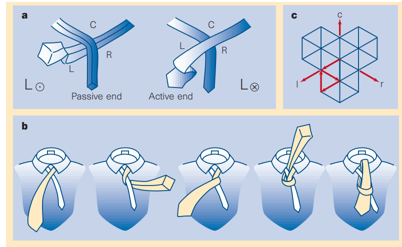



**Figure 1** All diagrams are drawn in the frame of reference of the mirror image of the actual tie. **a,** The two ways of beginning a knot, $L_{\odot}$ and $L_{\otimes}$. For knots beginning with $L_{\odot}$, the tie must begin inside-out. **b,** The four-in-hand, denoted by the sequence $L_{\otimes} R_{\odot} L_{\otimes} C_{\odot} T$. **c,** A knot may be represented by a persistent random walk on a triangular lattice. The example shown is the four-in-hand, indicated by the walk $\hat{l}\hat{r}\hat{l}\hat{c}$.





## Applied knot theory:

Table 1 **Aesthetic tie knots**

| $h$ | $\gamma$ | $\gamma/h$ | $K(h,\gamma)$ | $s$ | $b$ | Name | Sequence |
|---|---|---|---|---|---|---|---|
| 3 | 1 | 0.33 | 1 | 0 | 0 | | $L_{\odot}R_{\otimes}C_{\odot}T$ |
| 4 | 1 | 0.25 | 1 | −1 | 1 | Four-in-hand | $L_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ |
| 5 | 2 | 0.40 | 2 | −1 | 0 | Pratt knot | $L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ |
| 6 | 2 | 0.33 | 4 | 0 | 0 | Half-Windsor | $L_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$ |
| 7 | 2 | 0.29 | 6 | −1 | 1 | | $L_{\odot}R_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ |
| 7 | 3 | 0.43 | 4 | 0 | 1 | | $L_{\odot}C_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$ |
| 8 | 2 | 0.25 | 8 | 0 | 2 | | $L_{\otimes}R_{\odot}L_{\otimes}C_{\odot}R_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$ |
| 8 | 3 | 0.38 | 12 | −1 | 0 | Windsor | $L_{\otimes}C_{\odot}R_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ |
| 9 | 3 | 0.33 | 24 | 0 | 0 | | $L_{\odot}R_{\otimes}C_{\odot}L_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$ |
| 9 | 4 | 0.44 | 8 | −1 | 2 | | $L_{\odot}C_{\otimes}R_{\odot}C_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ |

Knots are characterized by half-winding number $h$, centre number $\gamma$, centre fraction $\gamma/h$, knots per class $K(h, \gamma)$, symmetry $s$, balance $b$, name and sequence.

- $h$ = number of moves
- $\gamma$ = number of center moves
- $K(h,\gamma) = 2^{\gamma-1}\binom{h-\gamma-2}{\gamma-1}$
- $s = \sum_{i=1}^{h} x_i$ where $x$ = -1 for $L$ and +1 for $R$.
- $b = \frac{1}{2}\sum_{i=2}^{h-1} |\omega_i + \omega_{i-1}|$ where $\omega = \pm 1$ represents winding direction.





## Random walks #crazytownbananapants

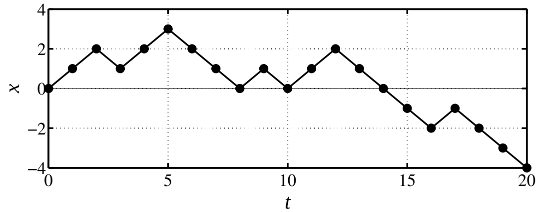
### The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after $t$ steps?
- Will our zombie texter always return to the origin?
- What about higher dimensions?

### Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

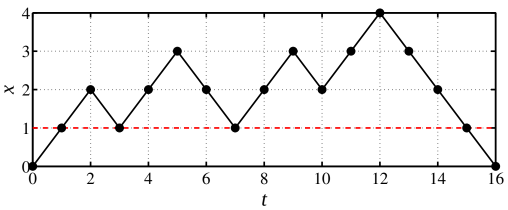
## For random walks in 1-$d$:




- A return to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8$, 10, and 14.
- Call $P_{\text{fr}(2n)}$ the probability of first return at $t = 2n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.








- Can assume zombie texter first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \le t \le 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) = 2 \cdot \frac{1}{2} Pr(x_t \ge 1, 1 \le t \le 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for texters that first lurch to $x = -1$.





## Counting first returns:

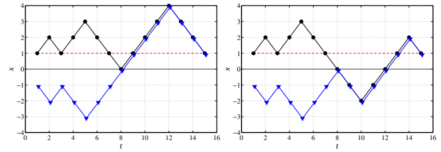
### Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking $t$ steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \ge 1$.
- Call walks that drop below $x = 1$ excluded walks.
- We'll use a method of images to identify these excluded walks.





## Examples of excluded walks:



## Key observation for excluded walks:

- For any path starting at $x$=1 that hits 0, there is a unique matching path starting at $x$=−1.
- Matching path first mirrors and then tracks after first reaching $x$=0.
- # of $t$-step paths starting and ending at $x$=1 and hitting $x$=0 at least once
  = # of $t$-step paths starting at $x$=-1 and ending at $x$=1 = $N(-1, 1, t)$
- So $N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$





## Probability of first return:

Insert question from assignment 5 :

- Find

$$\boxed{N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi} n^{3/2}}.}$$

- Normalized number of paths gives probability.
- Total number of possible paths = $2^{2n}$.
- $$P_{\text{fr}}(2n) = \frac{1}{2^{2n}} N_{\text{fr}}(2n)$$
$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi} n^{3/2}}$$
$$= \frac{1}{\sqrt{2\pi}} (2n)^{-3/2} \propto t^{-3/2}.$$





- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \ge 3$ dimensions
- Associated human ~~genius~~: George Pólya





## Random walks

### On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

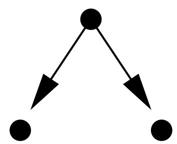
### On networks:

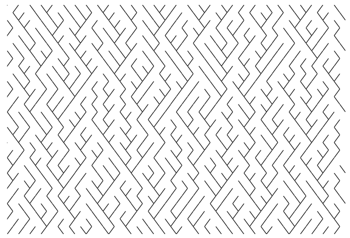
- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
- Equal probability still present: walkers traverse edges with equal frequency. #totallygroovy





## Scheidegger Networks [17, 4]





- Random directed network on triangular lattice.
- Toy model of real networks.
- 'Flow' is southeast or southwest with equal probability.





## Scheidegger networks

- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length $\ell$ distribution: $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

## Connections between exponents:

- For a basin of length $\ell$, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- $\mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3 a^{-1/3}\mathrm{d}a$
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-3/2}\mathrm{d}\ell$
  $\propto (a^{2/3})^{-3/2} a^{-1/3}\mathrm{d}a$
  $= a^{-4/3}\mathrm{d}a$
  $= a^{-\tau}\mathrm{d}a$




## Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

### Generalize relationship between area and length:

- Hack's law [10]:

$$\ell \propto a^h.$$

- For real, large networks [13] $h \simeq 0.5$ (isometric scaling)
- Smaller basins possibly $h > 1/2$ (allometric scaling).
- Models exist with interesting values of $h$.
- Plan: Redo calc with $\gamma$, $\tau$, and $h$.




## Connections between exponents:

- Given

$$\ell \propto a^h, \ P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1}\mathrm{d}a$
- Find $\tau$ in terms of $\gamma$ and $h$.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h(\gamma-1))}\mathrm{d}a$
- $$\boxed{\tau = 1 + h(\gamma - 1)}$$
- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.




## Connections between exponents:

### With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies to: [3]

$$\boxed{\tau = 2 - h}$$

and

$$\boxed{\gamma = 1/h}$$

- Only one exponent is independent (take $h$).
- Simplifies system description.
- Expect Scaling Relations where power laws are found.
- Need only characterize Universality class with independent exponents.




## Death ...

### Failure:

- A very simple model of failure/death
- $x_t$ = entity's 'health' at time $t$
- Start with $x_0 > 0$.
- Entity fails when $x$ hits 0.

"Explaining mortality rate plateaus"
Weitz and Fraser,
Proc. Natl. Acad. Sci., **98**, 15383–15386, 2001. [18]




## ... and the NBA:

### Basketball and other sports [2]:

- Three arcsine laws (Lévy [12]) for continuous-time random walk last time $T$:

$$\frac{1}{\pi} \frac{1}{\sqrt{t(T-t)}}.$$

  The arcsine distribution applies for:
  (1) fraction of time positive, (2) the last time the walk changes sign,
  and (3) the time the maximum is achieved.
- Well approximated by basketball score lines [8, 2].
- Australian Rules Football has some differences [11].



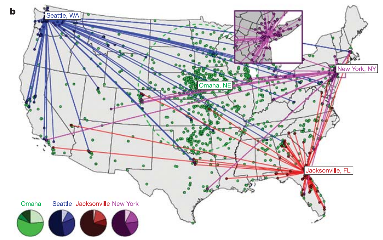

## More than randomness

- Can generalize to Fractional Random Walks [15, 16, 14]
- Fractional Brownian Motion, Lévy flights
- See Montroll and Shlesinger for example: [14]
  "On $1/f$ noise and other distributions with long tails."
  Proc. Natl. Acad. Sci., 1982.
- In 1-d, standard deviation $\sigma$ scales as

$$\sigma \sim t^\alpha$$

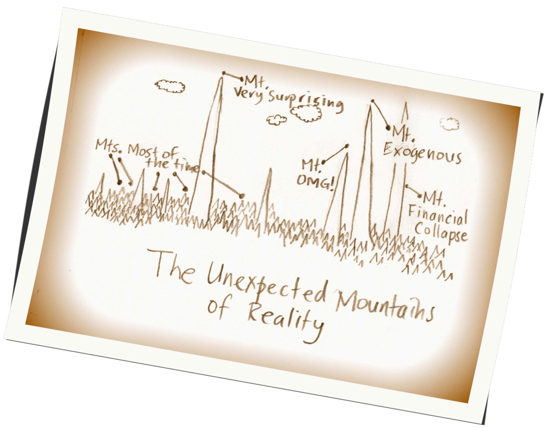
  $\alpha = 1/2$ — diffusive
  $\alpha > 1/2$ — superdiffusive
  $\alpha < 1/2$ — subdiffusive
- Extensive memory of path now matters...

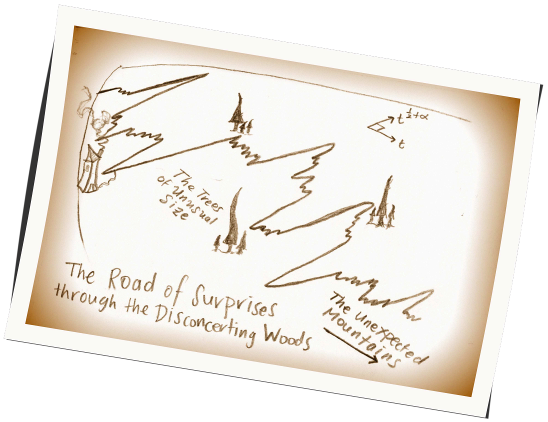



- First big studies of movement and interactions of people.
- Brockmann *et al.* [1] "Where's George" study.
- Beyond Lévy: Superdiffusive in space but with long waiting times.
- Tracking movement via cell phones [9] and Twitter [7].







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