Mixed, correlated random networks

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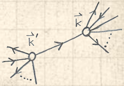
Principles of Complex Systems, Vols. 1 & 2
CSYS/MATH 300 and 303, 2021–2022 | @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont



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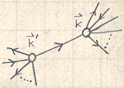

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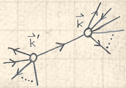
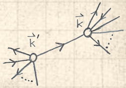


Outline




THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
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THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
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THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
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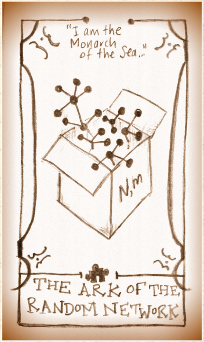
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


"I am the Monarch of the Sea.."
N,m
THE ARK OF THE RANDOM NETWORK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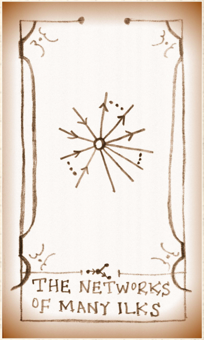
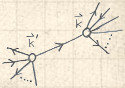

THE NETWORKS OF MANY ILKS

Random directed networks:

- So far, we've largely studied networks with undirected, unweighted edges.



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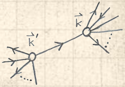

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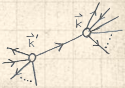
- Nodes have k_{i} and k_{o} incoming and outgoing edges, otherwise random.



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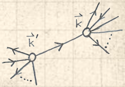

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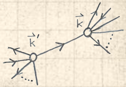

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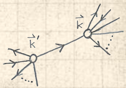
- Marginal in-degree and out-degree distributions:

$$P_{k_{\mathrm{i}}} = \sum_{k_{\mathrm{o}}=0}^{\infty} P_{k_{\mathrm{i}},k_{\mathrm{o}}} \text{ and } P_{k_{\mathrm{o}}} = \sum_{k_{\mathrm{i}}=0}^{\infty} P_{k_{\mathrm{i}},k_{\mathrm{o}}}$$



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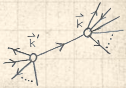
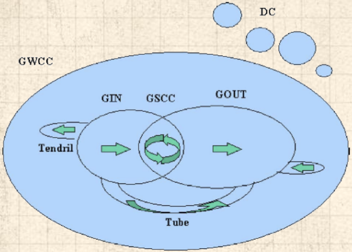
- Required balance:

$$\langle k_{\mathrm{i}} \rangle = \sum_{k_{\mathrm{i}}=0}^{\infty}\sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{i}} P_{k_{\mathrm{i}},k_{\mathrm{o}}} = \sum_{k_{\mathrm{i}}=0}^{\infty}\sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{o}} P_{k_{\mathrm{i}},k_{\mathrm{o}}} = \langle k_{\mathrm{o}} \rangle$$

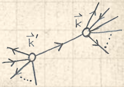


Directed network structure:



From Boguñá and Serano. [1]

- GWCC = Giant Weakly Connected Component (directions removed);
- GIN = Giant In-Component;
- GOUT = Giant Out-Component;
- GSCC = Giant Strongly Connected Component;
- DC = Disconnected Components (finite).



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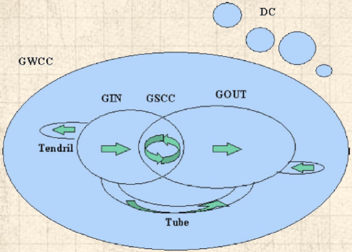

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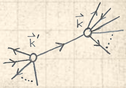
- When moving through a family of increasingly connected directed random networks, GWCC usually appears before GIN, GOUT, and GSCC which tend to appear together. [4, 1]



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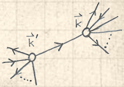
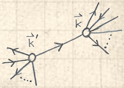


Important observation:

- Directed and undirected random networks are separate families ...



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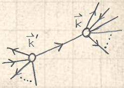

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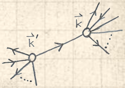

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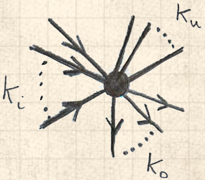

- Consider nodes with three types of edges:
  1. k_{u} undirected edges,
  2. k_{i} incoming directed edges,
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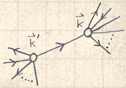

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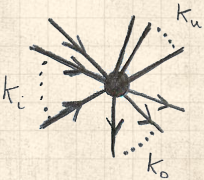
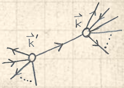


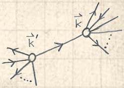
- Consider nodes with three types of edges:
  1. k_{u} undirected edges,
  2. k_{i} incoming directed edges,
  3. k_{o} outgoing directed edges.
- Define a node by generalized degree:

$$\vec{k} = [\ k_{\mathrm{u}}\ \ k_{\mathrm{i}}\ \ k_{\mathrm{o}}\ ]^{\mathsf{T}}.$$



Joint degree distribution:

$$P_{\vec{k}} \text{ where } \vec{k} = [\, k_{\mathrm{u}} \;\; k_{\mathrm{i}} \;\; k_{\mathrm{o}} \,]^{\mathsf{T}}.$$



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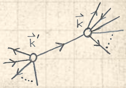
$$\langle k_{\text{i}} \rangle = \sum_{k_{\text{u}}=0}^{\infty} \sum_{k_{\text{i}}=0}^{\infty} \sum_{k_{\text{o}}=0}^{\infty} k_{\text{i}} P_{\vec{k}} = \sum_{k_{\text{u}}=0}^{\infty} \sum_{k_{\text{i}}=0}^{\infty} \sum_{k_{\text{o}}=0}^{\infty} k_{\text{o}} P_{\vec{k}} = \langle k_{\text{o}} \rangle$$



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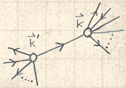
- Otherwise, no other restrictions and connections are random.



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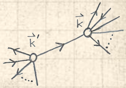
- Otherwise, no other restrictions and connections are random.
- Directed and undirected random networks are disjoint subfamilies:

$$\text{Undirected: } P_{\vec{k}} = P_{k_{\mathsf{u}}} \delta_{k_{\mathsf{i}},0} \delta_{k_{\mathsf{o}},0},$$

$$\text{Directed: } P_{\vec{k}} = \delta_{k_{\mathsf{u}},0} P_{k_{\mathsf{i}},k_{\mathsf{o}}}.$$



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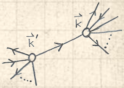
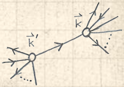


Correlations:

- Now add correlations (two point or Markovian) ☐:



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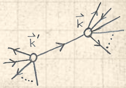
  1. $P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')$ = probability that an undirected edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node.



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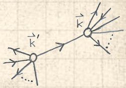

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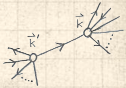

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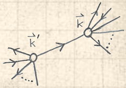

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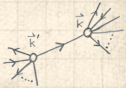

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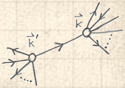

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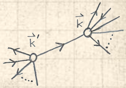

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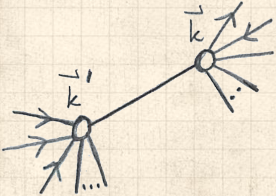

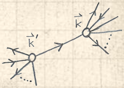

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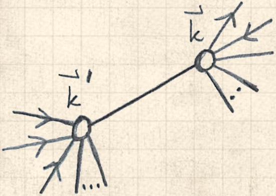

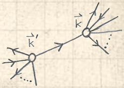

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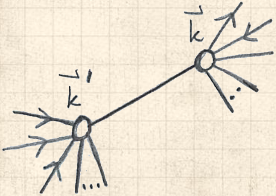

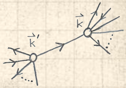

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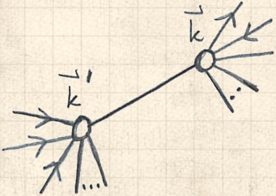

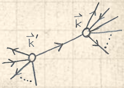

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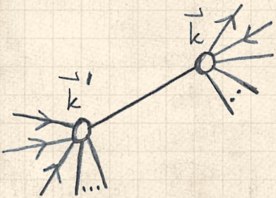

- Conditional probability connection:

$$P^{(\mathrm{u})}(\vec{k}, \vec{k}') = P^{(\mathrm{u})}(\vec{k} \mid \vec{k}') \frac{k'_{\mathrm{u}} P(\vec{k}')}{\langle k'_{\mathrm{u}} \rangle}$$

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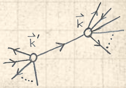

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$$\frac{k_{\mathrm{o}} P(\vec{k})}{\langle k_{\mathrm{o}} \rangle} \text{ and } \frac{k_{\mathrm{i}} P(\vec{k})}{\langle k_{\mathrm{i}} \rangle}$$

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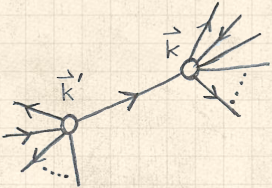

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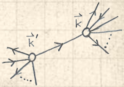

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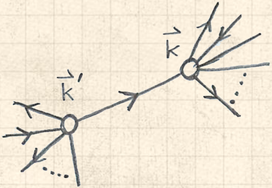

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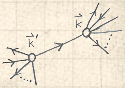

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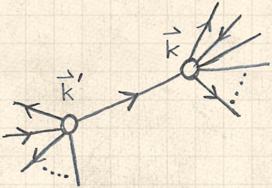

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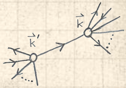
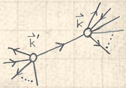


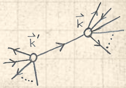
- Note that $P^{(\mathrm{dir})}(\vec{k}, \vec{k}')$ and $P^{(\mathrm{dir})}(\vec{k}', \vec{k})$ are in general not related if $\vec{k} \neq \vec{k}'$.

Outline



Global spreading condition: [2]

When are cascades possible?:



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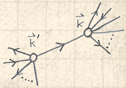

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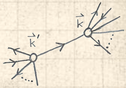

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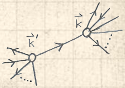

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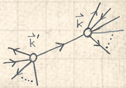
- Both are composed of (1) probability of connection to a node of a given type; (2) number of newly infected edges if successful; and (3) probability of infection.



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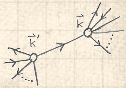
Local growth equation:

- Define number of infected edges leading to nodes a distance d away from the original seed as $f(d)$.



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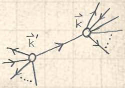

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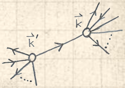

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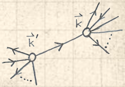

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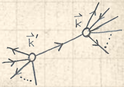
- Applies for discrete time and continuous time contagion processes.
- Now see $B_{k_{\mathrm{u}},1}$ is the probability that an infected edge eventually infects a node.
- Also allows for recovery of nodes (SIR).



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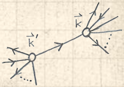
- Now have two types of edges spreading infection: directed and undirected.



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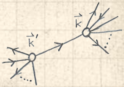

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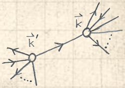
- Define $f^{(u)}(d)$ and $f^{(o)}(d)$ as the expected number of infected undirected and directed edges leading to nodes a distance d from seed.



Gain ratio now has a matrix form:

$$\begin{bmatrix} f^{(\mathsf{u})}(d+1) \\ f^{(\mathsf{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathsf{u})}(d) \\ f^{(\mathsf{o})}(d) \end{bmatrix}$$

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$$f^{(o)}(d+1) = \sum_{\vec{k}} \left[\frac{k_u P_{\vec{k}}}{\langle k_u \rangle} \bullet k_o B_{k_u+k_i,1} f^{(u)}(d) + \frac{k_i P_{\vec{k}}}{\langle k_i \rangle} \bullet k_o \bullet B_{k_u+k_i,1} f^{(o)}(d) \right]$$

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$$f^{(\mathsf{o})}(d+1) = \sum_{\vec{k}} \left[\frac{k_{\mathsf{u}} P_{\vec{k}}}{\langle k_{\mathsf{u}} \rangle} \bullet k_{\mathsf{o}} B_{k_{\mathsf{u}}+k_{\mathsf{i}},1} f^{(\mathsf{u})}(d) + \frac{k_{\mathsf{i}} P_{\vec{k}}}{\langle k_{\mathsf{i}} \rangle} \bullet k_{\mathsf{o}} \bullet B_{k_{\mathsf{u}}+k_{\mathsf{i}},1} f^{(\mathsf{o})}(d) \right]$$

- Gain ratio matrix:

$$\mathbf{R} = \sum_{\vec{k}} \left[\begin{array}{cc} \frac{k_{\mathsf{u}} P_{\vec{k}}}{\langle k_{\mathsf{u}} \rangle} \bullet (k_{\mathsf{u}} - 1) & \frac{k_{\mathsf{i}} P_{\vec{k}}}{\langle k_{\mathsf{i}} \rangle} \bullet k_{\mathsf{u}} \\ \frac{k_{\mathsf{u}} P_{\vec{k}}}{\langle k_{\mathsf{u}} \rangle} \bullet k_{\mathsf{o}} & \frac{k_{\mathsf{i}} P_{\vec{k}}}{\langle k_{\mathsf{i}} \rangle} \bullet k_{\mathsf{o}} \end{array} \right] \bullet B_{k_{\mathsf{u}}+k_{\mathsf{i}},1}$$

- Gain ratio now has a matrix form:

$$\begin{bmatrix} f^{(\mathsf{u})}(d+1) \\ f^{(\mathsf{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathsf{u})}(d) \\ f^{(\mathsf{o})}(d) \end{bmatrix}$$

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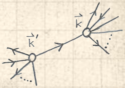
- Spreading condition: max eigenvalue of $\mathbf{R} > 1$.

Global spreading condition:

- Useful change of notation for making results more general: write $P^{(\mathrm{u})}(\vec{k}\,|\,*) = \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle}$ and $P^{(\mathrm{i})}(\vec{k}\,|\,*) = \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle}$ where $*$ indicates the starting node's degree is irrelevant (no correlations).



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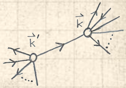

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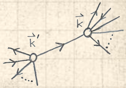
- Now have, for the example of mixed, uncorrelated random networks:

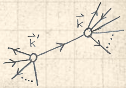
$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet (k_{\mathrm{u}} - 1) & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{u}} \\ P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$$



Summary of contagion conditions for uncorrelated networks:

- I. Undirected, Uncorrelated—$f(d+1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_u} P^{(u)}(k_u \,|\, *) \bullet (k_u - 1) \bullet B_{k_u,*}$$



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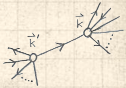

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- III. Mixed Directed and Undirected, Uncorrelated—

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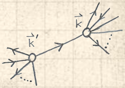
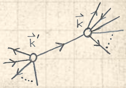
$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathsf{u})}(\vec{k} \,|\, *) \bullet (k_{\mathsf{u}} - 1) & P^{(\mathsf{i})}(\vec{k} \,|\, *) \bullet k_{\mathsf{u}} \\ P^{(\mathsf{u})}(\vec{k} \,|\, *) \bullet k_{\mathsf{o}} & P^{(\mathsf{i})}(\vec{k} \,|\, *) \bullet k_{\mathsf{o}} \end{bmatrix} \bullet B_{k_{\mathsf{u}} k_{\mathsf{i}},*}$$



Correlated version:

- Now have to think of transfer of infection from edges emanating from degree $\vec{k}'$ nodes to edges emanating from degree $\vec{k}$ nodes.



Correlated version:

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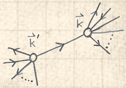

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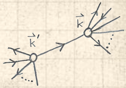
- Edge types are now more diverse beyond directed and undirected as originating node type matters.



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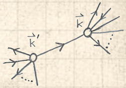
- Sums are now over $\vec{k}'$.



Summary of contagion conditions for correlated networks:

- IV. Undirected, Correlated—$f_{k_u}(d+1) = \sum_{k'_u} R_{k_u k'_u} f_{k'_u}(d)$

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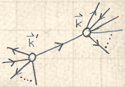

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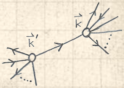
$$R_{k_\mathrm{i} k_\mathrm{o} k'_\mathrm{i} k'_\mathrm{o}} = P^{(\mathrm{i})}(k_\mathrm{i}, k_\mathrm{o} \mid k'_\mathrm{i}, k'_\mathrm{o}) \bullet k_\mathrm{o} \bullet B_{k_\mathrm{i} k_\mathrm{o} k'_\mathrm{i} k'_\mathrm{o}}$$



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- VI. Mixed Directed and Undirected, Correlated—

$$\begin{bmatrix} f^{(\mathrm{u})}_{\vec{k}}(d+1) \\ f^{(\mathrm{o})}_{\vec{k}}(d+1) \end{bmatrix} = \sum_{k'} \mathbf{R}_{\vec{k}\vec{k}'} \begin{bmatrix} f^{(\mathrm{u})}_{\vec{k}'}(d) \\ f^{(\mathrm{o})}_{\vec{k}'}(d) \end{bmatrix}$$

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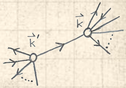
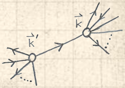
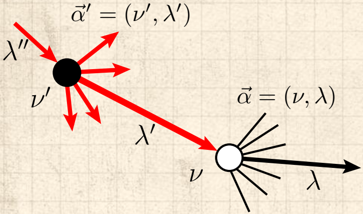


Outline



Full generalization:



$$f_{\vec{\alpha}}(d+1) = \sum_{\vec{\alpha}'} R_{\vec{\alpha}\vec{\alpha}'} f_{\vec{\alpha}'}(d)$$

$R_{\vec{\alpha}\vec{\alpha}'}$ is the gain ratio matrix and has the form:

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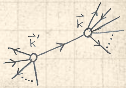

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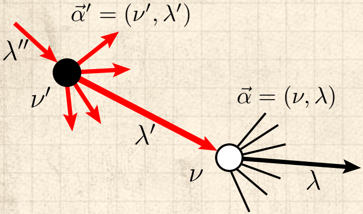

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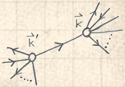

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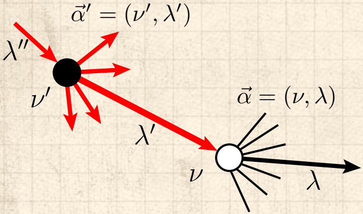

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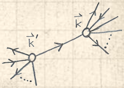

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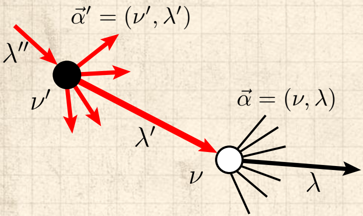

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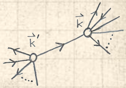

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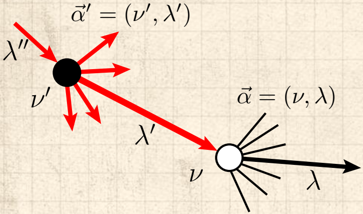

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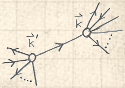
- Generalized contagion condition:

$$\max|\mu| : \mu \in \sigma(\mathbf{R}) > 1$$



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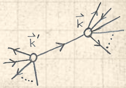

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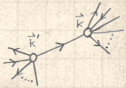

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- Two good things:

$$Q_{\text{trig}} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet B_{k1} \bullet \left[1 - \left(1 - Q_{\text{trig}}\right)^{k-1}\right],$$

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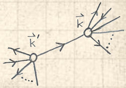

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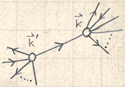
- Equivalent to result found via the eldritch route of generating functions.



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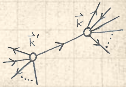
- Equivalent to result found via the eldritch route of generating functions.
- Generating functions arguably make some kinds of calculations easier (but perhaps we don't care about component sizes that much).



- As we saw earlier, the triggering probability for simple contagion on random networks can be determined with a straightforward physical argument.
- Two good things:

$$Q_{\text{trig}} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet B_{k1} \bullet \left[1 - \left(1 - Q_{\text{trig}}\right)^{k-1}\right],$$

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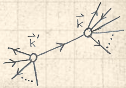
- Equivalent to result found via the eldritch route of generating functions.
- Generating functions arguably make some kinds of calculations easier (but perhaps we don't care about component sizes that much).
- On the other hand, a plainspoken physical argument helps us generalize to correlated networks more easily.



Summary of triggering probabilities for uncorrelated networks: [3]

- I. Undirected, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_{\text{u}}} P^{(\text{u})}(k'_{\text{u}} \,|\, \cdot) B_{k'_{\text{u}}1} \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{u}}-1}\right]$$

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- II. Directed, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_{\text{i}}, k'_{\text{o}}} P^{(\text{u})}(k'_{\text{i}}, k'_{\text{o}} | \cdot) B_{k'_{\text{i}}1} \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{o}}} \right]$$

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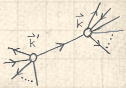
Summary of triggering probabilities for uncorrelated networks:

- III. Mixed Directed and Undirected, Uncorrelated—

$$Q^{(\mathsf{u})}_{\mathsf{trig}} = \sum_{\vec{k}'} P^{(\mathsf{u})}(\vec{k}'\,|\,\cdot) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathsf{u})}_{\mathsf{trig}})^{k'_{\mathsf{u}}-1} (1 - Q^{(\mathsf{o})}_{\mathsf{trig}})^{k'_{\mathsf{o}}}\right]$$

$$Q^{(\mathsf{o})}_{\mathsf{trig}} = \sum_{\vec{k}'} P^{(\mathsf{i})}(\vec{k}'\,|\,\cdot) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathsf{u})}_{\mathsf{trig}})^{k'_{\mathsf{u}}} (1 - Q^{(\mathsf{o})}_{\mathsf{trig}})^{k'_{\mathsf{o}}}\right]$$

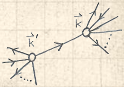
$$S_{\mathsf{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q^{(\mathsf{u})}_{\mathsf{trig}})^{k'_{\mathsf{u}}} (1 - Q^{(\mathsf{o})}_{\mathsf{trig}})^{k'_{\mathsf{o}}}\right]$$



Summary of triggering probabilities for correlated networks:

- IV. Undirected, Correlated— $Q_{\text{trig}}(k_{\text{u}}) = \sum_{k'_{\text{u}}} P^{(\text{u})}(k'_{\text{u}} \,|\, k_{\text{u}}) B_{k'_{\text{u}}1} \left[1 - (1 - Q_{\text{trig}}(k'_{\text{u}}))^{k'_{\text{u}}-1}\right]$

$$S_{\text{trig}} = \sum_{k'_{\text{u}}} P(k'_{\text{u}}) \left[1 - (1 - Q_{\text{trig}}(k'_{\text{u}}))^{k'_{\text{u}}}\right]$$



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- IV. Undirected, Correlated— $Q_{\text{trig}}(k_{\text{u}}) = \sum_{k'_{\text{u}}} P^{(\text{u})}(k'_{\text{u}} \,|\, k_{\text{u}}) B_{k'_{\text{u}}1} \left[1 - (1 - Q_{\text{trig}}(k'_{\text{u}}))^{k'_{\text{u}}-1}\right]$

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- V. Directed, Correlated— $Q_{\text{trig}}(k_{\text{i}}, k_{\text{o}}) = \sum_{k'_{\text{i}}, k'_{\text{o}}} P^{(\text{u})}(k'_{\text{i}}, k'_{\text{o}} | \, k_{\text{i}}, k_{\text{o}}) B_{k'_{\text{i}}1} \left[1 - (1 - Q_{\text{trig}}(k'_{\text{i}}, k'_{\text{o}}))^{k'_{\text{o}}}\right]$

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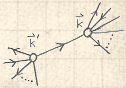


Summary of triggering probabilities for correlated networks:

- VI. Mixed Directed and Undirected, Correlated—

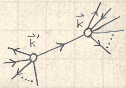
$$Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}-1} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}} \right]$$

$$Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}} \right]$$

$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}} \right]$$

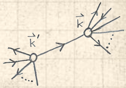
Nutshell:

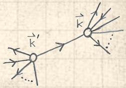
- Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.



Nutshell:

- Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.
- Spreading conditions and triggering probabilities of contagion processes can be determined using a direct, physical approach.





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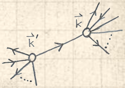
- These conditions can be generalized to arbitrary random networks with arbitrary node and edge types.
- More generalizations: bipartite affiliation graphs and multilayer networks.



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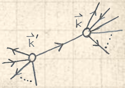

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