

Generating Functions and Networks

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Principles of Complex Systems, Vols. 1 & 2
CSYS/MATH 300 and 303, 2021-2022 | @pocsvox

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Computational Story Lab | Vermont Complex Systems Center
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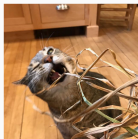
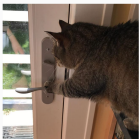
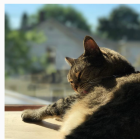
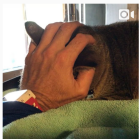
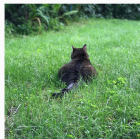
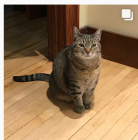
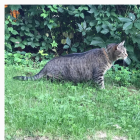
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- Basic Properties
- Giant Component
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- Component sizes
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- A few examples
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 On Instagram at [pratchett_the_cat](https://www.instagram.com/pratchett_the_cat) 

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Generating functionology ^[1]



Idea: Given a sequence a_0, a_1, a_2, \dots , associate each element with a distinct function or other mathematical object.

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 **Idea:** Given a sequence a_0, a_1, a_2, \dots , associate each element with a distinct function or other mathematical object.

 Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

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 The **generating function** (g.f.) for a sequence $\{a_n\}$ is


$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$




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-  Related to Fourier, Laplace, Mellin, ...

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Simple examples:

Rolling dice and flipping coins:


 $p_k^{(\square)} = \mathbf{Pr}(\text{throwing a } k) = 1/6$ where $k = 1, 2, \dots, 6$.

$$F^{(\square)}(x) = \sum_{k=1}^6 p_k^{(\square)} x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$




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 $p_0^{(\text{coin})} = \mathbf{Pr}(\text{head}) = 1/2, p_1^{(\text{coin})} = \mathbf{Pr}(\text{tail}) = 1/2$.

$$F^{(\text{coin})}(x) = p_0^{(\text{coin})} x^0 + p_1^{(\text{coin})} x^1 = \frac{1}{2}(1 + x).$$




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 A generating function for a probability distribution is called a **Probability Generating Function (p.g.f.)**.




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 A generating function for a probability distribution is called a **Probability Generating Function (p.g.f.)**.

 We'll come back to these simple examples as we derive various delicious properties of generating functions.



Example

 Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where geometrically, we have $c = 1 - e^{-\lambda}$

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Check die and coin p.g.f.'s.



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Properties:

 Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k$$

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Properties:

 Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \Big|_{x=1}$$

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Useful pieces for probability distributions:

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Useful pieces for probability distributions:



Normalization:

$$F(1) = 1$$

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 Higher moments:

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 Higher moments:

$$\langle k^n \rangle = \left(x \frac{d}{dx} \right)^n F(x) \Big|_{x=1}$$

 k th element of sequence (general):

$$P_k = \frac{1}{k!} \frac{d^k}{dx^k} F(x) \Big|_{x=0}$$

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A beautiful, fundamental thing:



The generating function for the sum of two
random variables


$$W = U + V$$

is

$$F_W(x) = F_U(x)F_V(x).$$





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 Conolve yourself with Convolutions:
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 1. Add two coins (tail=0, head=1).

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-  Try with die and coin p.g.f.'s.
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 2. Add two dice.
 3. Add a coin flip to one die roll.



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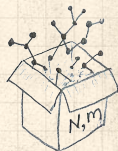
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Edge-degree distribution



Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

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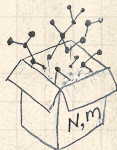
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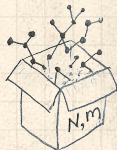


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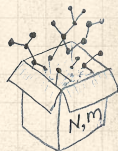
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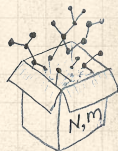
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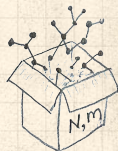
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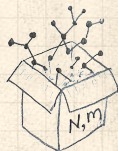
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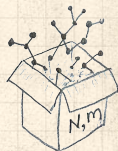
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Giant component condition in terms of g.f. is:


$$\langle k \rangle_R = F'_R(1) > 1.$$





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 We first need the g.f. for R_k .


 We'll now use this notation:

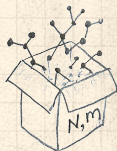
$F_P(x)$ is the g.f. for P_k .

$F_R(x)$ is the g.f. for R_k .

 Giant component condition in terms of g.f. is:

$$\langle k \rangle_R = F'_R(1) > 1.$$

 Now find how F_R is related to F_P ...



Edge-degree distribution



We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k$$

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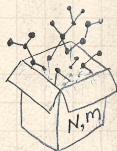
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Edge-degree distribution



We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

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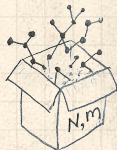
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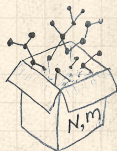
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$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to $j = k + 1$ and pull out $\frac{1}{\langle k \rangle}$:



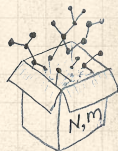
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Shift index to $j = k + 1$ and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1}$$



Edge-degree distribution

 We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

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$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{d}{dx} x^j$$

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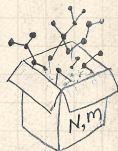
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Edge-degree distribution

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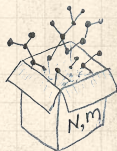
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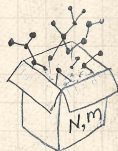
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Edge-degree distribution



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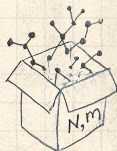
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Finally, since $\langle k \rangle = F'_P(1)$,

$$F_R(x) = \frac{F'_P(x)}{F'_P(1)}$$



Edge-degree distribution



Recall giant component condition is

$$\langle k \rangle_R = F'_R(1) > 1.$$

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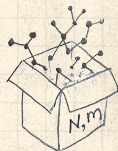
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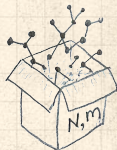


Recall giant component condition is

$$\langle k \rangle_R = F'_R(1) > 1.$$



Since we have $F_R(x) = F'_P(x)/F'_P(1)$,



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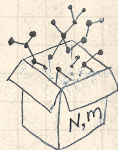
Recall giant component condition is

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Since we have $F_R(x) = F'_P(x)/F'_P(1)$,

$$F'_R(x) = \frac{F''_P(x)}{F'_P(1)}.$$



Edge-degree distribution



Recall giant component condition is

$$\langle k \rangle_R = F'_R(1) > 1.$$



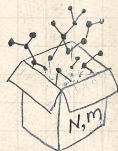
Since we have $F_R(x) = F'_P(x)/F'_P(1)$,

$$F'_R(x) = \frac{F''_P(x)}{F'_P(1)}.$$



Setting $x = 1$, our condition becomes

$$\frac{F''_P(1)}{F'_P(1)} > 1$$



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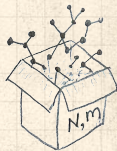
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Size distributions

To figure out the **size of the largest component** (S_1), we need more resolution on component sizes.

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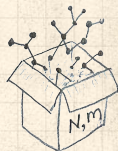
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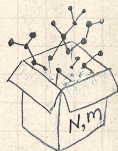


Size distributions

To figure out the **size of the largest component** (S_1), we need more resolution on component sizes.

Definitions:


 π_n = probability that a random node belongs to a finite component of size $n < \infty$.




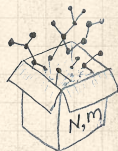
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Size distributions

To figure out the **size of the largest component** (S_1), we need more resolution on component sizes.

Definitions:

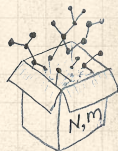
 π_n = probability that a random node belongs to a finite component of size $n < \infty$.

 ρ_n = probability that a random end of a random link leads to a finite subcomponent of size $n < \infty$.

Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

neighbors \Leftrightarrow components

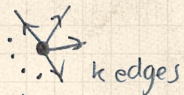


Connecting probabilities:

n nodes

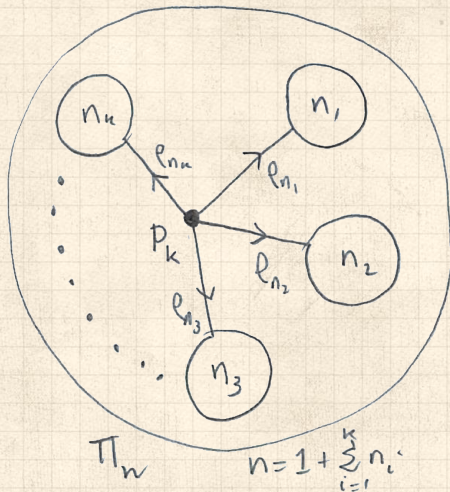


π_n



P_k

k edges



π_n

$$n = 1 + \sum_{i=1}^k n_i$$

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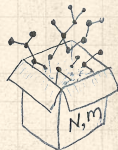
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Markov property of random networks connects

π_n , ρ_n , and P_k .

Connecting probabilities:

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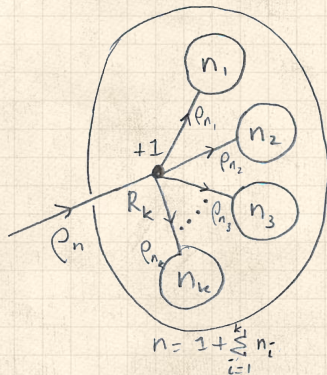
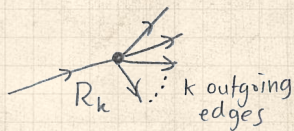
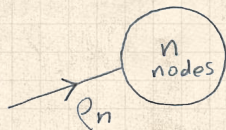
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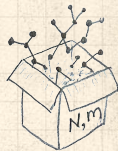
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 Markov property of random networks connects ρ_n and R_k .



G.f.'s for component size distributions:

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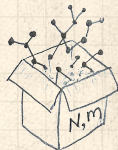
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G.f.'s for component size distributions:



$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

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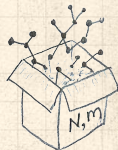
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The largest component:



Subtle key: $F_{\pi}(1)$ is the probability that a node belongs to a **finite** component.

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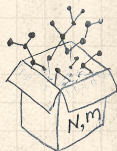
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G.f.'s for component size distributions:

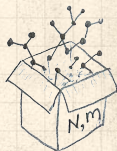


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 **Subtle key:** $F_{\pi}(1)$ is the probability that a node belongs to a **finite** component.

 Therefore: $S_1 = 1 - F_{\pi}(1)$.





G.f.'s for component size distributions:




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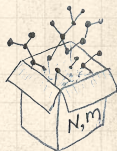
 **Subtle key:** $F_{\pi}(1)$ is the probability that a node belongs to a **finite** component.

 Therefore: $S_1 = 1 - F_{\pi}(1)$.

Our mission, which we accept:

 Determine and connect the four generating functions

$$F_P, F_R, F_{\pi}, \text{ and } F_{\rho}.$$



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Useful results we'll need for g.f.'s

Sneaky Result 1:

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Sneaky Result 1:

 Consider two random variables U and V whose values may be 0, 1, 2, ...

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Useful results we'll need for g.f.'s

Sneaky Result 1:

- Consider two random variables U and V whose values may be $0, 1, 2, \dots$
- Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V .



Useful results we'll need for g.f.'s

Sneaky Result 1:

- Consider two random variables U and V whose values may be 0, 1, 2, ...
- Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V .
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^U V^{(i)} \text{ with each } V^{(i)} \stackrel{d}{=} V$$

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Sneaky Result 1:

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- SR1: If a third random variable is defined as

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then

$$F_W(x) = F_U(F_V(x))$$

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Proof of SR1:

Write probability that variable W has value k as W_k .

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Proof of SR1:

Write probability that variable W has value k as W_k .

$$W_k = \sum_{j=0}^{\infty} U_j \times \Pr(\text{sum of } j \text{ draws of variable } V = k)$$

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$$= \sum_{j=0}^{\infty} U_j \sum_{\{i_1, i_2, \dots, i_j\}} V_{i_1} V_{i_2} \cdots V_{i_j}$$

$i_1 + i_2 + \dots + i_j = k$

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$i_1 + i_2 + \dots + i_j = k$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k$$

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$$W_k = \sum_{j=0}^{\infty} U_j \times \Pr(\text{sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} V_{i_2} \dots V_{i_j}$$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} V_{i_2} \dots V_{i_j} x^k$$

$$= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty}$$

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Proof of SR1:

Write probability that variable W has value k as W_k .

$$W_k = \sum_{j=0}^{\infty} U_j \times \Pr(\text{sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} V_{i_2} \dots V_{i_j}$$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} V_{i_2} \dots V_{i_j} x^k$$

$$= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j}$$

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Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j}$$

x^k piece of $\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^j$

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Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \underbrace{\sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j}$$
$$\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j = (F_V(x))^j$$

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Proof of SR1:

With some concentration, observe:

$$\begin{aligned} F_W(x) &= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \underbrace{\sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j} \\ &= \sum_{j=0}^{\infty} U_j \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j = (F_V(x))^j \\ &= \sum_{j=0}^{\infty} U_j (F_V(x))^j \end{aligned}$$

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Proof of SR1:

With some concentration, observe:

$$\begin{aligned} F_W(x) &= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j} \\ &= \underbrace{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j}_{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j = (F_V(x))^j} \\ &= \sum_{j=0}^{\infty} U_j (F_V(x))^j \\ &= F_U(F_V(x)) \end{aligned}$$

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Proof of SR1:

With some concentration, observe:

$$\begin{aligned} F_W(x) &= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \underbrace{\sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j} \\ &= \sum_{j=0}^{\infty} U_j \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j = (F_V(x))^j \\ &= \sum_{j=0}^{\infty} U_j (F_V(x))^j \\ &= F_U(F_V(x)) \end{aligned}$$

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Proof of SR1:

With some concentration, observe:

$$\begin{aligned} F_W(x) &= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j} \\ &= \underbrace{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j}_{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j = (F_V(x))^j} \\ &= \sum_{j=0}^{\infty} U_j (F_V(x))^j \\ &= F_U(F_V(x)) \end{aligned}$$



Alternate, groovier proof in the accompanying assignment.

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Sneaky Result 2:

 Start with a random variable U with distribution U_k ($k = 0, 1, 2, \dots$)

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Useful results we'll need for g.f.'s

Sneaky Result 2:

 Start with a random variable U with distribution U_k ($k = 0, 1, 2, \dots$)


 SR2: If a second random variable is defined as


$$V = U + 1$$



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Sneaky Result 2:

 Start with a random variable U with distribution U_k ($k = 0, 1, 2, \dots$)

 SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } F_V(x) = xF_U(x)$$



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Sneaky Result 2:

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Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.



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Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k$$



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Sneaky Result 2:

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Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.


$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$




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Sneaky Result 2:

 Start with a random variable U with distribution U_k ($k = 0, 1, 2, \dots$)

 SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

 Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.





$$\begin{aligned} \therefore F_V(x) &= \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k \\ &= x \sum_{j=0}^{\infty} U_j x^j \end{aligned}$$




Useful results we'll need for g.f.'s

Sneaky Result 2:

 Start with a random variable U with distribution U_k ($k = 0, 1, 2, \dots$)

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 Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.



$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

$$= x \sum_{j=0}^{\infty} U_j x^j = xF_U(x).$$



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Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

$$= x \sum_{j=0}^{\infty} U_j x^j = xF_U(x).$$



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Generalization of SR2:

 (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

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Generalization of SR2:

 (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

 (2) If $V = U - i$ then

$$F_V(x) = x^{-i} F_U(x)$$

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Generalization of SR2:

 (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

 (2) If $V = U - i$ then

$$F_V(x) = x^{-i} F_U(x)$$

$$= x^{-i} \sum_{k=0}^{\infty} U_k x^k$$

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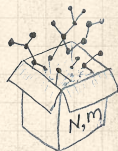
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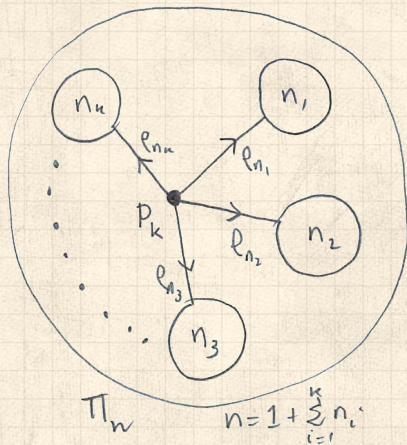
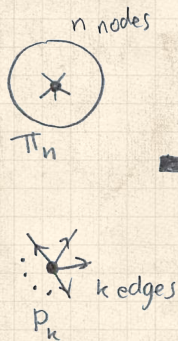
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Connecting generating functions:

 **Goal:** figure out forms of the component generating functions, F_π and F_ρ .



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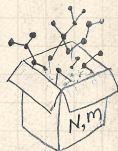
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 Relate π_n to P_k and ρ_n through one step of recursion.

Connecting generating functions:

 π_n = probability that a random node belongs to a finite component of size n

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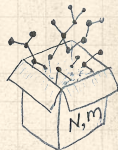
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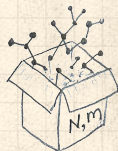
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 π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$



Connecting generating functions:



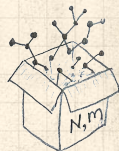
π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$



Therefore:

$$F_{\pi}(x) =$$



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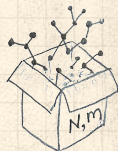
π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$



Therefore:

$$F_{\pi}(x) = \underbrace{F_P(F_{\rho}(x))}_{\text{SR1}}$$



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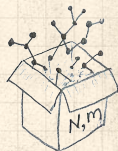
π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$



Therefore:

$$F_{\pi}(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_P(F_{\rho}(x))}_{\text{SR1}}$$




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 π_n = probability that a random node belongs to a finite component of size n

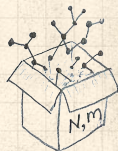
$$= \sum_{k=0}^{\infty} P_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$



Therefore:

$$F_{\pi}(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_P(F_{\rho}(x))}_{\text{SR1}}$$

 Extra factor of x accounts for random node itself.



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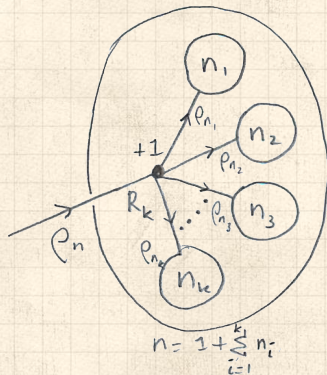
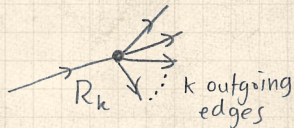
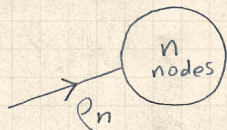
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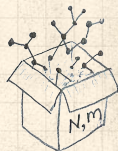
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Relate ρ_n to R_k and ρ_n through one step of recursion.



Connecting generating functions:

 ρ_n = probability that a random link leads to a finite subcomponent of size n .

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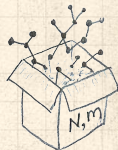
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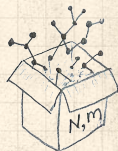


Connecting generating functions:


 ρ_n = probability that a random link leads to a finite subcomponent of size n .


 Invoke one step of recursion:

ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n - 1$,



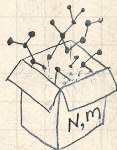
Connecting generating functions:

 ρ_n = probability that a random link leads to a finite subcomponent of size n .


 Invoke one step of recursion:


ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n - 1$,

$$= \sum_{k=0}^{\infty} R_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$



Connecting generating functions:

 ρ_n = probability that a random link leads to a finite subcomponent of size n .

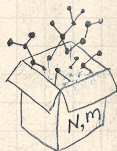
 Invoke one step of recursion:

ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n - 1$,


$$= \sum_{k=0}^{\infty} R_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$




Therefore: $F_{\rho}(x) =$



Connecting generating functions:

 ρ_n = probability that a random link leads to a finite subcomponent of size n .

 Invoke one step of recursion:

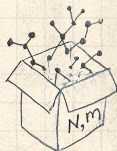
ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n - 1$,

$$= \sum_{k=0}^{\infty} R_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$





Therefore:

$$F_{\rho}(x) = \underbrace{F_R(F_{\rho}(x))}_{\text{SR1}}$$



Connecting generating functions:

 ρ_n = probability that a random link leads to a finite subcomponent of size n .

 Invoke one step of recursion:

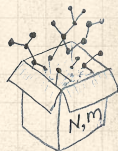
ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n - 1$,

$$= \sum_{k=0}^{\infty} R_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$





Therefore:

$$F_{\rho}(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_R(F_{\rho}(x))}_{\text{SR1}}$$



Connecting generating functions:

 ρ_n = probability that a random link leads to a finite subcomponent of size n .

 Invoke one step of recursion:


ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n - 1$,

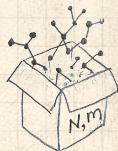
$$= \sum_{k=0}^{\infty} R_k \times \Pr \left(\text{sum of sizes of subcomponents at end of } k \text{ random links} = n - 1 \right)$$




Therefore:

$$F_{\rho}(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_R(F_{\rho}(x))}_{\text{SR1}}$$

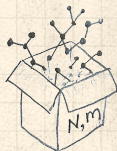
 Again, extra factor of x accounts for random node itself.




Connecting generating functions:

 We now have two functional equations connecting our generating functions:


$$F_{\pi}(x) = xF_P(F_{\rho}(x)) \quad \text{and} \quad F_{\rho}(x) = xF_R(F_{\rho}(x))$$

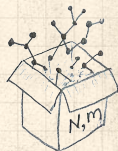


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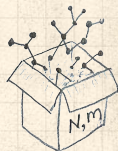
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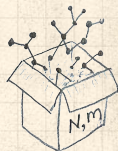
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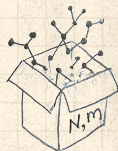
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-  We can do this because it **only involves** F_{ρ} and $F_{\mathcal{R}}$.

-  The first equation then immediately gives us F_{π} in terms of F_{ρ} and $F_{\mathcal{R}}$.



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Component**

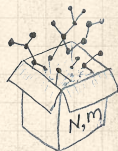
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Remembering vaguely what we are doing:

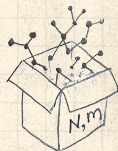


Component sizes



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Finding F_π to obtain the **fractional size of the largest component** $S_1 = 1 - F_\pi(1)$.



Component sizes

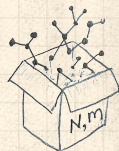


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Component sizes



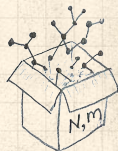
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Component sizes

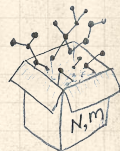
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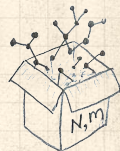
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 Solve second equation numerically for $F_\rho(1)$.

 Plug $F_\rho(1)$ into first equation to obtain $F_\pi(1)$.



Component sizes

Example: Standard random graphs.

 We can show $F_P(x) = e^{-kx(1-x)}$

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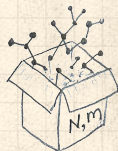
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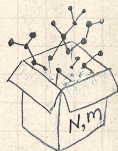


Component sizes

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 We can show $F_P(x) = e^{-kx}$

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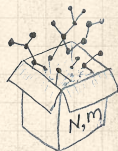
Component sizes

Example: Standard random graphs.

 We can show $F_P(x) = e^{-\langle k \rangle(1-x)}$


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Component sizes

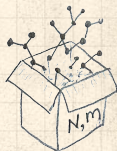
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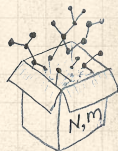
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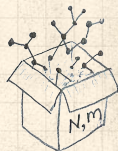
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Component sizes


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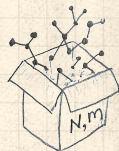
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 So $F_\pi(x) = F_\rho(x) = xF_R(F_\rho(x)) = xF_R(F_\pi(x))$



Component sizes


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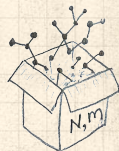
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 Consistent with how our dirty (but wrong) trick worked earlier ...



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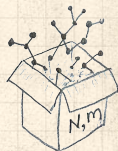
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 Consistent with how our dirty (but wrong) trick worked earlier ...

 $\pi_n = \rho_n$ just as $P_k = R_k$.



Component sizes



We are down to

$$F_{\pi}(x) = xF_R(F_{\pi}(x)) \text{ and } F_R(x) = e^{-\langle k \rangle(1-x)}.$$

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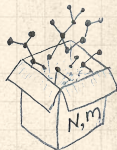
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$$\therefore F_{\pi}(x) = xe^{-\langle k \rangle(1-F_{\pi}(x))}$$

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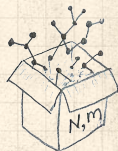
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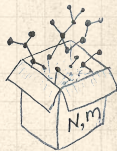
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Component sizes



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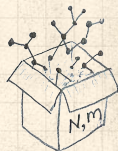
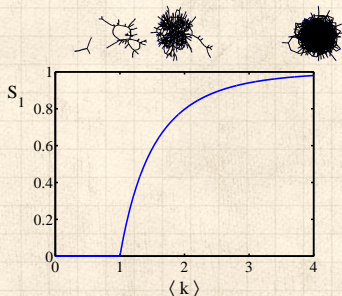
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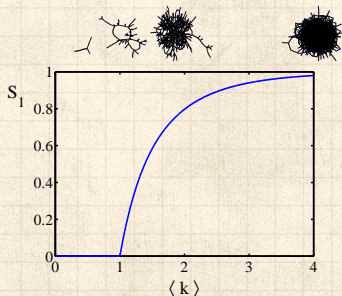
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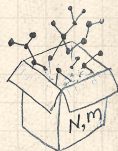
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Just as we found with our dirty trick ...



Component sizes



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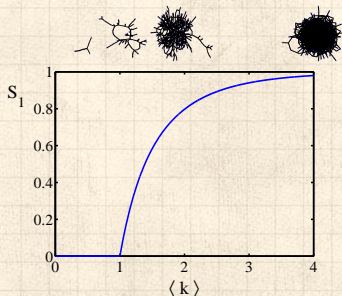
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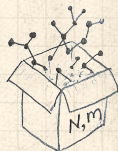
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Just as we found with our dirty trick ...



Again, we (usually) have to resort to numerics ...



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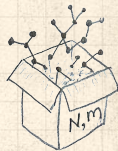
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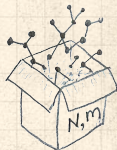
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A few simple random networks to contemplate
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 **Notation:** The Kronecker delta function $\delta_{i,j} = 1$
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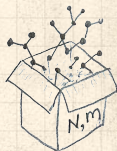
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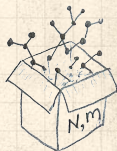


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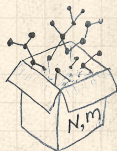
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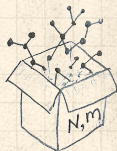
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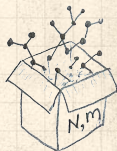
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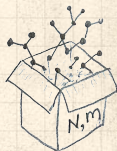
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 $P_k = a\delta_{k1} + (1 - a)\delta_{kk'}$ for some fixed $k' \geq 2$ with $0 \leq a \leq 1$.



A joyful example \square :

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

 We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}$.

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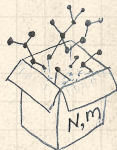
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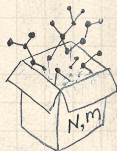
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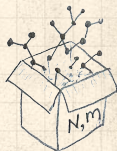
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 Check for goodness:

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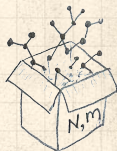
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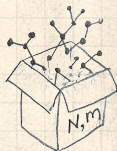
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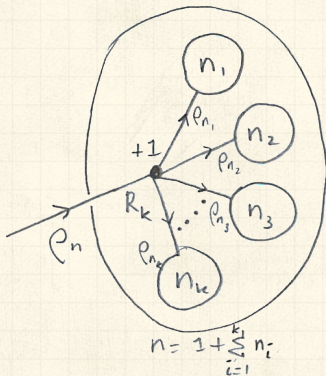
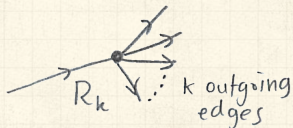
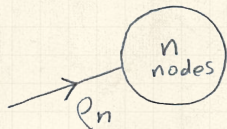
 Things to figure out: Component size generating functions for π_n and ρ_n , and the size of the giant component.



Find $F_\rho(x)$ first:

 We know:

$$F_\rho(x) = xF_R(F_\rho(x)).$$



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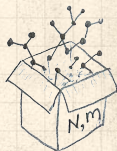
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Sticking things in things, we have:

$$F_{\rho}(x) = x \left(\frac{1}{4} + \frac{3}{4} [F_{\rho}(x)]^2 \right).$$

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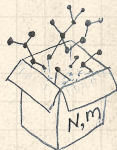
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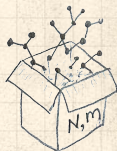
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Rearranging:

$$3x [F_\rho(x)]^2 - 4F_\rho(x) + x = 0.$$




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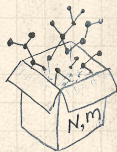
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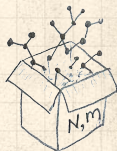
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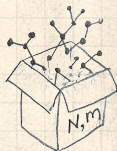
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 Time for a Taylor series expansion.

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 First: which sign do we take?





Because ρ_n is a probability distribution, we know $F_\rho(1) \leq 1$ and $F_\rho(x) \leq 1$ for $0 \leq x \leq 1$.

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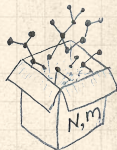



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we see that the positive sign solution blows to smithereens, and the negative one is okay.




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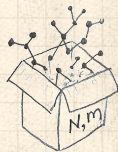
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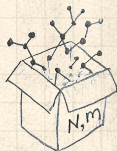
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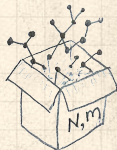
We can now deploy the Taylor expansion:


$$(1+z)^\theta = \binom{\theta}{0}z^0 + \binom{\theta}{1}z^1 + \binom{\theta}{2}z^2 + \binom{\theta}{3}z^3 + \dots$$




 Let's define a binomial for arbitrary θ and $k = 0, 1, 2, \dots$:

$$\binom{\theta}{k} = \frac{\Gamma(\theta + 1)}{\Gamma(k + 1)\Gamma(\theta - k + 1)}$$

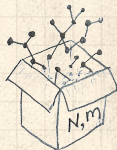


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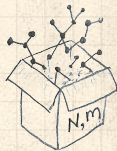


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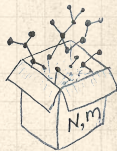
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where we've used $\Gamma(x + 1) = x\Gamma(x)$ and noted that $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.



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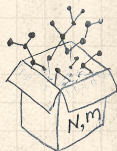
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Note: $(1 + z)^\theta \sim 1 + \theta z$ always.



 Totally psyched, we go back to here:

$$F_{\rho}(x) = \frac{2}{3x} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right).$$


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$$F_{\rho}(x) =$$

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 Giving:

$$F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n =$$

$$\frac{1}{4}x + \frac{3}{64}x^3 + \frac{9}{512}x^5 + \dots + \frac{2}{3} \left(\frac{3}{4} \right)^k \frac{(-1)^{k+1} \Gamma\left(\frac{3}{2}\right)}{\Gamma(k+1) \Gamma\left(\frac{3}{2} - k\right)} x^{2k-1} + \dots$$


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 Do odd powers make sense?

 We can now find $F_\pi(x)$ with:

$$F_\pi(x) = xF_P(F_\rho(x))$$

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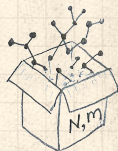
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 We can now find $F_\pi(x)$ with:

$$\begin{aligned} F_\pi(x) &= xF_P(F_\rho(x)) \\ &= x\frac{1}{2}\left((F_\rho(x))^1 + (F_\rho(x))^3\right) \end{aligned}$$

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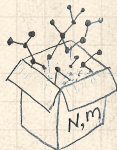
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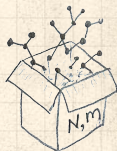
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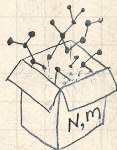
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 Delicious.

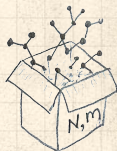



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$$\begin{aligned} F_\pi(x) &= xF_P(F_\rho(x)) \\ &= x\frac{1}{2}\left(\left(F_\rho(x)\right)^1 + \left(F_\rho(x)\right)^3\right) \\ &= x\frac{1}{2}\left[\frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right) + \frac{2^3}{(3x)^3}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right)^3\right]. \end{aligned}$$


 Delicious.


 In principle, we can now extract all the π_n .




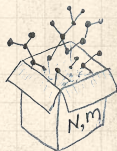
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 Delicious.

 In principle, we can now extract all the π_n .

 But let's just find the size of the giant component.





First, we need $F_\rho(1)$:

$$F_\rho(x)|_{x=1} = \frac{2}{3 \cdot 1} \left(1 - \sqrt{1 - \frac{3}{4}1^2} \right) = \frac{1}{3}.$$

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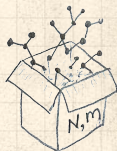
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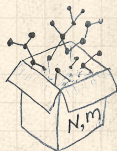



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This is the probability that a random edge leads to a sub-component of finite size.



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$$F_\pi(1) = 1 \cdot F_P(F_\rho(1))$$



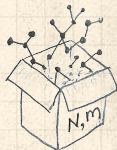
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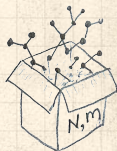
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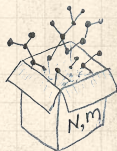
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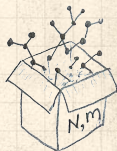
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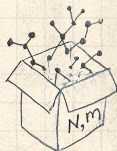
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 This is the probability that a random chosen node belongs to a finite component.

 Finally, we have

$$S_1 = 1 - F_\pi(1) = 1 - \frac{5}{27} = \frac{22}{27}.$$



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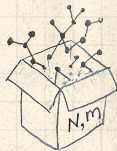
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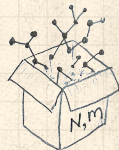
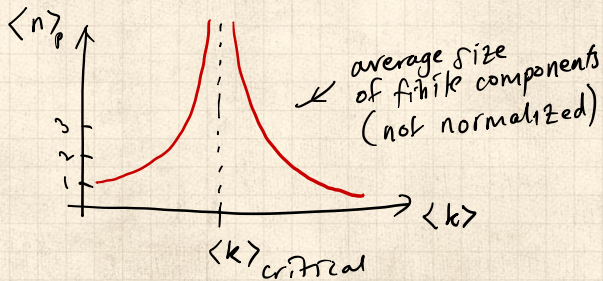
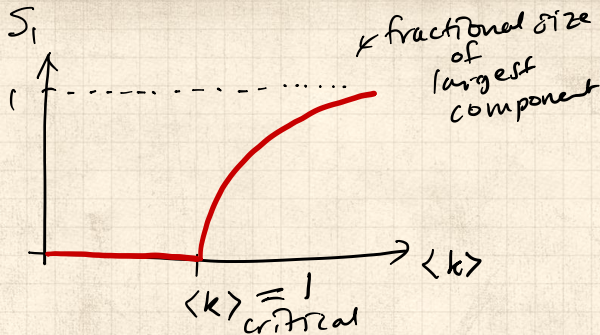
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Average component size

 Next: find **average size** of **finite** components $\langle n \rangle$.

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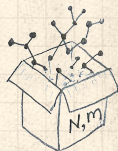
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Average component size

 Next: find **average size** of **finite** components $\langle n \rangle$.

 Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.

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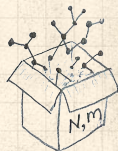
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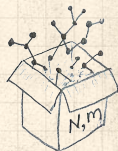
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Average component size


- Next: find **average size** of **finite** components $\langle n \rangle$.
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Average component size

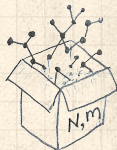
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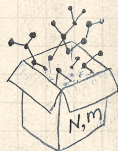
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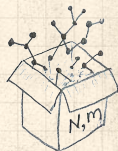
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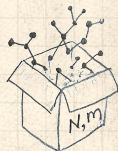
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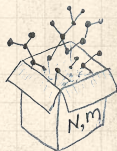
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Now set $x = 1$ in both equations.

We solve the second equation for $F'_\rho(1)$ (we must already have $F_\rho(1)$).

Plug $F'_\rho(1)$ and $F_\rho(1)$ into first equation to find $F'_\pi(1)$.



Average component size

Example: Standard random graphs.

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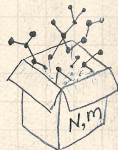
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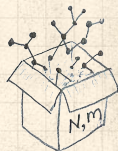
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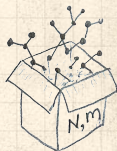
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 Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.


 Two differentiated equations reduce to only one:


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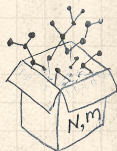
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Average component size


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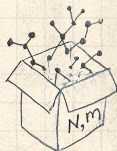
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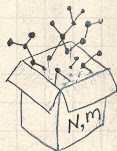
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 Set $x = 1$ and replace $F_\pi(1)$ with $1 - S_1$.



Average component size

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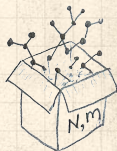


Replace $F_P(F_\pi(x))$ using $F_\pi(x) = xF_P(F_\pi(x))$.



Set $x = 1$ and replace $F_\pi(1)$ with $1 - S_1$.

End result:
$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$



Average component size



Our result for standard random networks:

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$

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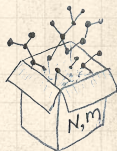
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Average component size


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 Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.




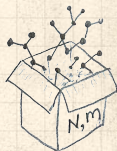
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 Look at what happens when we increase $\langle k \rangle$ to 1 from below.





Average component size

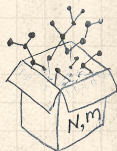
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Average component size

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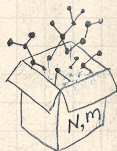
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$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$


 This blows up as $\langle k \rangle \rightarrow 1$.





Average component size

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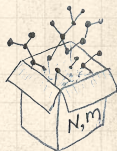
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 **Reason:** we have a power law distribution of component sizes at $\langle k \rangle = 1$.





Average component size

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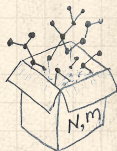
 We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

 This blows up as $\langle k \rangle \rightarrow 1$.

 **Reason:** we have a power law distribution of component sizes at $\langle k \rangle = 1$.

 Typical critical point behavior ...



Average component size




Limits of $\langle k \rangle = 0$ and ∞ make sense for


$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$



Average component size

 Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$

 As $\langle k \rangle \rightarrow 0$, $S_1 = 0$, and $\langle n \rangle \rightarrow 1$.

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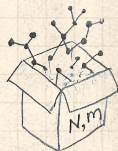
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 Limits of $\langle k \rangle = 0$ and ∞ make sense for

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 As $\langle k \rangle \rightarrow 0$, $S_1 = 0$, and $\langle n \rangle \rightarrow 1$.

 All nodes are isolated.

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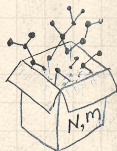
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 All nodes are isolated.

 As $\langle k \rangle \rightarrow \infty$, $S_1 \rightarrow 1$ and $\langle n \rangle \rightarrow 0$.

 No nodes are outside of the giant component.

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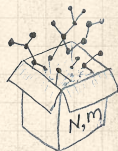
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No nodes are outside of the giant component.

Extra on largest component size:

For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}/N$.

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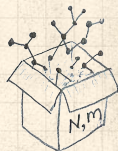
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Extra on largest component size:

For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}/N$.

For $\langle k \rangle < 1$, $S_1 \sim (\log N)/N$.

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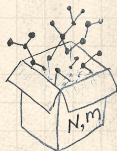
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Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.

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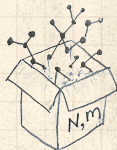
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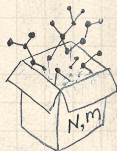
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 Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.

 We're after:

$$\langle n \rangle = F'_\pi(1) = F'_P(F_\rho(1)) + F'_\rho(1)F'_P(F_\rho(1))$$



 Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.


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$$\langle n \rangle = F'_\pi(1) = F_P(F_\rho(1)) + F'_\rho(1)F'_P(F_\rho(1))$$

where we first need to compute

$$F'_\rho(1) = F_R(F_\rho(1)) + F'_\rho(1)F'_R(F_\rho(1)).$$




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$$F'_\rho(1) = F_R(F_\rho(1)) + F'_\rho(1)F'_R(F_\rho(1)).$$

 Place stick between teeth, and recall that we have:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2.$$



Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.

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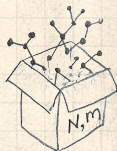
$$F'_\rho(1) = F_R(F_\rho(1)) + F'_\rho(1)F'_R(F_\rho(1)).$$

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Differentiation gives us:

$$F'_P(x) = \frac{1}{2} + \frac{3}{2}x^2 \text{ and } F'_R(x) = \frac{3}{2}x.$$





We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

$$F'_\rho(1) = F_R(F_\rho(1)) + F'_\rho(1)F'_R(F_\rho(1))$$

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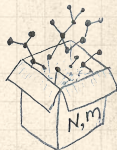
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We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

$$\begin{aligned} F'_\rho(1) &= F_R(F_\rho(1)) + F'_\rho(1)F'_R(F_\rho(1)) \\ &= F_R\left(\frac{1}{3}\right) + F'_\rho(1)F'_R\left(\frac{1}{3}\right) \end{aligned}$$

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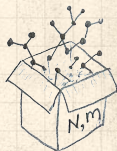
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After some reallocation of objects, we have $F'_\rho(1) = \frac{13}{2}$.





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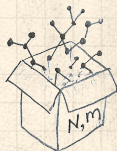
$$\begin{aligned} F'_\rho(1) &= F_R(F_\rho(1)) + F'_\rho(1)F'_R(F_\rho(1)) \\ &= F_R\left(\frac{1}{3}\right) + F'_\rho(1)F'_R\left(\frac{1}{3}\right) \\ &= \frac{1}{4} + \frac{3}{4} \frac{1}{3^2} + F'_\rho(1) \frac{3}{2} \frac{1}{3}. \end{aligned}$$



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$$\text{Finally: } \langle n \rangle = F'_\pi(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2} F'_P\left(\frac{1}{3}\right)$$





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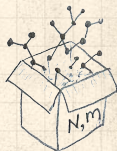
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$$\begin{aligned} F'_\rho(1) &= F_R(F_\rho(1)) + F'_\rho(1)F'_R(F_\rho(1)) \\ &= F_R\left(\frac{1}{3}\right) + F'_\rho(1)F'_R\left(\frac{1}{3}\right) \\ &= \frac{1}{4} + \frac{\cancel{3}}{4} \frac{1}{\cancel{3^2}} + F'_\rho(1) \frac{\cancel{3}}{2} \frac{1}{\cancel{3}}. \end{aligned}$$



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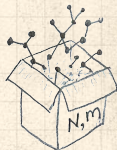
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So, kinda small.



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Generating functions allow us to strangely calculate features of random networks.



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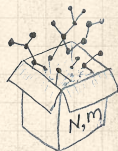
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Generating functions allow us to strangely calculate features of random networks.



They're a bit scary and magical.



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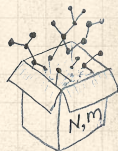
Generating functions allow us to strangely calculate features of random networks.



They're a bit scary and magical.



We'll find generating functions useful for contagion.



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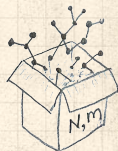
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- Generating functions allow us to strangely calculate features of random networks.
- They're a bit scary and magical.
- We'll find generating functions useful for contagion.
- But we'll also see that more direct, physics-bearing calculations are possible.



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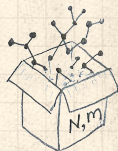
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<https://www.youtube.com/watch?v=bGBoZbT7cR8?rel=0> 

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References

[1] H. S. Wilf.

Generatingfunctionology.

A K Peters, Natick, MA, 3rd edition, 2006. pdf 