

Measures of centrality

Last updated: 2021/10/07, 17:43:00 EDT

Principles of Complex Systems, Vols. 1 & 2
CSYS/MATH 300 and 303, 2021–2022 | @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont



Licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License.

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



1 of 31

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



2 of 31

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



4 of 31

Centrality

- One possible reflection of importance is **centrality**.
- Presumption is that nodes or edges that are (in some sense) in the middle of a network are important for the network's function.
- Idea of centrality comes from social networks literature [7].
- Many flavors of centrality ...
 - Many are topological and quasi-dynamical;
 - Some are based on dynamics (e.g., traffic).
- We will define and examine a few ...
- (Later: see centrality useful in identifying communities in networks.)

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



5 of 31

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



7 of 31

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



9 of 31

Centrality

Degree centrality

- Naively estimate importance by **node degree**. [7]
- Doh**: assumes linearity (If node i has twice as many friends as node j , it's twice as important.)
- Doh**: doesn't take in any non-local information.

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



7 of 31

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



9 of 31

Closeness centrality

- Idea**: Nodes are more central if they can reach other nodes 'easily.'
- Measure average shortest path from a node to all other nodes.
- Define **Closeness Centrality** for node i as

$$\frac{N - 1}{\sum_{j, j \neq i} (\text{shortest distance from } i \text{ to } j)}$$
- Range is 0 (no friends) to 1 (single hub).
- Unclear what the exact values of this measure tells us because of its ad-hocness.
- General problem with simple centrality measures: what do they exactly mean?
- Perhaps, at least, we obtain an ordering of nodes in terms of 'importance.'

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



7 of 31

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



9 of 31

Betweenness centrality

- Betweenness centrality** is based on coherence of shortest paths in a network.
- Idea**: If the quickest way between any two nodes on a network disproportionately involves certain nodes, then they are 'important' in terms of global cohesion.
- For each node i , **count how many shortest paths pass through i** .
- In the case of ties, divide counts between paths.
- Call frequency of shortest paths passing through node i the betweenness of i , B_i .
- Note: Exclude shortest paths between i and other nodes.
- Note: works for weighted and unweighted networks.

- Consider a network with N nodes and m edges (possibly weighted).
- Computational goal**: Find $\binom{N}{2}$ shortest paths between all pairs of nodes.
- Traditionally use Floyd-Warshall algorithm.
- Computation time grows as $O(N^3)$.
- See also:
 - Dijkstra's algorithm for finding shortest path between two specific nodes,
 - and Johnson's algorithm which outperforms Floyd-Warshall for sparse networks: $O(mN + N^2 \log N)$.
- Newman (2001) [4, 5] and Brandes (2001) [1] independently derive equally fast algorithms that also compute betweenness.
- Computation times grow as:
 - $O(mN)$ for unweighted graphs;
 - and $O(mN + N^2 \log N)$ for weighted graphs.

Shortest path between node i and all others:

- Consider unweighted networks.
- Use **breadth-first search**:
 - Start at node i , giving it a distance $d = 0$ from itself.
 - Create a list of all of i 's neighbors and label them being at a distance $d = 1$.
 - Go through list of most recently visited nodes and find all of their neighbors.
 - Exclude any nodes already assigned a distance.
 - Increment distance d by 1.
 - Label newly reached nodes as being at distance d .
 - Repeat steps 3 through 6 until all nodes are visited.
- Record which nodes link to which nodes moving out from i (former are 'predecessors' with respect to i 's shortest path structure).
- Runs in $O(m)$ time and gives $N - 1$ shortest paths.
- Find all shortest paths in $O(mN)$ time

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



11 of 31

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



12 of 31

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References



13 of 31

Outline

Background

Centrality measures

- Degree centrality
- Closeness centrality
- Betweenness
- Eigenvalue centrality
- Hubs and Authorities

Nutshell

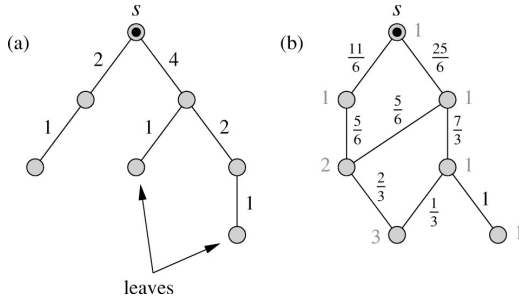
References

How big is my node?

- Basic question**: how 'important' are specific nodes and edges in a network?
- An **important node** or **edge** might:
 - handle** a relatively large amount of the network's traffic (e.g., cars, information);
 - bridge** two or more distinct groups (e.g., liason, interpreter);
 - be a **source** of important ideas, knowledge, or judgments (e.g., supreme court decisions, an employee who 'knows where everything is').
- So how do we quantify such a slippery concept as importance?
- We generate ad hoc, reasonable measures, and examine their utility ...

Newman's Betweenness algorithm: [4]

PoCS
@pocsvox
Measures of
centrality

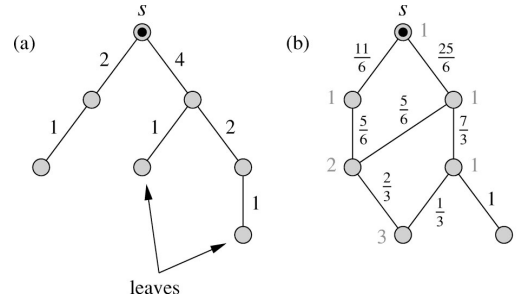


Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References

14 of 31

Newman's Betweenness algorithm: [4]

PoCS
@pocsvox
Measures of
centrality



Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References

17 of 31

Perron-Frobenius theorem: \square If an $N \times N$ matrix A has non-negative entries then:

PoCS
@pocsvox
Measures of
centrality

1. A has a real eigenvalue $\lambda_1 \geq |\lambda_i|$ for $i = 2, \dots, N$.
2. λ_1 corresponds to left and right 1-d eigenspaces for which we can choose a basis vector that has non-negative entries.
3. The dominant real eigenvalue λ_1 is bounded by the minimum and maximum row sums of A :

$$\min_i \sum_{j=1}^N a_{ij} \leq \lambda_1 \leq \max_i \sum_{j=1}^N a_{ij}$$

4. All other eigenvectors have one or more negative entries.
5. The matrix A can make toast.
6. Note: Proof is relatively short for symmetric matrices that are strictly positive [6] and just non-negative [3].

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References

21 of 31

Newman's Betweenness algorithm: [4]

PoCS
@pocsvox
Measures of
centrality

1. Set all nodes to have a value $c_{ij} = 0$, $j = 1, \dots, N - 1$ (c for count).
2. Select one node i and find **shortest paths** to all other $N - 1$ nodes using breadth-first search.
3. Record # equal shortest paths reaching each node.
4. Move through nodes according to their distance from i , starting with the furthest.
5. Travel **back towards i** from each starting node j , along shortest path(s), adding 1 to every value of $c_{i\ell}$ at each node ℓ along the way.
6. Whenever more than one possibility exists, apportion according to total number of short paths coming through predecessors.
7. Exclude starting node j and i from increment.
8. Repeat steps 2-8 for every node i and obtain betweenness as $B_j = \sum_{i=1}^N c_{ij}$.

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References

15 of 31

Important nodes have important friends:

- Define x_i as the 'importance' of node i .
- Idea: x_i depends (somehow) on x_j if j is a neighbor of i .
- Recursive:** importance is transmitted through a network.
- Simplest possibility is a linear combination:

$$x_i \propto \sum_j a_{ji} x_j$$

- Assume further that constant of proportionality, c , is independent of i .
- Above gives $\vec{x} = c \mathbf{A}^T \vec{x}$ or $\mathbf{A}^T \vec{x} = c^{-1} \vec{x} = \lambda \vec{x}$.
- Eigenvalue equation based on adjacency matrix ...
- Note: Lots of despair over size of the largest eigenvalue. [7] Lose sight of original assumption's non-physicality.

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References

19 of 31

Other Perron-Frobenius aspects:

- Assuming our network is **irreducible** \square , meaning there is only one component, is reasonable: just consider one component at a time if more than one exists.
- Irreducibility means largest eigenvalue's eigenvector has strictly non-negative entries.
- Analogous to notion of ergodicity: every state is reachable.
- (Another term: **Primitive** graphs and matrices.)

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References

22 of 31

Newman's Betweenness algorithm: [4]

PoCS
@pocsvox
Measures of
centrality

- For a **pure tree network**, c_{ij} is the number of nodes beyond j from i 's vantage point.
- Same algorithm for computing drainage area in river networks (with 1 added across the board).
- For **edge betweenness**, use exact same algorithm but now
 1. j indexes edges,
 2. and we add one to each edge as we traverse it.
- For both algorithms, computation time grows as

$$O(mN).$$

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References

16 of 31

Important nodes have important friends:

- So: solve $\mathbf{A}^T \vec{x} = \lambda \vec{x}$.
- But which eigenvalue and eigenvector?
- We, the people, would like:
 1. A unique solution. \checkmark
 2. λ to be real. \checkmark
 3. Entries of \vec{x} to be real. \checkmark
 4. Entries of \vec{x} to be non-negative. \checkmark
 5. λ to actually mean something ... (maybe too much)
 6. Values of x_i to mean something (what does an observation that $x_3 = 5x_7$ mean?) (maybe only ordering is informative ...) (maybe too much)
 7. λ to equal 1 would be nice ... (maybe too much)
 8. Ordering of \vec{x} entries to be robust to reasonable modifications of linear assumption (maybe too much)

- We rummage around in bag of tricks and pull out the Perron-Frobenius theorem ...

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References

20 of 31

Hubs and Authorities

- Generalize eigenvalue centrality to allow nodes to have two attributes:
 1. **Authority:** how much knowledge, information, etc., held by a node on a topic.
 2. **Hubness (or Hubosity or Hubbishness or Hubtasticness):** how well a node 'knows' where to find information on a given topic.
- Original work due to the legendary Jon Kleinberg. [2]
- Best hubs point to best authorities.
- Recursive:** Hubs authoritatively link to hubs, authorities hubbishly link to other authorities.
- More:** look for dense links between sets of 'good' hubs pointing to sets of 'good' authorities.
- Known as the **HITS algorithm** \square (Hyperlink-Induced Topics Search).

PoCS
@pocsvox
Measures of
centrality

Background
Centrality
measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities
Nutshell
References

24 of 31

Hubs and Authorities

Give each node two scores:

1. x_i = **authority score** for node i
2. y_i = **hubtasticness score** for node i

As for eigenvector centrality, we connect the scores of neighboring nodes.

New story I: a good authority is linked to by good hubs.

Means x_i should increase as $\sum_{j=1}^N a_{ji} y_j$ increases.

Note: indices are ji meaning j has a directed link to i .

New story II: good hubs point to good authorities.

Means y_i should increase as $\sum_{j=1}^N a_{ij} x_j$ increases.

Linearity assumption:

$$\vec{x} \propto A^T \vec{y} \text{ and } \vec{y} \propto A \vec{x}$$

Nutshell:

Measuring centrality is well motivated if hard to carry out well.

We've only looked at a few major ones.

Methods are often taken to be more sophisticated than they really are.

Centrality can be used pragmatically to perform diagnostics on networks (see structure detection).

Focus on nodes rather than groups or modules is a homo narrativus constraint.

Possible that better approaches will be developed.

References II

[4] M. E. J. Newman. Scientific collaboration networks. II. Shortest paths, weighted networks, and centrality. [Phys. Rev. E, 64\(1\):016132, 2001. pdf](#)

[5] M. E. J. Newman and M. Girvan. Finding and evaluating community structure in networks. [Phys. Rev. E, 69\(2\):026113, 2004. pdf](#)

[6] F. Ninio. A simple proof of the Perron-Frobenius theorem for positive symmetric matrices. [J. Phys. A.: Math. Gen., 9:1281-1282, 1976. pdf](#)

Hubs and Authorities

So let's say we have

$$\vec{x} = c_1 A^T \vec{y} \text{ and } \vec{y} = c_2 A \vec{x}$$

where c_1 and c_2 must be positive.

Above equations combine to give

$$\vec{x} = c_1 A^T c_2 A \vec{x} = \lambda A^T A \vec{x}.$$

where $\lambda = c_1 c_2 > 0$.

It's all good: we have the heart of singular value decomposition before us ...

References I

[1] U. Brandes. A faster algorithm for betweenness centrality. [J. Math. Sociol., 25:163-177, 2001. pdf](#)

[2] J. M. Kleinberg. Authoritative sources in a hyperlinked environment. [Proc. 9th ACM-SIAM Symposium on Discrete Algorithms, 1998. pdf](#)

[3] K. Y. Lin. An elementary proof of the perron-frobenius theorem for non-negative symmetric matrices. [Chinese Journal of Physics, 15:283-285, 1977. pdf](#)

References III

[7] S. Wasserman and K. Faust. [Social Network Analysis: Methods and Applications.](#) Cambridge University Press, Cambridge, UK, 1994.

We can do this:

$A^T A$ is symmetric.

$A^T A$ is semi-positive definite so its eigenvalues are all ≥ 0 .

$A^T A$'s eigenvalues are the square of A 's singular values.

$A^T A$'s eigenvectors form a joyful orthogonal basis.

Perron-Frobenius tells us that only the dominant eigenvalue's eigenvector can be chosen to have non-negative entries.

So: linear assumption leads to a solvable system.

What would be very good: find networks where we have independent measures of node 'importance' and see how importance is actually distributed.