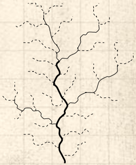

Branching Networks II

Last updated: 2021/10/02, 00:15:03 EDT

Principles of Complex Systems, Vols. 1 & 2
CSYS/MATH 300 and 303, 2021–2022 | @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

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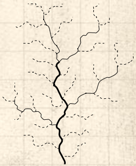

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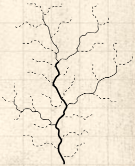
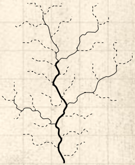


Outline

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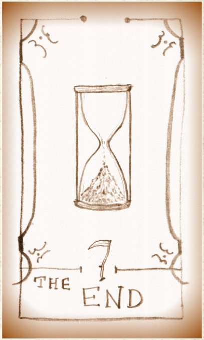
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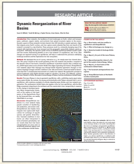
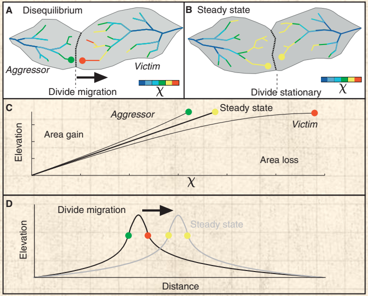
Piracy on the high χ's:



"Dynamic Reorganization of River Basins"
Willett et al.,
Science, **343**, 1248765, 2014. [21]



$$\frac{\partial z(x,t)}{\partial t} = U - K A^m \left| \frac{\partial z(x,t)}{\partial x} \right|^n$$

$$z(x) = z_{\mathrm{b}} + \left(\frac{U}{K A_0^m} \right)^{1/n} \chi$$

$$\chi = \int_{x_{\mathrm{b}}}^{x} \left(\frac{A_0}{A(x')} \right)^{m/n} \mathrm{d}x'$$

Piracy on the high χ's:

http://www.youtube.com/watch?v=FnroL1_-l2c?rel=0

More: How river networks move across a landscape (Science Daily)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Can Horton and Tokunaga be happy?

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- In terms of network achitecture, Horton's laws appear to contain less detailed information than Tokunaga's law.

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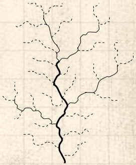

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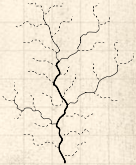

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  Insert question from assignment 1

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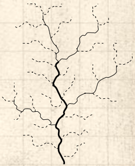

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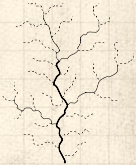
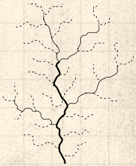
- Known result: Tokunaga $\rightarrow$ Horton [18, 19, 20, 9, 2]



Let us make them happy

We need one more ingredient:



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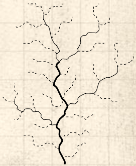
We need one more ingredient:

Space-fillingness



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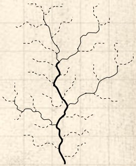
- A network is space-filling if the average distance between adjacent streams is roughly constant.



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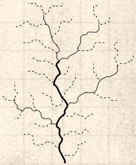
- Reasonable for river and cardiovascular networks



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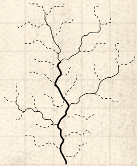
- For river networks:
  Drainage density ρ_{dd} = inverse of typical distance between channels in a landscape.



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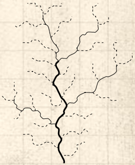
$$\rho_{\mathrm{dd}} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}}$$



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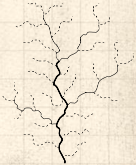
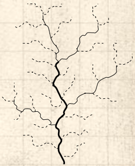
$$\rho_{\mathrm{dd}} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \frac{\sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega}{a_\Omega}$$



More with the happy-making thing

Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

More with the happy-making thing



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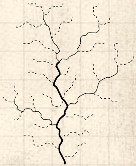

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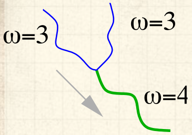
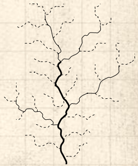


1. Running into another stream of order ω and generating a stream of order $\omega + 1$

   ...



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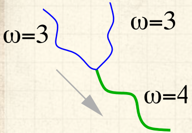

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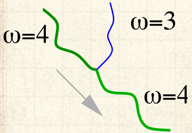
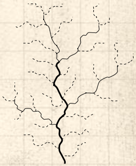


2. Running into and being absorbed by a stream of higher order $\omega' > \omega$ ...



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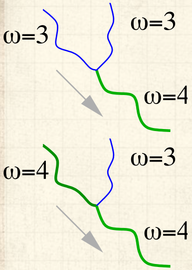

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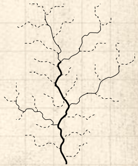

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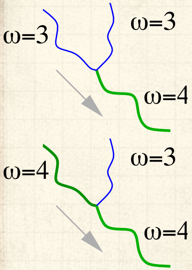

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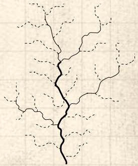
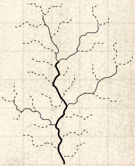


More with the happy-making thing

Putting things together:

$$n_\omega = \underbrace{2 n_{\omega+1}}_{\text{generation}} +$$



More with the happy-making thing

Putting things together:

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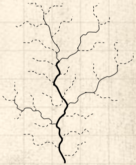

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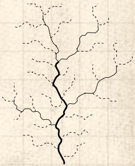

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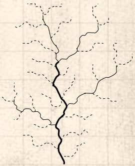
- Insert question from assignment 1
- Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

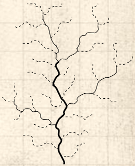
(The larger value is the one we want.)



Finding other Horton ratios

Connect Tokunaga to R_s

- Now use uniform drainage density ρ_{dd}.



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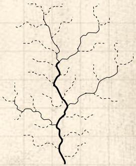

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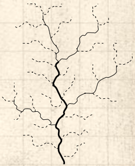

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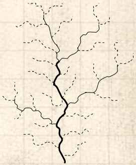

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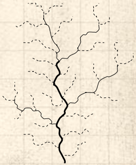

Horton and Tokunaga are happy

Altogether then:

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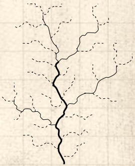

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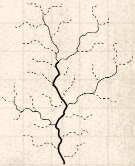

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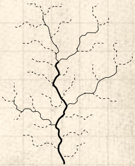

Some observations:

- R_n and R_ℓ depend on T_1 and R_T.

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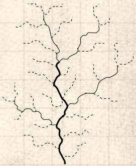

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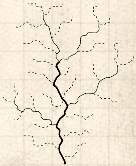

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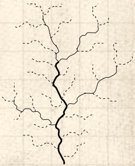

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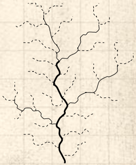
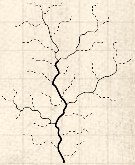
- Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between non-trivial statistical distributions. [3, 4]



Horton and Tokunaga are happy

The other way round

- Note: We can invert the expresssions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.



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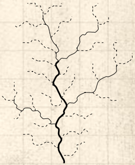

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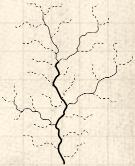
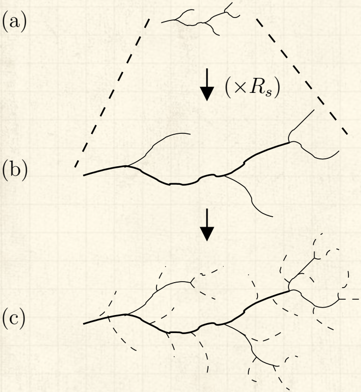
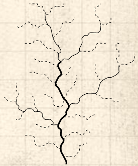
- $$R_T = R_\ell,$$
- $$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n.$$
- Suggests we should be able to argue that Horton's laws imply Tokunaga's laws (if drainage density is uniform) ...



Horton and Tokunaga are friends

From Horton to Tokunaga [2]





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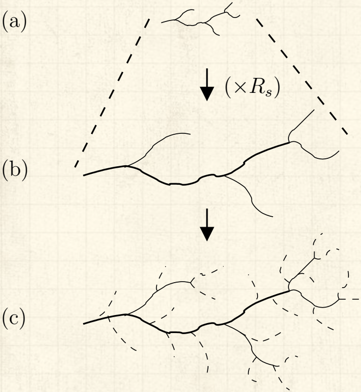
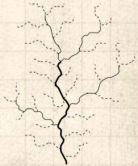


- Assume Horton's laws hold for number and length



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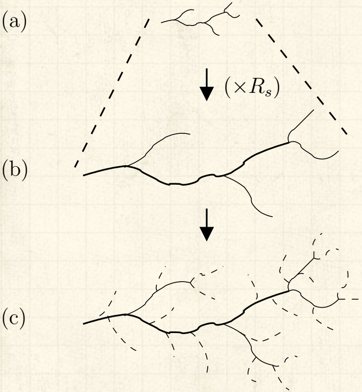
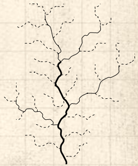


- Assume Horton's laws hold for number and length
- Start with picture showing an order ω stream and order $\omega - 1$ generating and side streams.



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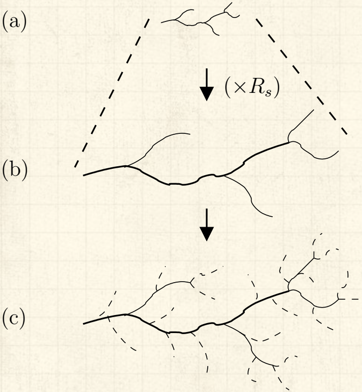

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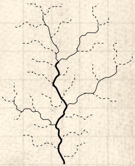

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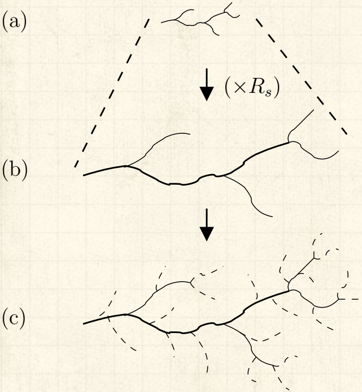

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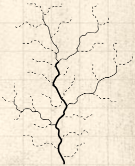
- Maintain drainage density by adding new order $\omega - 1$ streams



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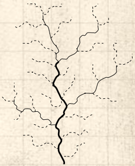

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Horton and Tokunaga are friends

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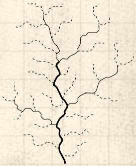

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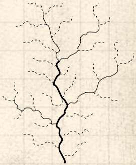

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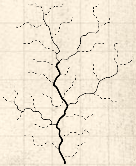

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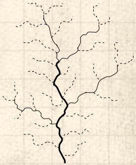
- For large ω, Tokunaga's law is the solution—let's check ...



Horton and Tokunaga are friends

Just checking:

- Substitute Tokunaga's law $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$ into

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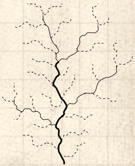

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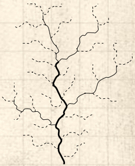

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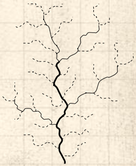

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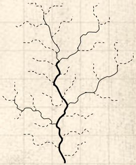
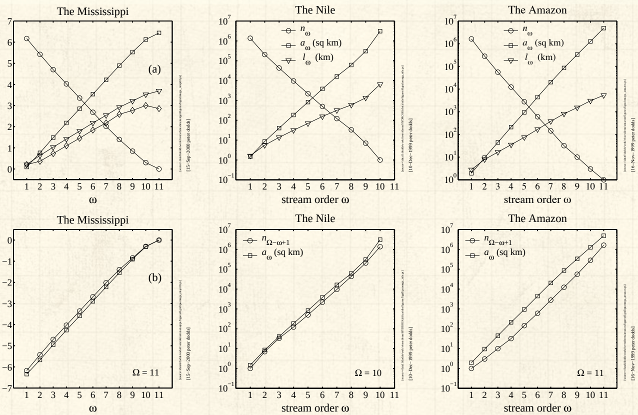
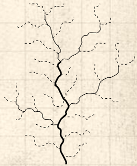
$$\simeq (R_\ell - 1) T_1 \frac{R_\ell^{\,k-1}}{R_\ell - 1} = T_1 R_\ell^{k-1}$$ ...yep.



Horton's laws of area and number:

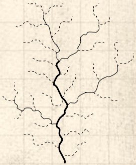




- In bottom plots, stream number graph has been flipped vertically.
- Highly suggestive that $R_n \equiv R_a$ ...

Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?

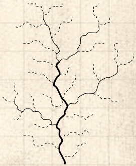


Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?
- Rule of thumb: discard data for two smallest and two largest orders.

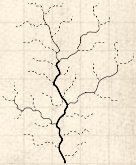
Mississippi:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
$[2, 3]$	5.27	5.26	2.48	2.30	1.00
$[2, 5]$	4.86	4.96	2.42	2.31	1.02
$[2, 7]$	4.77	4.88	2.40	2.31	1.02
$[3, 4]$	4.72	4.91	2.41	2.34	1.04
$[3, 6]$	4.70	4.83	2.40	2.35	1.03
$[3, 8]$	4.60	4.79	2.38	2.34	1.04
$[4, 6]$	4.69	4.81	2.40	2.36	1.02
$[4, 8]$	4.57	4.77	2.38	2.34	1.05
$[5, 7]$	4.68	4.83	2.36	2.29	1.03
$[6, 7]$	4.63	4.76	2.30	2.16	1.03
$[7, 8]$	4.16	4.67	2.41	2.56	1.12
mean μ	4.69	4.85	2.40	2.33	1.04
std dev σ	0.21	0.13	0.04	0.07	0.03
σ/μ	0.045	0.027	0.015	0.031	0.024



Amazon:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
$[2, 3]$	4.78	4.71	2.47	2.08	0.99
$[2, 5]$	4.55	4.58	2.32	2.12	1.01
$[2, 7]$	4.42	4.53	2.24	2.10	1.02
$[3, 5]$	4.45	4.52	2.26	2.14	1.01
$[3, 7]$	4.35	4.49	2.20	2.10	1.03
$[4, 6]$	4.38	4.54	2.22	2.18	1.03
$[5, 6]$	4.38	4.62	2.22	2.21	1.06
$[6, 7]$	4.08	4.27	2.05	1.83	1.05
mean μ	4.42	4.53	2.25	2.10	1.02
std dev σ	0.17	0.10	0.10	0.09	0.02
σ/μ	0.038	0.023	0.045	0.042	0.019



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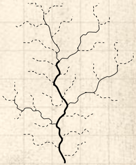

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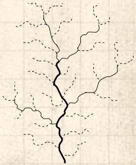

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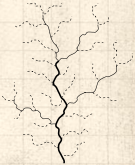

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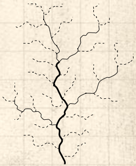

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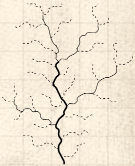

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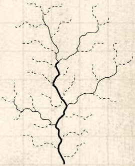

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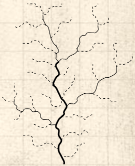

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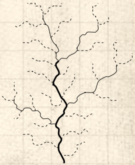

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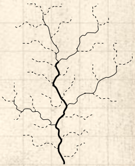

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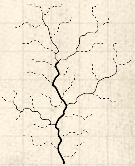

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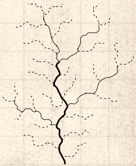

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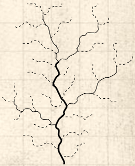

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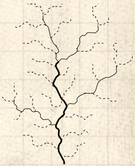

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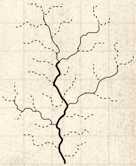

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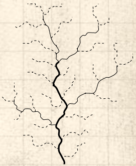

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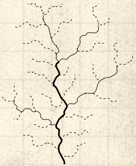

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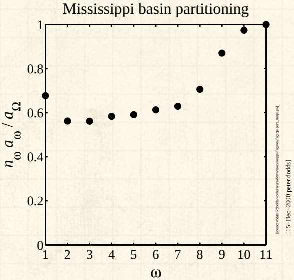
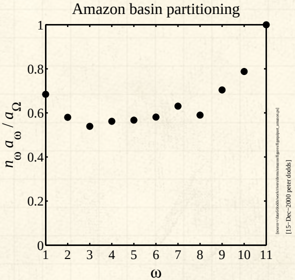
$$n_\omega \propto (R_n)^{-\omega}$$

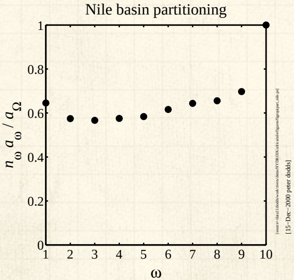
$$\bar{a}_\omega \propto (R_a)^{\omega} \propto n_\omega^{-1}$$

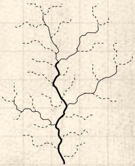
Equipartitioning:

Some examples:





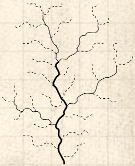




Neural Reboot: Fwoompf

http://www.youtube.com/watch?v=5mUs70SqD4o?rel=0

Scaling laws



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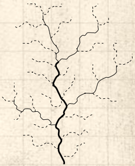
Scaling laws



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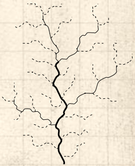

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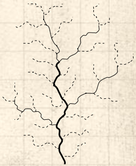

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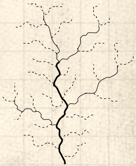

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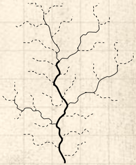

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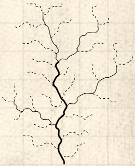
- Only two parameters are independent: $(T_1, R_T) \Leftrightarrow (R_n, R_s)$



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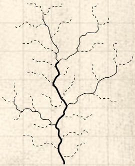

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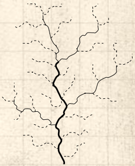

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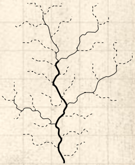

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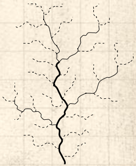

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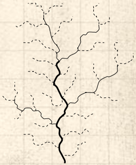

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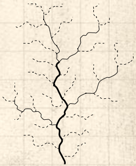

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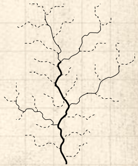

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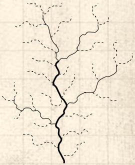

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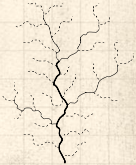

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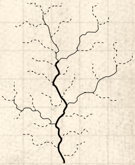

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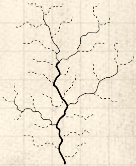

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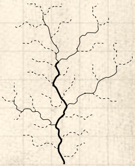
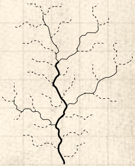


Scaling laws

Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)
  - Word frequency (Zipf's law) [22]
  - Wealth (maybe not—at least heavy tailed)
  - Statistical mechanics (phase transitions) [5]
- A big part of the story of complex systems
- Arise from mechanisms: growth, randomness, optimization, ...
- Our task is always to illuminate the mechanism ...



Scaling laws

Connecting exponents

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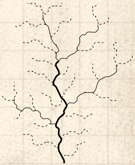

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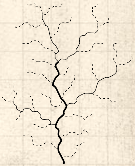

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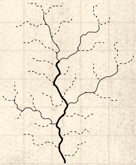

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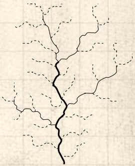

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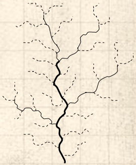

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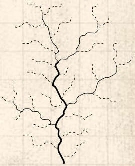

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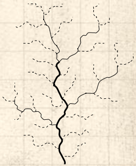

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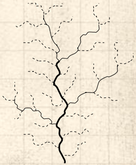

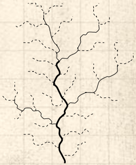

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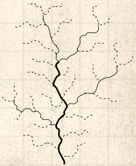

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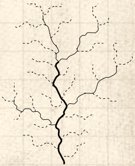

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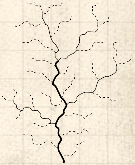

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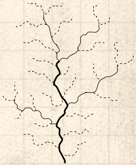

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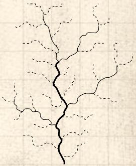

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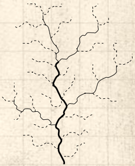

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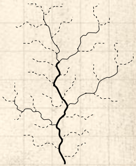

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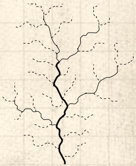

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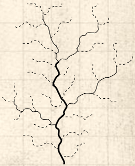

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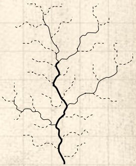

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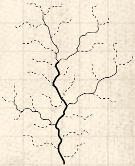

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- Use Horton's law of stream segments: $\bar{s}_\omega / \bar{s}_{\omega-1} = R_s$ ...

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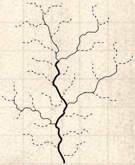

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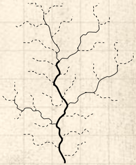

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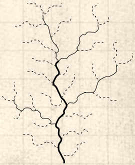

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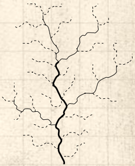

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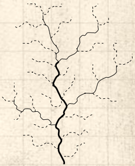

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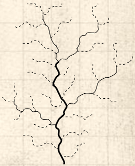

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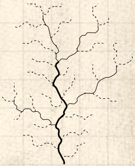

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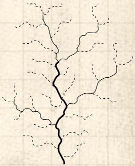

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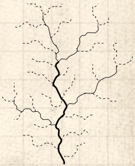

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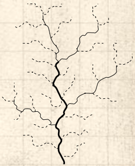

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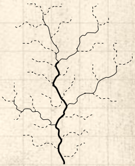

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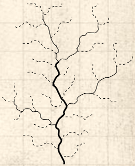

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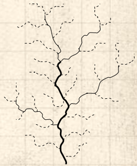

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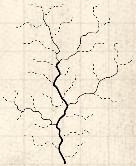
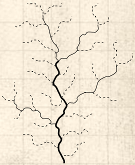


Scaling laws



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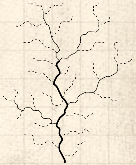

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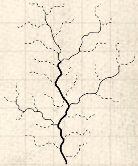

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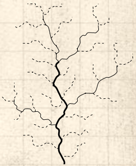

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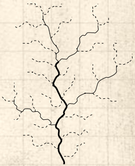

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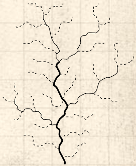

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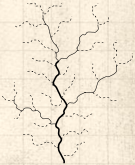

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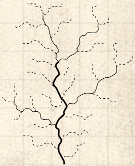

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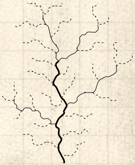

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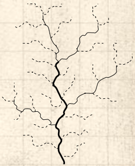

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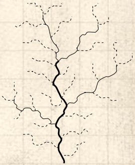

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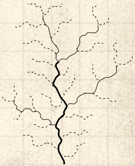

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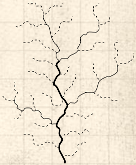

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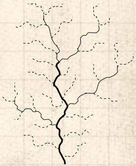

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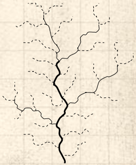

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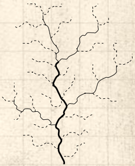

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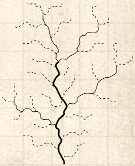
Insert question from assignment 2



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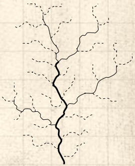

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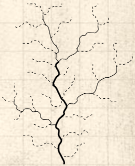
Insert question from assignment 2

- Such connections between exponents are called scaling relations
- Let's connect to one last relationship: Hack's law



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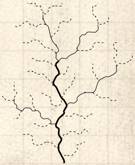

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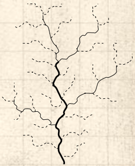

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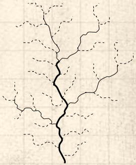

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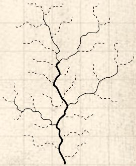

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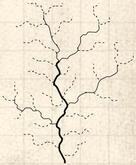

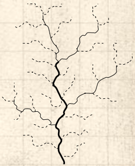

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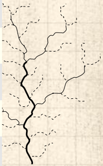
We mentioned there were a good number of ‘laws’: [2]

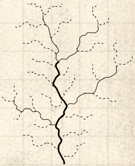
Relation:	Name or description:
$T_k = T_1(R_T)^{k-1}$	Tokunaga’s law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega / n_{\omega+1} = R_n$	Horton’s law of stream numbers
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	Horton’s law of main stream lengths
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	Horton’s law of basin areas
$\bar{s}_{\omega+1} / \bar{s}_\omega = R_s$	Horton’s law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack’s law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein’s law
$\lambda \sim L^\varphi$	variation of Langbein’s law



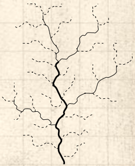
Connecting exponents

Only 3 parameters are independent:
e.g., take d, R_n, and R_s

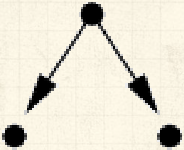
relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d
$T_k = T_1(R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_\omega/n_{\omega+1} = R_n$	R_n
$\bar{a}_{\omega+1}/\bar{a}_\omega = R_a$	$R_a = R_n$
$\bar{\ell}_{\omega+1}/\bar{\ell}_\omega = R_\ell$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \ln R_s / \ln R_n$
$a \sim L^D$	$D = d/h$
$L_\perp \sim L^H$	$H = d/h - 1$
$P(a) \sim a^{-\tau}$	$\tau = 2 - h$
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim a^\beta$	$\beta = 1 + h$
$\lambda \sim L^\varphi$	$\varphi = d$

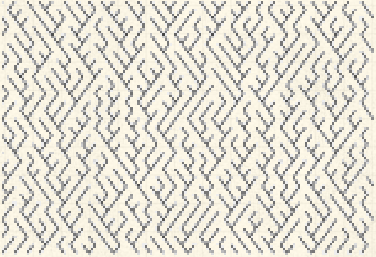


Scheidegger's model



Directed random networks [11, 12]



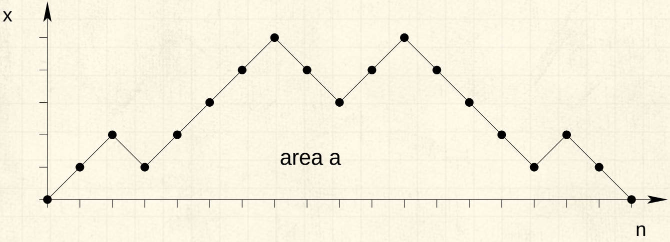


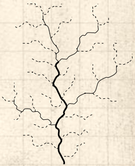
- $$P(\searrow) = P(\swarrow) = 1/2$$
- Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]
- Useful and interesting test case

A toy model—Scheidegger's model

Random walk basins:

- Boundaries of basins are random walks





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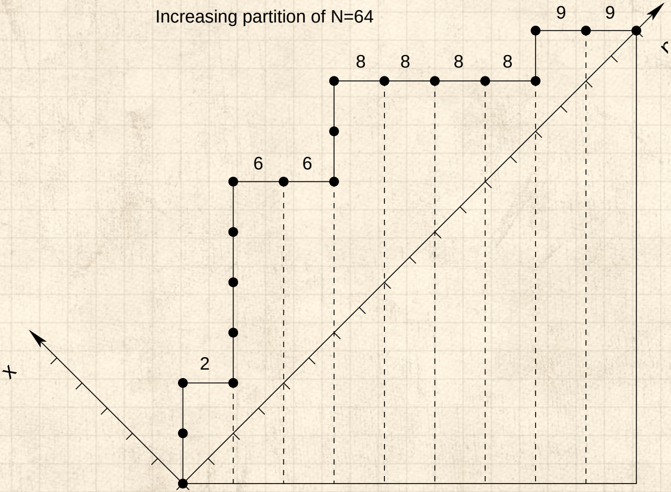

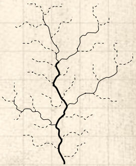

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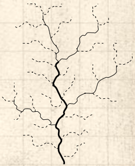

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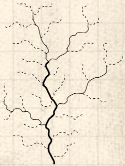

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- Find $\tau = 4/3$, $h = 2/3$, $\gamma = 3/2$, $d = 1$.

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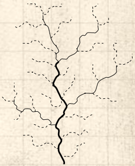
- Find $\tau = 4/3$, $h = 2/3$, $\gamma = 3/2$, $d = 1$.
- Note $\tau = 2 - h$ and $\gamma = 1/h$.



Scheidegger's model

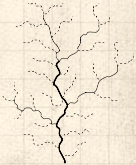
Prob for first return of a random walk in (1+1) dimensions (from CSYS/MATH 300):

- $$P(n) \sim \frac{1}{2\sqrt{\pi}} \, n^{-3/2}.$$

  and so $P(\ell) \propto \ell^{-3/2}$.
- Typical area for a walk of length n is $\propto n^{3/2}$:

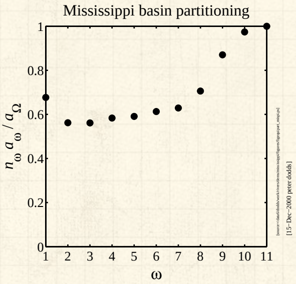
  $$\ell \propto a^{2/3}.$$

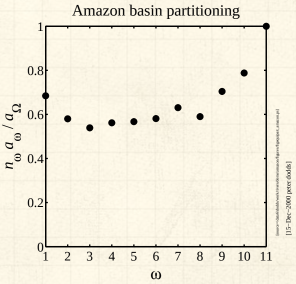
- Find $\tau = 4/3$, $h = 2/3$, $\gamma = 3/2$, $d = 1$.
- Note $\tau = 2 - h$ and $\gamma = 1/h$.
- R_n and R_ℓ have not been derived analytically.

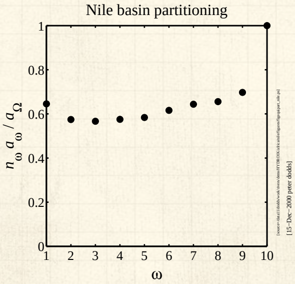


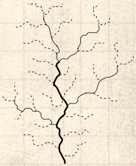
Equipartitioning reexamined:

Recall this story:





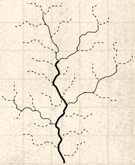




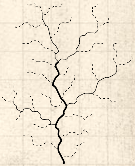
Equipartitioning

- What about

$$P(a) \sim a^{-\tau} \qquad ?$$



Equipartitioning



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$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$

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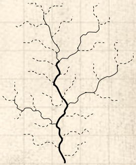

Equipartitioning

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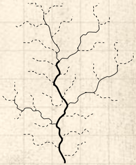
- while stream ordering separates basins ...



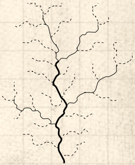
Hard neural reboot (sound matters):

https://twitter.com/round_boys/status/951873765964681216

Fluctuations

Moving beyond the mean:

Fluctuations

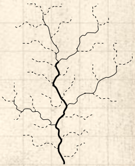


Moving beyond the mean:

- Both Horton's laws and Tokunaga's law relate average properties, e.g.,

$$\bar{s}_{\omega}/\bar{s}_{\omega-1} = R_s$$

Fluctuations



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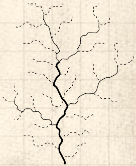

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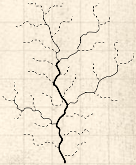
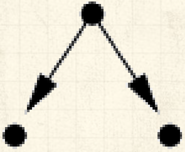
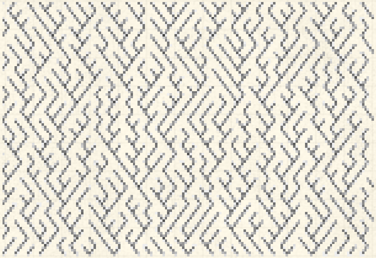
- Natural generalization to consider relationships between probability distributions
- Yields rich and full description of branching network structure
- See into the heart of randomness ...



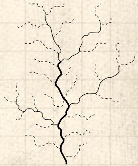
A toy model—Scheidegger's model

Directed random networks [11, 12]



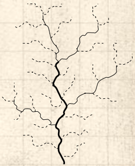


- $$P(\searrow) = P(\swarrow) = 1/2$$
- Flow is directed downwards



Generalizing Horton's laws

- $\bar{\ell}_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell|\omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell/R_\ell^\omega)$



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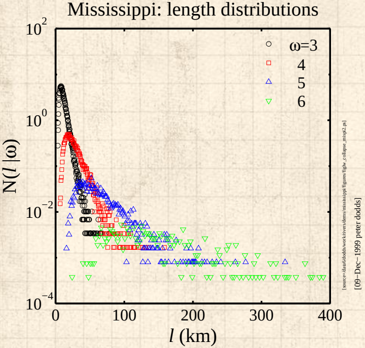

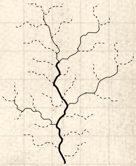

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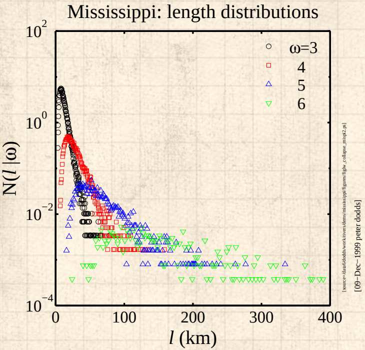

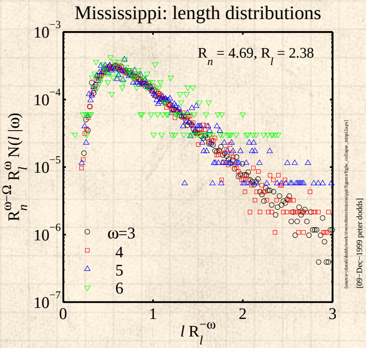

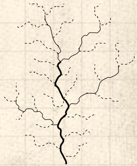

- Scaling collapse works well for intermediate orders

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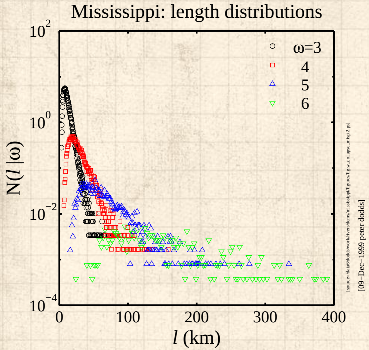
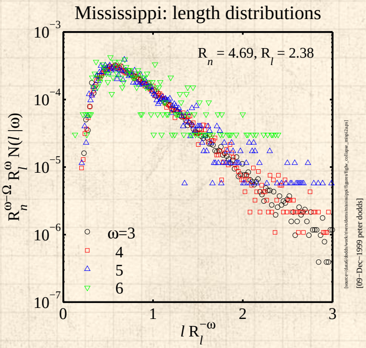
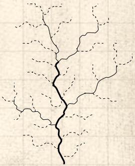


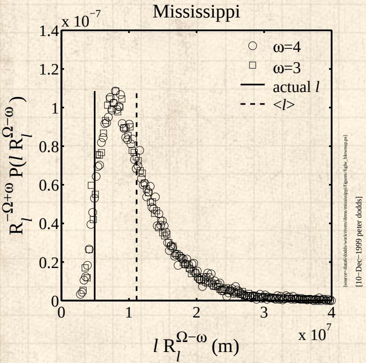


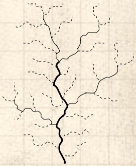


- Scaling collapse works well for intermediate orders
- All moments grow exponentially with order

Generalizing Horton's laws

- How well does overall basin fit internal pattern?





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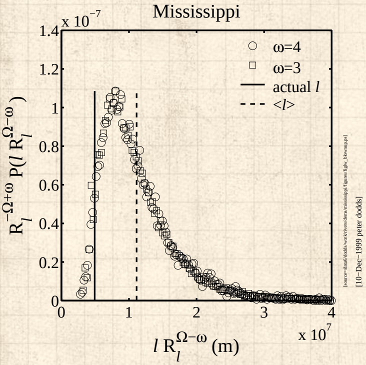
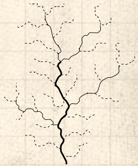


- Actual length = 4920 km (at 1 km res)



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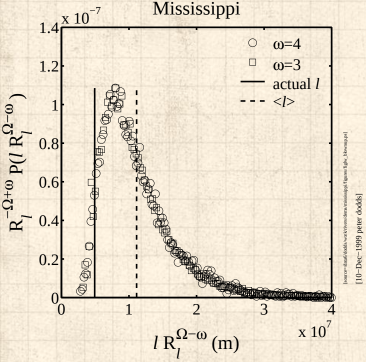
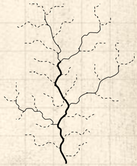


- Actual length = 4920 km (at 1 km res)
- Predicted Mean length = 11100 km



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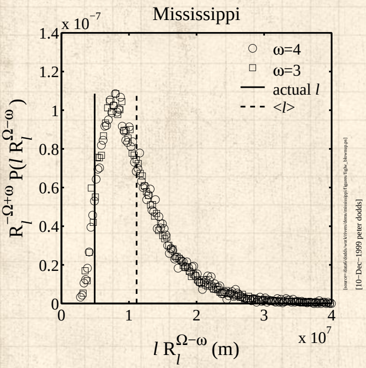
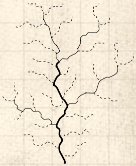


- Actual length = 4920 km (at 1 km res)
- Predicted Mean length = 11100 km
- Predicted Std dev = 5600 km



Generalizing Horton's laws

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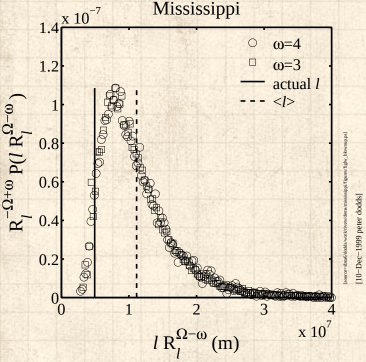

- Actual length = 4920 km (at 1 km res)
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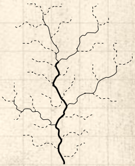
- Actual length/Mean length = 44 %



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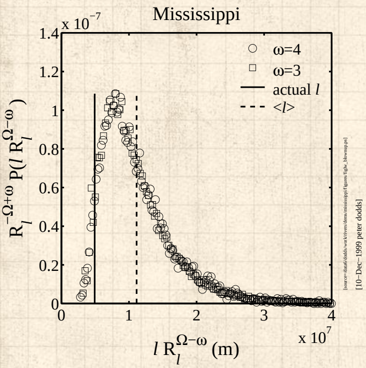

- Actual length = 4920 km (at 1 km res)
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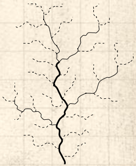
- Predicted Std dev = 5600 km
- Actual length/Mean length = 44 %
- Okay.

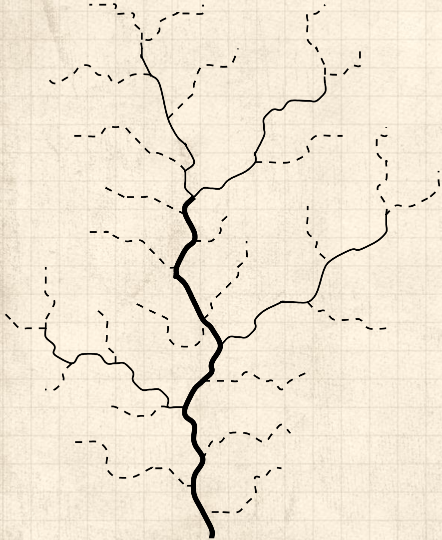


Generalizing Horton's laws

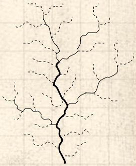
Comparison of predicted versus measured main stream lengths for large scale river networks (in 10^3 km):

basin:	ℓ_Ω	$\bar{\ell}_\Omega$	σ_ℓ	$\ell_\Omega/\bar{\ell}_\Omega$	$\sigma_\ell/\bar{\ell}_\Omega$
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73
	a_Ω	$\bar{a}_\Omega$	σ_a	$a_\Omega/\bar{a}_\Omega$	$\sigma_a/\bar{a}_\Omega$
Mississippi	2.74	7.55	5.58	0.36	0.74
Amazon	5.40	9.07	8.04	0.60	0.89
Nile	3.08	0.96	0.79	3.19	0.82
Congo	3.70	10.09	8.28	0.37	0.82
Kansas	0.14	0.49	0.42	0.28	0.86

Combining stream segments distributions:



- Stream segments sum to give main stream lengths
- $$\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$$



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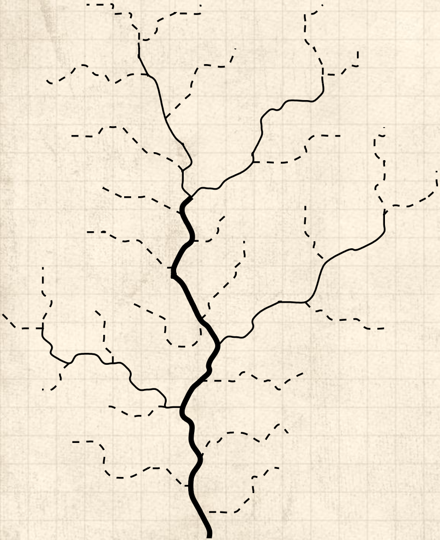
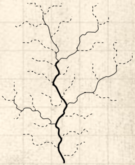


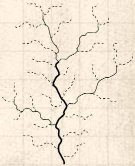
- Stream segments sum to give main stream lengths
- $$\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$$
- $P(\ell_\omega)$ is a convolution of distributions for the s_ω



Generalizing Horton's laws

- Sum of variables $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$ leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \cdots * N(s|\omega)$$



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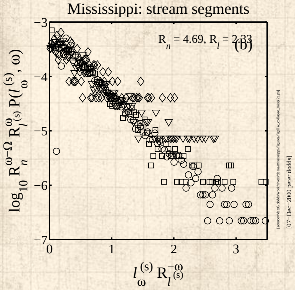
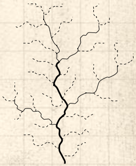


$$N(s|\omega) = \frac{1}{R_n^\omega R_\ell^\omega} F\left(s/R_\ell^\omega\right)$$

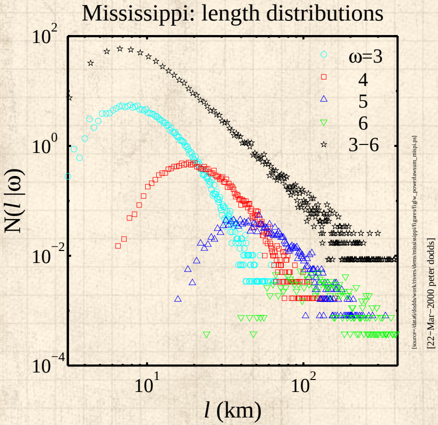
$$F(x) = e^{-x/\xi}$$

Mississippi: $\xi \simeq 900$ m.

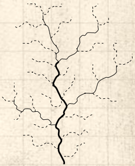


Generalizing Horton's laws

- Next level up: Main stream length distributions must combine to give overall distribution for stream length



- $P(\ell) \sim \ell^{-\gamma}$



Generalizing Horton's laws

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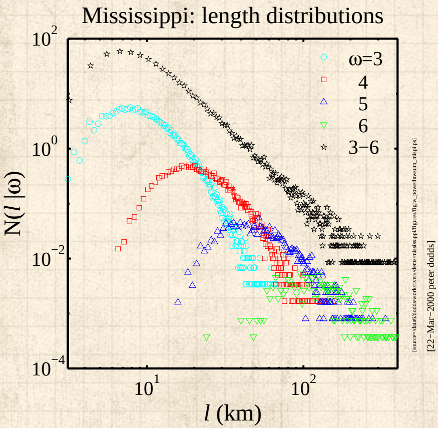
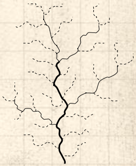
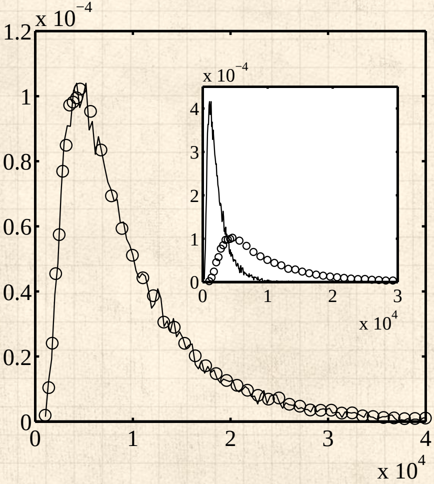


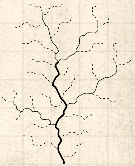
- $P(\ell) \sim \ell^{-\gamma}$
- Another round of convolutions [3]
- Interesting ...



Generalizing Horton's laws

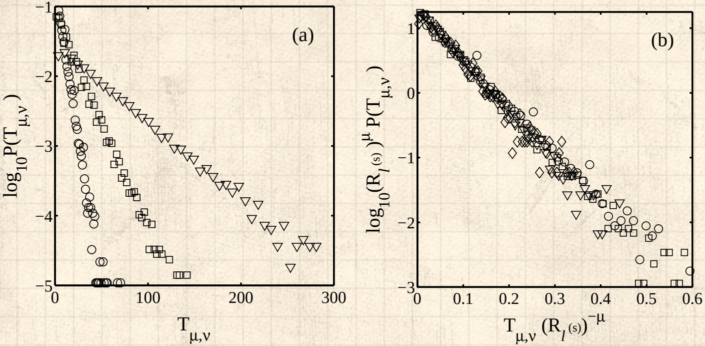
- Number and area distributions for the Scheidegger model [3]
- $P(n_{1,6})$ versus $P(a_6)$ for a randomly selected $\omega = 6$ basin.

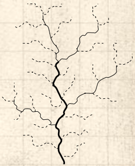




Generalizing Tokunaga's law

Scheidegger:

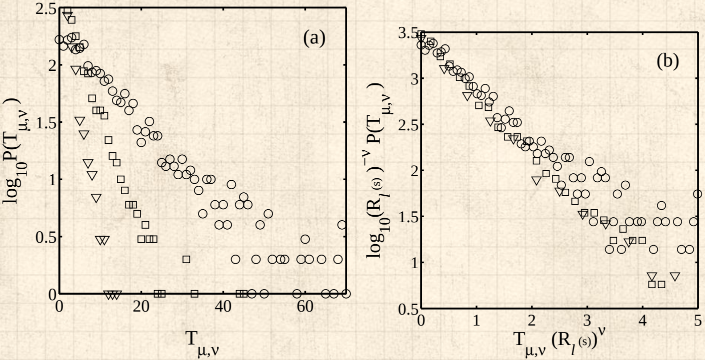




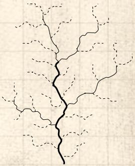
- Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using R_s

Generalizing Tokunaga's law

Mississippi:



- Same data collapse for Mississippi ...



Generalizing Tokunaga's law

So

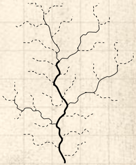
$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t \left[T_{\mu,\nu}/(R_s)^{\mu-\nu-1} \right]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

$$\boxed{P(s_\mu) \Leftrightarrow P(T_{\mu,\nu})}$$

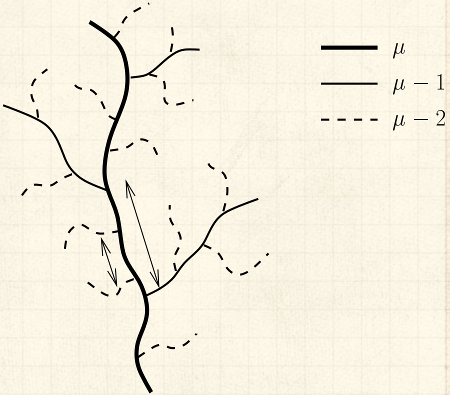
- Exponentials arise from randomness.
- Look at joint probability $P(s_\mu, T_{\mu,\nu})$.

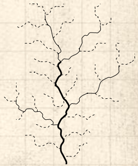


Generalizing Tokunaga's law

Network architecture:

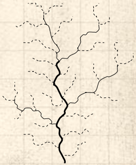
- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments





Generalizing Tokunaga's law

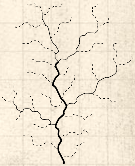
- Follow streams segments down stream from their beginning



Generalizing Tokunaga's law

- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is constant:

$$\tilde{p}_{\mu} \simeq 1/(R_s)^{\mu-1}\xi_s$$



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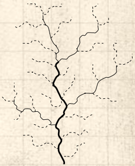

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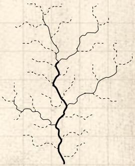

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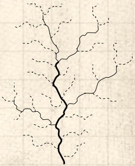
- $\Rightarrow$ random spatial distribution of stream segments



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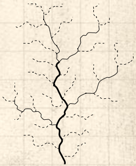

- Joint distribution for generalized version of Tokunaga's law:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

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    - $p_\nu =$ probability of absorbing an order ν side stream

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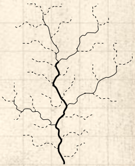

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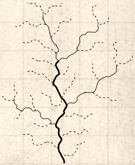
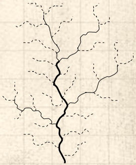
  where
  - p_ν = probability of absorbing an order ν side stream
  - $\tilde{p}_\mu$ = probability of an order μ stream terminating
- Approximation: depends on distance units of s_μ
- In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.



Generalizing Tokunaga's law

- Now deal with this thing:

$$P(s_{\mu}, T_{\mu,\nu}) = \tilde{p}_{\mu} \binom{s_{\mu} - 1}{T_{\mu,\nu}} p_{\nu}^{T_{\mu,\nu}} (1 - p_{\nu} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu,\nu} - 1}$$



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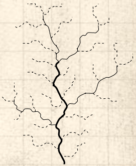

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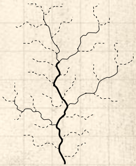
- Set $(x, y) = (s_\mu, T_{\mu,\nu})$ and $q = 1 - p_\nu - \tilde{p}_\mu$, approximate liberally.
- Obtain

$$P(x, y) = N x^{-1/2} \left[F(y/x)\right]^x$$

where

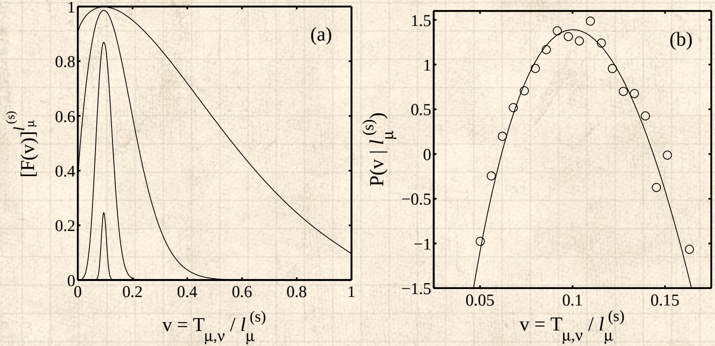
$$F(v) = \left(\frac{1-v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$

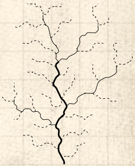


Generalizing Tokunaga's law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Scheidegger:





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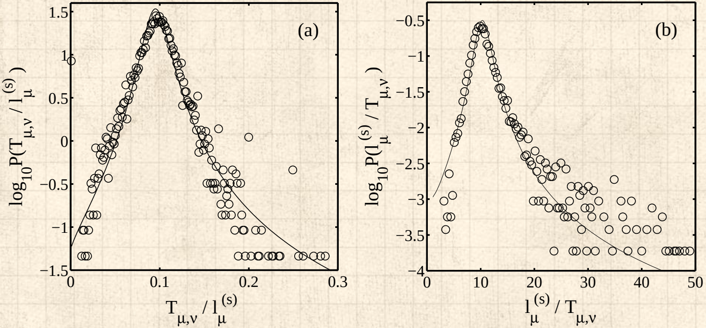

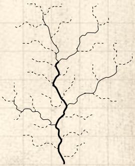

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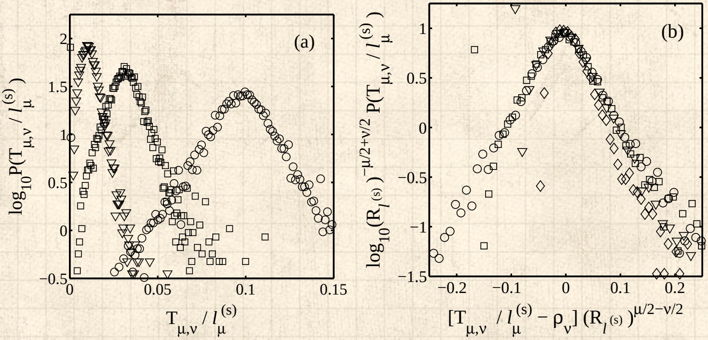

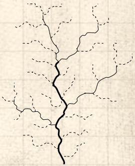

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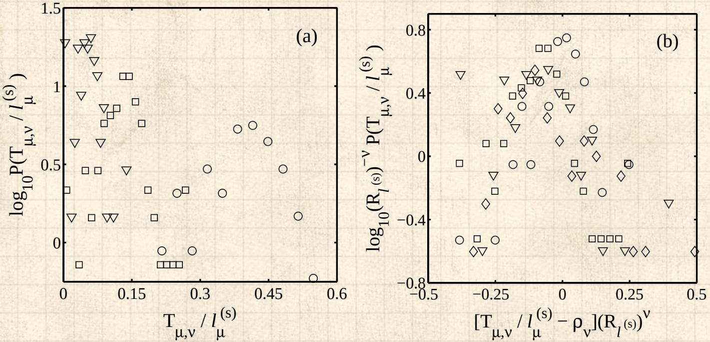

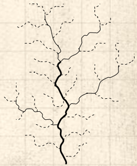


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
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THE EMERGENCE OF THINKING
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
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THE ILL-SCALING OF ALLOMETRIC BODIES
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THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
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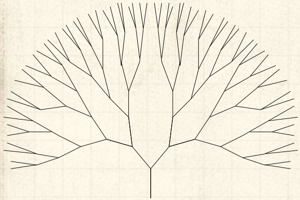
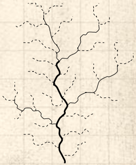
THE CHAOS OF FETTERED CONTAGION
THE END
THE UNKNOWN MECHANISM

Models

Random subnetworks on a Bethe lattice [13]





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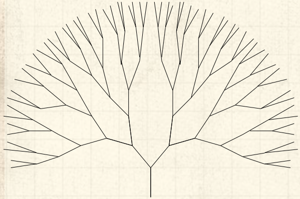

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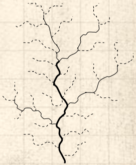

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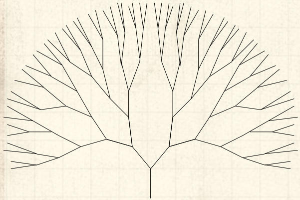

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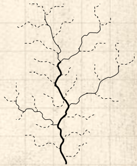

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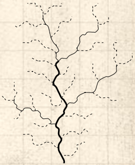

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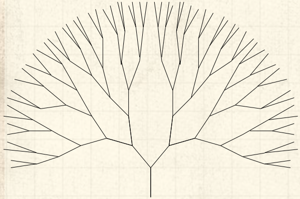

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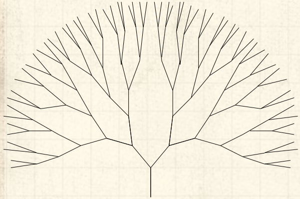

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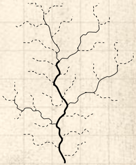

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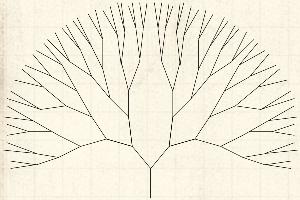

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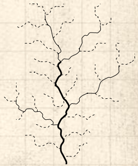

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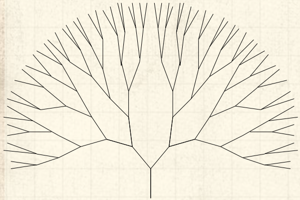

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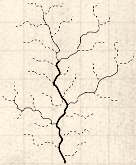
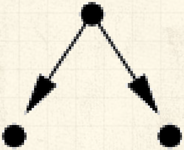
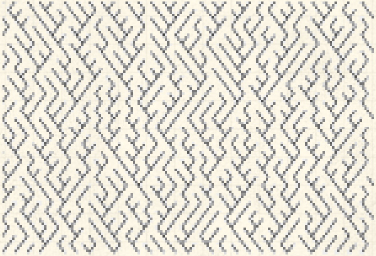


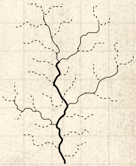
Scheidegger's model

Directed random networks [11, 12]





- $$P(\searrow) = P(\swarrow) = 1/2$$
- Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]



Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. [10]

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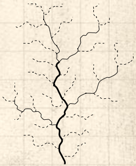

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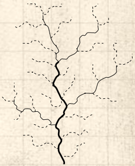

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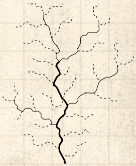

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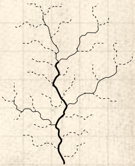
- Landscapes obtained numerically give exponents near that of real networks.



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$$\dot{\varepsilon} \propto \int \mathrm{d}\vec{r}\ (\text{flux}) \times (\text{force}) \sim \sum_i a_i \nabla h_i \sim \sum_i a_i^\gamma$$

- Landscapes obtained numerically give exponents near that of real networks.
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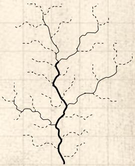
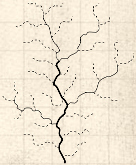


Optimal channel networks



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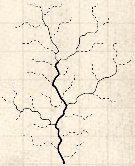
- Landscapes obtained numerically give exponents near that of real networks.
- But: numerical method used matters.
- And: Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network [8]

Theoretical networks



Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

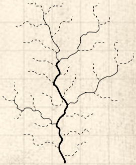
$h \Rightarrow \ell \propto a^h$ (Hack's law).

$d \Rightarrow \ell \propto L_{\parallel}^d$ (stream self-affinity).

Nutshell

Branching networks II Key Points:

- Horton's laws and Tokunaga law all fit together.



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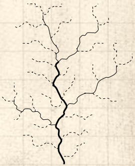

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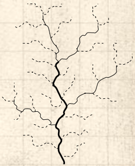

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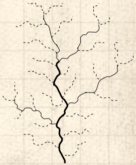

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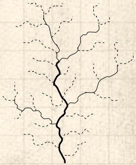
- For scaling laws, only $h = \ln R_\ell / \ln R_n$ and d are needed.
- Laws can be extended nicely to laws of distributions.
- Numerous models of branching network evolution exist: nothing rock solid yet.



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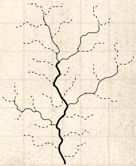

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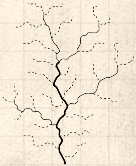

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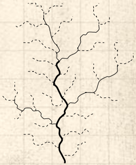

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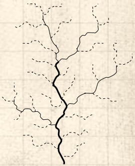

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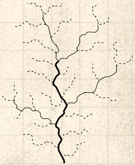

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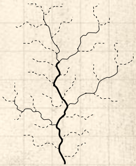

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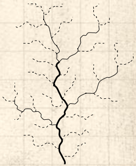

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