P o C S What's The Story?

Principles of Complex Systems, Vols. 1 & 2, CSYS/MATH 300 and 303
University of Vermont, Fall 2021
Assignment 30
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Due: never, 2021.
Relevant clips, episodes, and slides are listed on the assignment's page:
https://pdodds.w3.uvm.edu//teaching/courses/2021-2022principles-of-complex-systems//assignments/30/
Some useful reminders:
Deliverator: Prof. Peter Sheridan Dodds (contact through Teams)
Assistant Deliverator: Michael Arnold (contact through Teams)
Office: The Ether
Office hours: Tuesdays, 3:00 to 4:00 pm on Teams
Course website:
https://pdodds.w3.uvm.edu//teaching/courses/2021-2022principles-of-complex-systems

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All parts are worth 3 points unless marked otherwise. Please show all your workingses clearly and list the names of others with whom you collaborated.

For coding, we recommend you improve your skills with Python, R, and/or Julia. The Deliverator uses Matlab.

Graduate students are requested to use LaTeX (or related TeX variant). If you are new to LaTeX, please endeavor to submit at least n questions per assignment in LaTeX, where n is the assignment number.

Assignment submission:

1. Please send to both the Deliverator and Assistant Deliverator via direct message on Teams.
2. PDF only! Please name your file as follows (where the number is to be padded by a 0 if less than 10 and names are all lowercase): `CSYS300assignment%02d$firstname-$lastname.pdf` as in `CSYS300assignment06michael-palin.pdf`

The questions you don't have to do!

Some are open ended madnesses.

1. Computational Pareidolia.

   A peculiar class project.

   As a team, figure out how to gather, curate, and analyze pictures of the front of cars as they have evolved over time.

Upper limit of insanity: All cars ever sold in the US (types) combined with sales (tokens).

(a) Photos should be from the front.

(b) Ideally, we have photos

(c) Figure out how to assess the emotional content expressed by a car's 'face'.

(d) May be purely computational, may need to use people's assessments. We can use Mechanical Turk for example.

(e) Suggest setting up a single Github repository for the work.

Some articles:

- The faces thing: https://www.smithsonianmag.com/smart-news/for-experts-cars-really-do-have-faces-57005307/.
- Sinisterness: https://www.latimes.com/business/autos/la-hy-sinister-faces-pg-photogallery.html.
- Brain imaging: "High-resolution imaging of expertise reveals reliable object selectivity in the fusiform face area related to perceptual performance" https://www.pnas.org/content/early/2012/09/27/1116333109.abstract.

2. *"Any good idea can be stated in fifty words or less."—Stanisław Ulam.*[1]

   Things have sped up since Ulam made his claim.

   The top of the narrative hierarchy:

   Read through Anderson's seminal paper "More is different" [1] and generate three descriptions of complexification with exactly the following lengths:

   (a) 1–3 words,

   (b) 4–6 words,

   (c) and 7–12 words.

   The 1–3 words one: Try to improve on "More is different".

3. For class discussion, read "Will a large complex system be stable?" by Robert May [2].

   Put together three comments and/or questions.

[1] At the very least, Ulam's claim is self-consistent.

4. (3 + 3 + 3 + 3) This question is all about pure finite and infinite random networks

   We'll define a finite random network as follows. Take N labelled nodes and add links between each pair of nodes with probability p.

   (a) i. For a random node i, determine the probability distribution for its number of friends k, $P_k(p, N)$.

   ii. What kind of distribution is this?

   iii. What does this distribution tend toward in the limit of large N, if p is fixed?

   (No need to do calculations here; just invoke the right Rule of the Universe.)

   (b) Using $P_k(p, N)$, determine the average degree. Does your answer seem right intuitively?

   (c) Show that in the limit of $N \to \infty$ but with mean held constant, we obtain a Poisson degree distribution.

   Hint: to keep the mean constant, you will need to change p.

   (d) i. Compute the clustering coefficients C_1 and C_2 for standard finite random networks (N nodes).

   ii. Explain how your answers make sense.

   iii. What happens in the limit of an infinite random network with finite mean?

5. (3 + 3)

   Determine the clustering coefficient for toy model small-world networks [3] as a function of the rewiring probability p. Find C_1, the average local clustering coefficient:

$$C_1(p) = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i = \frac{1}{N} \sum_{i=1}^{N} \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2}$$

   where N is the number of nodes, $a_{ij} = 1$ if nodes i and j are connected, and $\mathcal{N}_i$ indicates the neighborhood of i.

   As per the original model, assume a ring network with each node connected to a fixed, even number m local neighbors ($m/2$ on each side). Take the number of nodes to be $N \gg m$.

   Start by finding $C_1(0)$ and argue for a $(1-p)^3$ correction factor to find an approximation of $C_1(p)$.

   Hint 1: you can think of finding C_1 as averaging over the possibilities for a single node.

Hint 2: assume that the degree of individual nodes does not change with rewiring but rather stays fixed at m. In other words, take the average degree of individuals as the degree of a randomly selected individual.

For what value of p is $C_1(p)/C_1(0) \simeq 1/2$?

Does this seem reasonable given your simulation?

(3 points for set up, 3 for solving.)

6. (3 + 3):

   Consider a modified version of the Barabàsi-Albert (BA) model [4] where two possible mechanisms are now in play. As in the original model, start with m_0 nodes at time $t = 0$. Let's make these initial guys connected such that each has degree 1. The two mechanisms are:

   M1: With probability p, a new node of degree 1 is added to the network. At time $t + 1$, a node connects to an existing node j with probability

   $$P(\text{connect to node } j) = \frac{k_j}{\sum_{i=1}^{N(t)} k_i} \tag{1}$$

   where k_j is the degree of node j and $N(t)$ is the number of nodes in the system at time t.

   M2: With probability $q = 1 - p$, a randomly chosen node adds a new edge, connecting to node j with the same preferential attachment probability as above.

   Note that in the limit $q = 0$, we retrieve the original BA model (with the difference that we are adding one link at a time rather than m here).

   In the long time limit $t \to \infty$, what is the expected form of the degree distribution P_k?

   Do we move out of the original model's universality class?

   Different analytic approaches are possible including a modification of the BA paper, or a Simon-like one (see also Krapivsky and Redner [5]).

   Hint: You can attempt to solve the problem exactly and you'll find an integrating factor story.

   Another hint, moment of mercy: Approximate the differential equation by considering large t (this will simplify the denominators).

   (3 points for set up, 3 for solving.)

7. (3 + 3)

   Using Gleeson and Calahane's iterative equations below, derive the contagion condition for a vanishing seed by taking the limit $\phi_0 \to 0$ and $t \to \infty$. In lectures, we derived the discrete evolution equations for the fraction of infected nodes ϕ_t and the fraction of infected edges θ_t as follows:

$$\phi_{t+1} = \phi_0 + (1-\phi_0)\sum_{k=0}^{\infty} P_k \sum_{j=0}^{k} \binom{k}{j} \theta_t^j (1-\theta_t)^{k-j} B_{kj},$$

$$\theta_{t+1} = G(\theta_t;\phi_0) = \phi_0 + (1-\phi_0)\sum_{k=1}^{\infty} \frac{kP_k}{\langle k \rangle} \sum_{j=0}^{k-1} \binom{k-1}{j} {\theta_t}^j (1-\theta_t)^{k-1-j} B_{kj},$$

   where $\theta_0 = \phi_0$, and B_{kj} is the probability that a degree k node becomes active when j of its neighbors are active.

   Recall that by contagion condition, we mean the requirements of a random network for macroscopic spreading to occur.

   To connect the paper's model and notation to those of our lectures, given a specific response function F and a threshold model, the B_{kj} are given by $B_{kj} = F(j/k)$.

   Allow B_{k0} to be arbitrary (i.e., not necessarily 0 as for simple threshold functions).

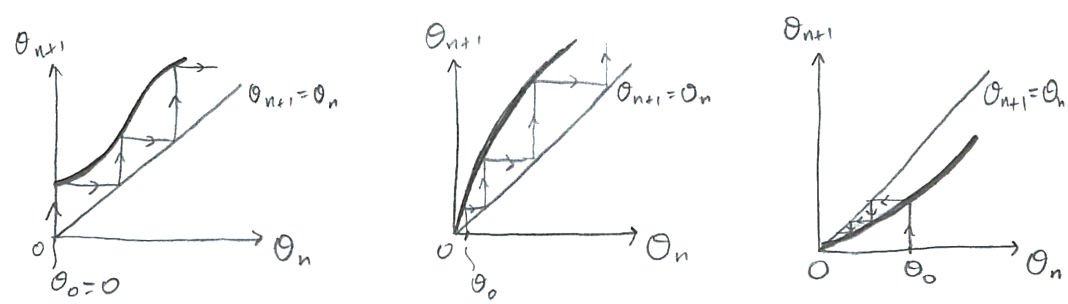
   We really only need to understand how θ_t behaves. Write the corresponding equation as $\theta_{t+1} = G(\theta_t;\phi_0)$ and determine when

   (a) $G(0;0) > 0$ (spreading is for free).

   (b) $G(0;0) = 0$ and $G'(0;\phi_0) > 1$ meaning $\phi = 0$ is a unstable fixed point.

   Here's a graphical hint for the three cases you need to consider as $\theta_0 \to 0$:

   Success: Sucesss: Fail:



8. (3 + 3 + 3) Optional:

   Solve Krapivsky-Redner's model for the pure linear attachment kernel $A_k = k$.

Starting point:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

with $n_0 = 0$.

(a) Determine n_1.

(b) Find a recursion relation for n_k in terms of n_{k-1}.

(c) Now find

$$n_k = \frac{4}{k(k+1)(k+2)}$$

for all k and hence determine γ.

9. (3 + 3) Optional:

From lectures:

(a) Starting from the recursion relation

$$n_k = \frac{A_{k-1}}{\mu + A_k} n_{k-1},$$

and $n_1 = \mu/(\mu + A_1)$, show that the expression for n_k for the Krapivsky-Redner model with an asymptotically linear attachment kernel A_k is:

$$\frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}.$$

(b) Now show that if $A_k \to k$ for $k \to \infty$ (or for large k), we obtain $n_k \to k^{-\mu-1}$.

10. (3 + 3 + 3)

From lectures, complete the analysis for the Krapivsky-Redner model with attachment kernel:

$$A_1 = \alpha \text{ and } A_k = k \text{ for } k \geq 2.$$

Find the scaling exponent $\gamma = \mu + 1$ by finding μ. From lectures, we assumed a linear growth in the sum of the attachment kernel weights $\mu t = \sum_{k=1}^{\infty} N_k(t) A_k$, with $\mu = 2$ for the standard kernel $A_k = k$.

We arrived at this expression for μ which you can use as your starting point:

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

(a) Show that the above expression leads to

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

Hint: you'll want to separate out the $j = 1$ case for which $A_j = \alpha$.

(b) Now use result that [5]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

to find the connection

$$\mu(\mu-1) = 2\alpha,$$

and show this leads to

$$\mu = \frac{1+\sqrt{1+8\alpha}}{2}.$$

(c) Interpret how varying α affects the exponent γ, explaining why $\alpha < 1$ and $\alpha > 1$ lead to the particular values of γ that they do.

11. Yes, even more on power law size distributions. It's good for you.

    For the probability distribution $P(x) = cx^{-\gamma}$, $0 < a \le x \le b$, compute the mean absolute displacement (MAD), which is given by $\langle |X - \langle X \rangle| \rangle$ where $\langle \cdot \rangle$ represents expected value. As always, simplify your expression as much as possible.

    MAD is a more reasonable estimate for the width of a distribution, but we like variance σ^2 because the calculations are much prettier. Really.

12. In the limit of $b \to \infty$, show that MAD asymptotically behave as:

$$\langle |X - \langle X \rangle| \rangle = \frac{2(\gamma-2)^{(\gamma-3)}}{(\gamma-1)^{(\gamma-2)}} a.$$

    How does this compare with the behavior of the variance? (See the last question of Assignment todo???.)

13. *Simon's model II:*

    A missing piece from the lectures: Obtain γ in terms of ρ by expanding Eq. **??** in terms of $1/k$. In the end, you will need to express n_k/n_{k-1} as $(1-1/k)^{\theta}$; from here, you will be able to identify γ. Taylor expansions and Procrustean truncations will be in order.

    This (dirty) method avoids finding the exact form for n_k.

14. A spectacularly optional extra.

    Warning:

    - Only attempt if using registered safety equipment including welding goggles and a lead apron.
    - Make sure to back up your brain in at least two geographically distant places beforehand (e.g., on different planets).

    Dangerous feature:

    - If you make it out, you will be very happy.

    In lectures on lognormals and other heavy-tailed distributions, we came across a super fun and interesting integral when considering organization size distributions arising from growth processes with variable lifespans.

    Show that

    $$P(x) = \int_{t=0}^{\infty} \lambda e^{-\lambda t} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln \frac{x}{m})^2}{2t}\right) \mathrm{d}t$$

    leads to:

    $$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln \frac{x}{m})^2}},$$

    and therefore, surprisingly, two different scaling regimes. Enjoyable suffering may be involved. Really enjoyable suffering. But many monks have found a way so you should follow their path laid out below.

    Hints and steps:

    - Make the substitution $t = u^2$ to find an integral of the form (excluding a constant of proportionality)

      $$I_1(a, b) = \int_0^{\infty} \exp\left(-au^2 - b/u^2\right) \mathrm{d}u$$

      where in our case $a = \lambda$ and $b = (\ln \frac{x}{m})^2/2$.
    - Substitute $au^2 = t^2$ into the above to find

      $$I_1(a, b) = \frac{1}{\sqrt{a}} \int_0^{\infty} \exp\left(-t^2 - ab/t^2\right) \mathrm{d}t$$

    - Now work on this integral:

      $$I_2(r) = \int_0^{\infty} \exp\left(-t^2 - r/t^2\right) \mathrm{d}t$$

      where $r = ab$.

- Differentiate I_2 with respect to r to create a simple differential equation for I_2. You will need to use the substitution $u = \sqrt{r}/t$ and your differential equation should be of the (very simple) form

$$\frac{\mathrm{d}I_2(r)}{\mathrm{d}r} = -(\text{something})I_2(r).$$

- Solve the differential equation you find. To find the constant of integration, you can evaluate $I_2(0)$ separately:

$$I_2(0) = \int_0^\infty \exp(-t^2)\mathrm{d}t,$$

where our friend $\Gamma(frac12)$ comes into play.

References

[1] P. W. Anderson. More is different. *Science*, 177(4047):393–396, 1972. pdf

[2] R. M. May. Will a large complex system be stable? *Nature*, 238:413–414, 1972. pdf

[3] D. J. Watts and S. J. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393:440–442, 1998. pdf

[4] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *Science*, 286:509–511, 1999. pdf

[5] P. L. Krapivsky and S. Redner. Organization of growing random networks. *Phys. Rev. E*, 63:066123, 2001. pdf