# Properties of Complex Networks

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## Principles of Complex Systems, Vols. 1 & 2
## CSYS/MATH 300 and 303, 2021–2022 | @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

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# Outline


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
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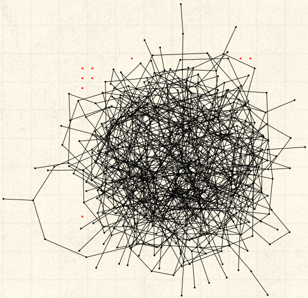

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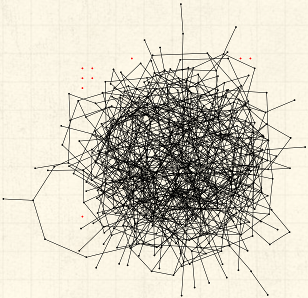

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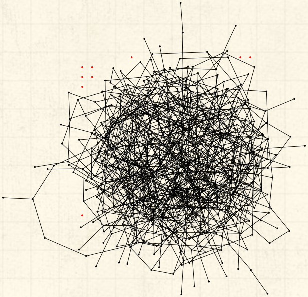

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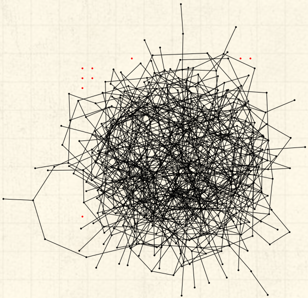

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  said Ponder [Stibbons] —*Making Money*, T. Pratchett.
- We need to extract digestible, meaningful aspects.

## Some key aspects of real complex networks:

- degree distribution*
- assortativity
- homophily
- clustering
- motifs
- modularity
- concurrency
- hierarchical scaling
- network distances
- centrality
- efficiency
- interconnectedness
- robustness

- Plus coevolution of network structure and processes on networks.

∗ Degree distribution is the elephant in the room that we are now all very aware of...

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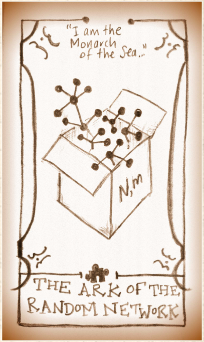
"I am the Monarch of the Sea.."
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- hubs may facilitate or impede contagion.





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- ‘Scale-free’ networks are growing networks that form according to a plausible mechanism.
- Randomness is out there, just not to the degree of a completely random network.

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- Disassortative network: high degree nodes connecting to low degree nodes.
  *Often techological or biological: Internet, WWW, protein interactions, neural networks, food webs.*





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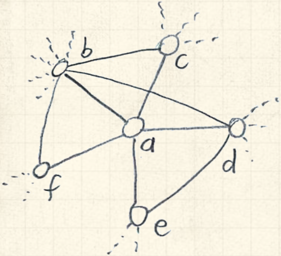
# Local socialness:

4. Clustering:





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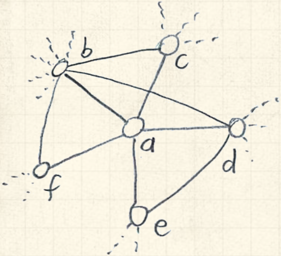


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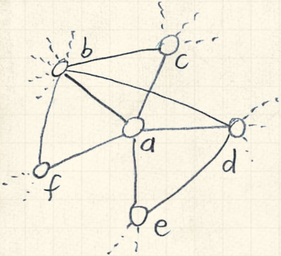


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- Two measures (explained on following slides):
  1. Watts & Strogatz [8]

$$C_1 = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i$$

  2. Newman [6]

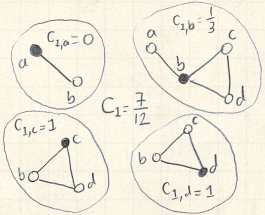
$$C_2 = \frac{3 \times \text{\#triangles}}{\text{\#triples}}$$





Example network:


Calculation of $C_1$:





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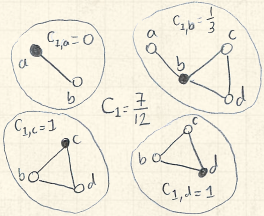


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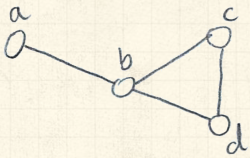


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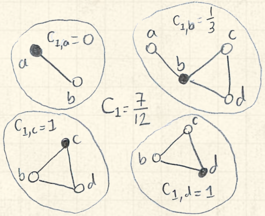


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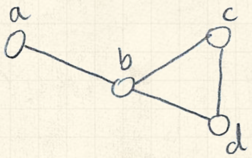


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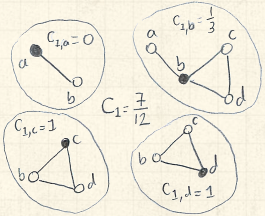


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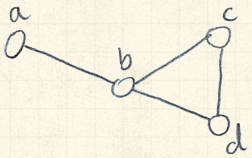


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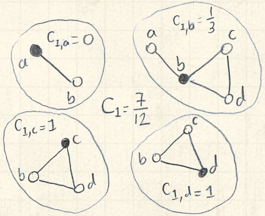


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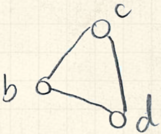
# Triples and triangles

Example network:


Triangles:

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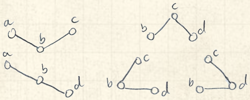


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# Triples and triangles






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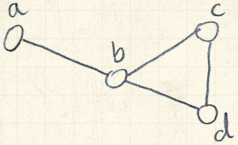


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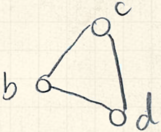


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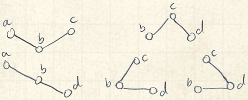

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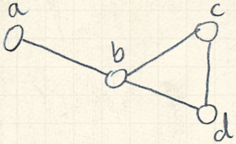


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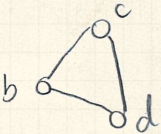


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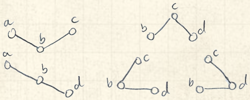

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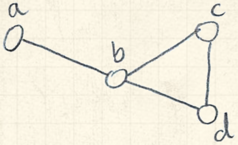


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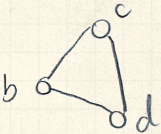


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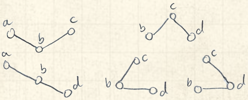

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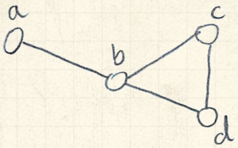


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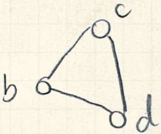


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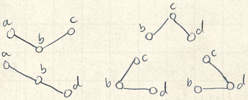

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- Social Network Analysis (SNA): fraction of transitive triples.



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# Outline

Properties of Complex Networks

A problem

Degree distributions

Assortativity

Clustering

Motifs

Concurrency

Branching ratios

Network distances

Interconnectedness

Nutshell

References



# Properties

5. motifs:





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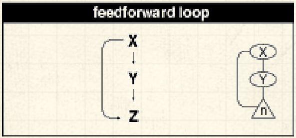
- e.g., Feed Forward Loop:

a 


Shen-Orr, Uri Alon, *et al.* [7]



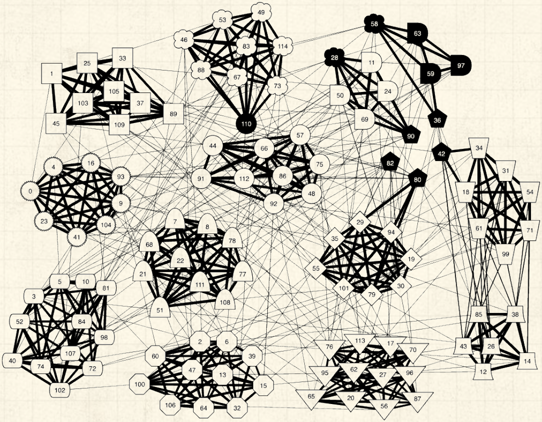


# Properties



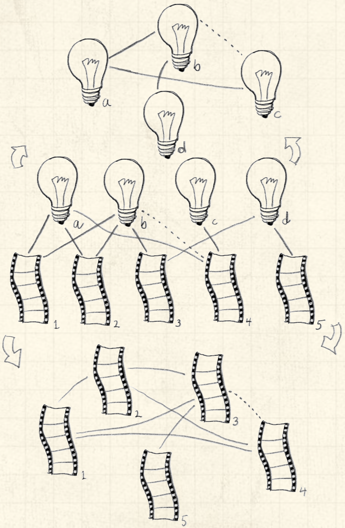



6. modularity and structure/community detection:




Clauset *et al.*, 2006 [2]: NCAA football

# Bipartite/multipartite affiliation structures:




- Many real-world networks have an underlying multi-partite structure.
  - Stories-tropes.
  - Boards and directors.
  - Films-actors-directors.
  - Classes-teachers-students.
  - Upstairs-downstairs.
- Unipartite networks may be induced or co-exist.





# Outline





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- Kretzschmar and Morris, 1996 [4]
- “Temporal networks” become a concrete area of study for Piranha Physicus in 2013.





# Outline







# Properties



## 8. Horton-Strahler ratios:

- Metrics for branching networks:



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  - Area/Volume: $R_a = \langle a_{\omega+1} \rangle / \langle a_\omega \rangle$

# Outline







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- $d_{\mathrm{cl}} = \infty$ only when all nodes are isolated.
- Closeness perhaps compresses too much into one number





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- ex 4: Recursive centrality: Hubs and Authorities (Jon Kleinberg [3])

# Outline





# Properties





## Interconnected networks and robustness (two for one deal):

“Catastrophic cascade of failures in interdependent networks”[1]. Buldyrev et al., Nature 2010.

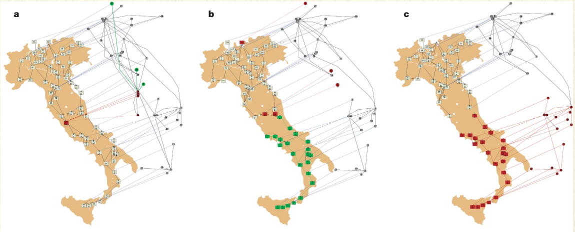


**Figure 1 | Modelling a blackout in Italy.** Illustration of an iterative process of a cascade of failures using real-world data from a power network (located on the map of Italy) and an Internet network (shifted above the map) that were implicated in an electrical blackout that occurred in Italy in September 2003[20]. The networks are drawn using the real geographical locations and every Internet server is connected to the geographically nearest power station. **a,** One power station is removed (red node on map) from the power network and as a result the Internet nodes depending on it are removed from the Internet network (red nodes above the map). The nodes that will be disconnected from the giant cluster (a cluster that spans the entire network) at the next step are marked in green. **b,** Additional nodes that were disconnected from the Internet communication network giant component are removed (red nodes above map). As a result the power stations depending on them are removed from the power network (red nodes on map). Again, the nodes that will be disconnected from the giant cluster at the next step are marked in green. **c,** Additional nodes that were disconnected from the giant component of the power network are removed (red nodes on map) as well as the nodes in the Internet network that depend on them (red nodes above map).

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- Specific focus on networks that are large-scale, sparse, natural or man-made, evolving and dynamic, and (crucially) measurable.
- Three main (blurred) categories:
  1. Physical (e.g., river networks),
  2. Interactional (e.g., social networks),
  3. Abstract (e.g., thesauri).





scale-free-networks,

# Neural reboot (NR):

## Mouse

https://www.youtube.com/watch?v=GpYY9oz9qnI?rel=0





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