



Mixed, correlated random networks

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Principles of Complex Systems, Vols. 1 & 2
CSYS/MATH 300 and 303, 2021-2022 | @pocsvox

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Mixed, correlated
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Productions



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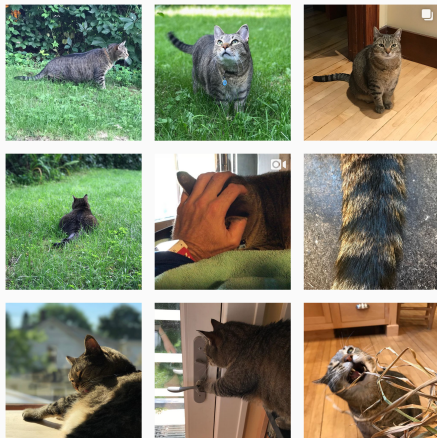


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 On Instagram at [pratchett_the_cat](https://www.instagram.com/pratchett_the_cat) 



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Random directed networks:



So far, we've largely studied networks with undirected, unweighted edges.

Now consider directed, unweighted edges.



Nodes have k_i and k_o incoming and outgoing edges, otherwise random.

Network defined by joint in- and out-degree distribution: P_{k_i, k_o}

Normalization: $\sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} P_{k_i, k_o} = 1$

Marginal in-degree and out-degree distributions:

$$P_{k_i} = \sum_{k_o=0}^{\infty} P_{k_i, k_o} \quad \text{and} \quad P_{k_o} = \sum_{k_i=0}^{\infty} P_{k_i, k_o}$$

Required balance:

$$\langle k_i \rangle = \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} k_i P_{k_i, k_o} = \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} k_o P_{k_i, k_o} = \langle k_o \rangle$$

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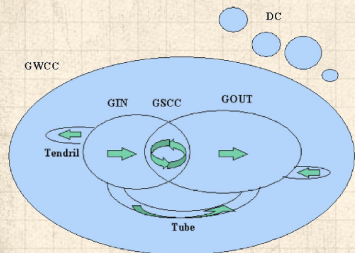
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Directed network structure:



 GWCC = Giant Weakly Connected Component (directions removed);

 GIN = Giant In-Component;

 GOUT = Giant Out-Component;

 GSCC = Giant Strongly Connected Component;

 DC = Disconnected Components (finite).

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From Boguñá and Serano. [1]

 When moving through a family of increasingly connected directed random networks, GWCC usually appears before GIN, GOUT, and GSCC which tend to appear together. [4, 1]



Important observation:

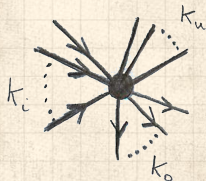
- Directed and undirected random networks are separate families ...
- ...and analyses are also disjoint.
- Need to examine a larger family of random networks with mixed directed and undirected edges.

Consider nodes with three types of edges:

- k_u undirected edges,
- k_i incoming directed edges,
- k_o outgoing directed edges.


Define a node by generalized degree:

$$\vec{k} = [k_u \ k_i \ k_o]^T.$$





Joint degree distribution:

$$P_{\vec{k}} \text{ where } \vec{k} = [k_u \ k_i \ k_o]^T.$$

 As for directed networks, require in- and out-degree averages to match up:

$$\langle k_i \rangle = \sum_{k_u=0}^{\infty} \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} k_i P_{\vec{k}} = \sum_{k_u=0}^{\infty} \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} k_o P_{\vec{k}} = \langle k_o \rangle$$

 Otherwise, no other restrictions and connections are random.

 Directed and undirected random networks are disjoint subfamilies:

$$\text{Undirected: } P_{\vec{k}} = P_{k_u} \delta_{k_i,0} \delta_{k_o,0},$$

$$\text{Directed: } P_{\vec{k}} = \delta_{k_u,0} P_{k_i, k_o}.$$



Correlations:



Now add correlations (two point or Markovian) □:

1. $P^{(u)}(\vec{k} | \vec{k}') =$ probability that an undirected edge leaving a degree \vec{k}' nodes arrives at a degree \vec{k} node.
2. $P^{(i)}(\vec{k} | \vec{k}') =$ probability that an edge leaving a degree \vec{k}' nodes arrives at a degree \vec{k} node is an in-directed edge relative to the destination node.
3. $P^{(o)}(\vec{k} | \vec{k}') =$ probability that an edge leaving a degree \vec{k}' nodes arrives at a degree \vec{k} node is an out-directed edge relative to the destination node.



Now require more refined (detailed) balance.



Conditional probabilities cannot be arbitrary.

1. $P^{(u)}(\vec{k} | \vec{k}')$ must be related to $P^{(u)}(\vec{k}' | \vec{k})$.
2. $P^{(o)}(\vec{k} | \vec{k}')$ and $P^{(i)}(\vec{k} | \vec{k}')$ must be connected.

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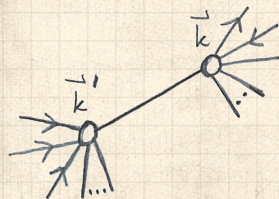
Correlations—Undirected edge balance:

- ☰ Randomly choose an edge, and randomly choose one end.
- ☰ Say we find a degree \vec{k} node at this end, and a degree \vec{k}' node at the other end.
- ☰ Define probability this happens as $P^{(u)}(\vec{k}, \vec{k}')$.
- ☰ Observe we must have $P^{(u)}(\vec{k}, \vec{k}') = P^{(u)}(\vec{k}', \vec{k})$.

☰ Conditional probability connection:

$$P^{(u)}(\vec{k}, \vec{k}') = P^{(u)}(\vec{k} | \vec{k}') \frac{k'_u P(\vec{k}')}{\langle k'_u \rangle}$$

$$P^{(u)}(\vec{k}', \vec{k}) = P^{(u)}(\vec{k}' | \vec{k}) \frac{k_u P(\vec{k})}{\langle k_u \rangle}$$



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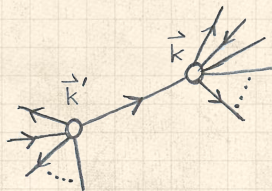
Correlations—Directed edge balance:

 The quantities

$$\frac{k_o P(\vec{k})}{\langle k_o \rangle} \text{ and } \frac{k_i P(\vec{k})}{\langle k_i \rangle}$$

give the probabilities that in starting at a random end of a randomly selected edge, we begin at a degree \vec{k} node and then find ourselves travelling:

1. along an outgoing edge, or
2. against the direction of an incoming edge.



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 We therefore have

$$P^{(\text{dir})}(\vec{k}, \vec{k}') = P^{(\text{i})}(\vec{k} | \vec{k}') \frac{k'_o P(\vec{k}')}{\langle k'_o \rangle} = P^{(\text{o})}(\vec{k}' | \vec{k}) \frac{k_i P(\vec{k})}{\langle k_i \rangle}.$$

 Note that $P^{(\text{dir})}(\vec{k}, \vec{k}')$ and $P^{(\text{dir})}(\vec{k}', \vec{k})$ are in general not related if $\vec{k} \neq \vec{k}'$.



Global spreading condition: [2]

When are cascades possible?:

- Consider uncorrelated mixed networks first.
- Recall our first result for undirected random networks, that edge gain ratio must exceed 1:

$$\mathbf{R} = \sum_{k_u=0}^{\infty} \frac{k_u P_{k_u}}{\langle k_u \rangle} \cdot (k_u - 1) \cdot B_{k_u,1} > 1.$$

- Similar form for purely directed networks:


$$\mathbf{R} = \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} \frac{k_i P_{k_i, k_o}}{\langle k_i \rangle} \cdot k_o \cdot B_{k_i,1} > 1.$$


- Both are composed of (1) probability of connection to a node of a given type; (2) number of newly infected edges if successful; and (3) probability of infection.




Global spreading condition:


Local growth equation:


 Define number of infected edges leading to nodes a distance d away from the original seed as $f(d)$.

 Infected edge growth equation:

$$f(d + 1) = \mathbf{R}f(d).$$

 Applies for discrete time and continuous time contagion processes.

 Now see $B_{k_u, 1}$ is the probability that an infected edge eventually infects a node.

 Also allows for recovery of nodes (SIR).

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Global spreading condition:

Mixed, uncorrelated random networks:

 Now have two types of edges spreading infection: directed and undirected.

 Gain ratio now more complicated:

1. Infected directed edges can lead to infected directed or undirected edges.
2. Infected undirected edges can lead to infected directed or undirected edges.

 Define $f^{(u)}(d)$ and $f^{(o)}(d)$ as the expected number of infected undirected and directed edges leading to nodes a distance d from seed.

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 Gain ratio now has a matrix form:

$$\begin{bmatrix} f^{(u)}(d+1) \\ f^{(o)}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(u)}(d) \\ f^{(o)}(d) \end{bmatrix}$$


 Two separate gain equations:

$$f^{(u)}(d+1) = \sum_{\bar{k}} \left[\frac{k_u P_{\bar{k}}}{\langle k_u \rangle} \bullet (k_u - 1) \bullet B_{k_u+k_i,1} f^{(u)}(d) + \frac{k_i P_{\bar{k}}}{\langle k_i \rangle} \bullet k_u \bullet B_{k_u+k_i,1} f^{(o)}(d) \right]$$

$$f^{(o)}(d+1) = \sum_{\bar{k}} \left[\frac{k_u P_{\bar{k}}}{\langle k_u \rangle} \bullet k_o B_{k_u+k_i,1} f^{(u)}(d) + \frac{k_i P_{\bar{k}}}{\langle k_i \rangle} \bullet k_o \bullet B_{k_u+k_i,1} f^{(o)}(d) \right]$$

 Gain ratio matrix:

$$\mathbf{R} = \sum_{\bar{k}} \begin{bmatrix} \frac{k_u P_{\bar{k}}}{\langle k_u \rangle} \bullet (k_u - 1) & \frac{k_i P_{\bar{k}}}{\langle k_i \rangle} \bullet k_u \\ \frac{k_u P_{\bar{k}}}{\langle k_u \rangle} \bullet k_o & \frac{k_i P_{\bar{k}}}{\langle k_i \rangle} \bullet k_o \end{bmatrix} \bullet B_{k_u+k_i,1}$$

 Spreading condition: max eigenvalue of $\mathbf{R} > 1$.

Global spreading condition:

- Useful change of notation for making results more general: write $P^{(u)}(\vec{k} | *) = \frac{k_u P_{\vec{k}}}{\langle k_u \rangle}$ and $P^{(i)}(\vec{k} | *) = \frac{k_i P_{\vec{k}}}{\langle k_i \rangle}$ where $*$ indicates the starting node's degree is irrelevant (no correlations).
- Also write $B_{k_u k_i, *}$ to indicate a more general infection probability, but one that does not depend on the edge's origin.
- Now have, for the example of mixed, uncorrelated random networks:

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(u)}(\vec{k} | *) \bullet (k_u - 1) & P^{(i)}(\vec{k} | *) \bullet k_u \\ P^{(u)}(\vec{k} | *) \bullet k_o & P^{(i)}(\vec{k} | *) \bullet k_o \end{bmatrix} \bullet B_{k_u k_i, *}$$



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Summary of contagion conditions for uncorrelated networks:

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 I. Undirected, Uncorrelated— $f(d + 1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_u} P^{(u)}(k_u | *) \bullet (k_u - 1) \bullet B_{k_u, *}$$

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 II. Directed, Uncorrelated— $f(d + 1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_i, k_o} P^{(i)}(k_i, k_o | *) \bullet k_o \bullet B_{k_i, *}$$

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 III. Mixed Directed and Undirected, Uncorrelated—

$$\begin{bmatrix} f^{(u)}(d + 1) \\ f^{(o)}(d + 1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(u)}(d) \\ f^{(o)}(d) \end{bmatrix}$$

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(u)}(\vec{k} | *) \bullet (k_u - 1) & P^{(i)}(\vec{k} | *) \bullet k_u \\ P^{(u)}(\vec{k} | *) \bullet k_o & P^{(i)}(\vec{k} | *) \bullet k_o \end{bmatrix} \bullet B_{k_u, k_i, *}$$



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Correlated version:

- Now have to think of transfer of infection from edges emanating from degree \vec{k}' nodes to edges emanating from degree \vec{k} nodes.
- Replace $P^{(i)}(\vec{k} | *)$ with $P^{(i)}(\vec{k} | \vec{k}')$ and so on.
- Edge types are now more diverse beyond directed and undirected as originating node type matters.
- Sums are now over \vec{k}' .

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Summary of contagion conditions for correlated networks:

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IV. Undirected,

Correlated— $f_{k_u}(d+1) = \sum_{k'_u} R_{k_u k'_u} f_{k'_u}(d)$

$$R_{k_u k'_u} = P^{(u)}(k_u | k'_u) \cdot (k_u - 1) \cdot B_{k_u k'_u}$$

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V. Directed,

Correlated— $f_{k_i k_o}(d+1) = \sum_{k'_i, k'_o} R_{k_i k_o k'_i k'_o} f_{k'_i k'_o}(d)$

$$R_{k_i k_o k'_i k'_o} = P^{(i)}(k_i, k_o | k'_i, k'_o) \cdot k_o \cdot B_{k_i k_o k'_i k'_o}$$

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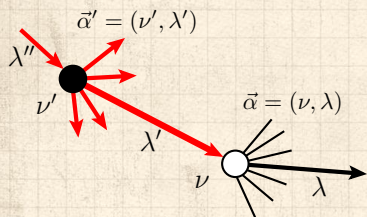
VI. Mixed Directed and Undirected, Correlated—

$$\begin{bmatrix} f_{\vec{k}}^{(u)}(d+1) \\ f_{\vec{k}}^{(o)}(d+1) \end{bmatrix} = \sum_{\vec{k}'} \mathbf{R}_{\vec{k} \vec{k}'} \begin{bmatrix} f_{\vec{k}'}^{(u)}(d) \\ f_{\vec{k}'}^{(o)}(d) \end{bmatrix}$$

$$\mathbf{R}_{\vec{k} \vec{k}'} = \begin{bmatrix} P^{(u)}(\vec{k} | \vec{k}') \cdot (k_u - 1) & P^{(i)}(\vec{k} | \vec{k}') \cdot k_u \\ P^{(u)}(\vec{k} | \vec{k}') \cdot k_o & P^{(i)}(\vec{k} | \vec{k}') \cdot k_o \end{bmatrix} \cdot B_{\vec{k} \vec{k}'}$$




Full generalization:





$$f_{\vec{\alpha}}(d+1) = \sum_{\vec{\alpha}'} R_{\vec{\alpha}\vec{\alpha}'} f_{\vec{\alpha}'}(d)$$


$R_{\vec{\alpha}\vec{\alpha}'}$ is the gain ratio matrix and has the form:

$$R_{\vec{\alpha}\vec{\alpha}'} = P_{\vec{\alpha}\vec{\alpha}'} \bullet k_{\vec{\alpha}\vec{\alpha}'} \bullet B_{\vec{\alpha}\vec{\alpha}'}$$

 $P_{\vec{\alpha}\vec{\alpha}'}$ = conditional probability that a type λ' edge emanating from a type ν' node leads to a type ν node.

 $k_{\vec{\alpha}\vec{\alpha}'}$ = potential number of newly infected edges of type λ emanating from nodes of type ν .

 $B_{\vec{\alpha}\vec{\alpha}'}$ = probability that a type ν node is eventually infected by a single infected type λ' link arriving from a neighboring node of type ν' .

 Generalized contagion condition:

$$\max|\mu| : \mu \in \sigma(\mathbf{R}) > 1$$



As we saw earlier, the triggering probability for simple contagion on random networks can be determined with a straightforward physical argument.

Two good things:

$$Q_{\text{trig}} = \sum_{k=0}^{\infty} \frac{k P_k}{\langle k \rangle} \cdot B_{k1} \cdot \left[1 - (1 - Q_{\text{trig}})^{k-1} \right],$$

$$P_{\text{trig}} = S_{\text{trig}} = \sum_k P_k \cdot \left[1 - (1 - Q_{\text{trig}})^k \right].$$

Equivalent to result found via the eldritch route of generating functions.

Generating functions arguably make some kinds of calculations easier (but perhaps we don't care about component sizes that much).

On the other hand, a plainspoken physical argument helps us generalize to correlated networks more easily.



Summary of triggering probabilities for uncorrelated networks: ^[3] □

I. Undirected, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_u} P^{(u)}(k'_u | \cdot) B_{k'_u 1} \left[1 - (1 - Q_{\text{trig}})^{k'_u - 1} \right]$$

$$P_{\text{trig}} = S_{\text{trig}} = \sum_{k'_u} P(k'_u) \left[1 - (1 - Q_{\text{trig}})^{k'_u} \right]$$

II. Directed, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_i, k'_o} P^{(u)}(k'_i, k'_o | \cdot) B_{k'_i 1} \left[1 - (1 - Q_{\text{trig}})^{k'_o} \right]$$

$$S_{\text{trig}} = \sum_{k'_i, k'_o} P(k'_i, k'_o) \left[1 - (1 - Q_{\text{trig}})^{k'_o} \right]$$

Summary of triggering probabilities for uncorrelated networks:

III. Mixed Directed and Undirected, Uncorrelated—

$$Q_{\text{trig}}^{(u)} = \sum_{\vec{k}'} P^{(u)}(\vec{k}' | \cdot) B_{\vec{k}'1} \left[1 - (1 - Q_{\text{trig}}^{(u)})^{k'_u - 1} (1 - Q_{\text{trig}}^{(o)})^{k'_o} \right]$$

$$Q_{\text{trig}}^{(o)} = \sum_{\vec{k}'} P^{(i)}(\vec{k}' | \cdot) B_{\vec{k}'1} \left[1 - (1 - Q_{\text{trig}}^{(u)})^{k'_u} (1 - Q_{\text{trig}}^{(o)})^{k'_o} \right]$$

$$S_{\text{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q_{\text{trig}}^{(u)})^{k'_u} (1 - Q_{\text{trig}}^{(o)})^{k'_o} \right]$$

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Summary of triggering probabilities for correlated networks:

IV. Undirected, Correlated— $Q_{\text{trig}}(k_u) =$

$$\sum_{k'_u} P^{(u)}(k'_u | k_u) B_{k'_u} \left[1 - (1 - Q_{\text{trig}}(k'_u))^{k'_u - 1} \right]$$

$$S_{\text{trig}} = \sum_{k'_u} P(k'_u) \left[1 - (1 - Q_{\text{trig}}(k'_u))^{k'_u} \right]$$

V. Directed, Correlated— $Q_{\text{trig}}(k_i, k_o) =$

$$\sum_{k'_i, k'_o} P^{(u)}(k'_i, k'_o | k_i, k_o) B_{k'_i} \left[1 - (1 - Q_{\text{trig}}(k'_i, k'_o))^{k'_o} \right]$$

$$S_{\text{trig}} = \sum_{k'_i, k'_o} P(k'_i, k'_o) \left[1 - (1 - Q_{\text{trig}}(k'_i, k'_o))^{k'_o} \right]$$

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Summary of triggering probabilities for correlated networks:

VI. Mixed Directed and Undirected, Correlated—

$$Q_{\text{trig}}^{(u)}(\vec{k}) = \sum_{\vec{k}'} P^{(u)}(\vec{k}' | \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q_{\text{trig}}^{(u)}(\vec{k}'))^{k'_u - 1} (1 - Q_{\text{trig}}^{(o)}(\vec{k}'))^{k'_o} \right]$$

$$Q_{\text{trig}}^{(o)}(\vec{k}) = \sum_{\vec{k}'} P^{(i)}(\vec{k}' | \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q_{\text{trig}}^{(u)}(\vec{k}'))^{k'_u} (1 - Q_{\text{trig}}^{(o)}(\vec{k}'))^{k'_o} \right]$$

$$S_{\text{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q_{\text{trig}}^{(u)}(\vec{k}'))^{k'_u} (1 - Q_{\text{trig}}^{(o)}(\vec{k}'))^{k'_o} \right]$$

Nutshell:

- ☰ Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.
- ☰ Spreading conditions and triggering probabilities of contagion processes can be determined using a direct, physical approach.
- ☰ These conditions can be generalized to arbitrary random networks with arbitrary node and edge types.
- ☰ More generalizations: bipartite affiliation graphs and multilayer networks.

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