Optimal Supply Networks III: Redistribution

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CSYS/MATH 300, Fall, 2020

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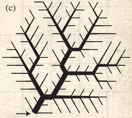
Computational Story Lab | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont



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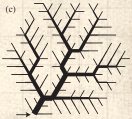

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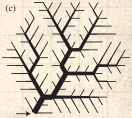
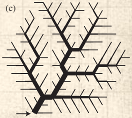


Outline

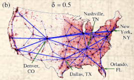
Distributed Sources
Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private

References



Many sources, many sinks

How do we distribute sources?



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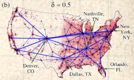

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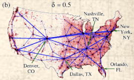

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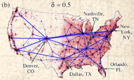

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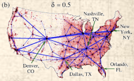
- Obvious: if density is uniform then sources are best distributed uniformly.



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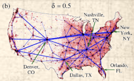

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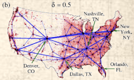
- Which lattice is optimal? The hexagonal lattice
- Q2: Given population density is uneven, what do we do?



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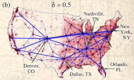
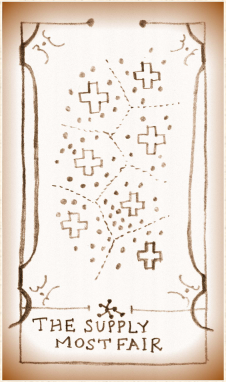
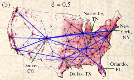
- Obvious: if density is uniform then sources are best distributed uniformly.
- Which lattice is optimal? The hexagonal lattice
- Q2: Given population density is uneven, what do we do?
- We'll follow work by Stephan (1977, 1984) [4, 5], Gastner and Newman (2006) [2], Um *et al.* (2009) [6], and work cited by them.



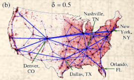

THE JOHN DORY
THE SUN
THE NETWORKS COMPLEX
THE ARK OF THE RANDOM NETWORK
THE NETWORKS FREE OF SCALE
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE UNKNOWN MECHANISM
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES


THE SUPPLY MOST FAIR


(b)
δ = 0.5
Nashville, TN
New York, NY
Denver, CO
Dallas, TX
Orlando, FL

Optimal source allocation

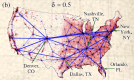
Solidifying the basic problem



Optimal source allocation

Solidifying the basic problem

- Given a region with some population distribution ρ, most likely uneven.



Optimal source allocation

Solidifying the basic problem

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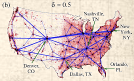
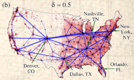


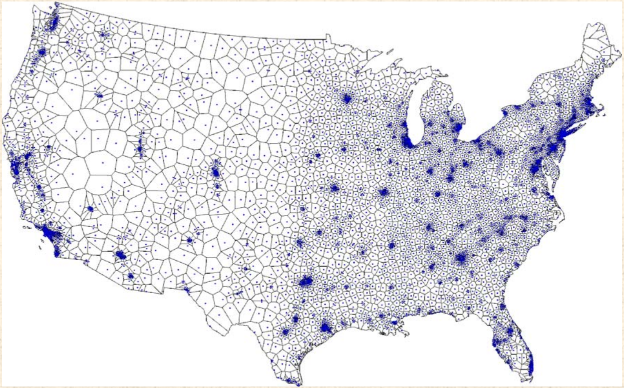
Optimal source allocation

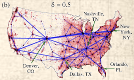
Solidifying the basic problem

- Given a region with some population distribution ρ, most likely uneven.
- Given resources to build and maintain N facilities.
- Q: How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?



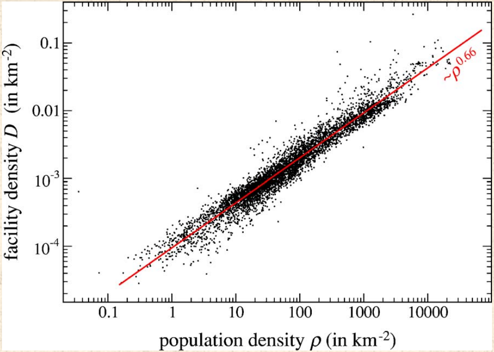
"Optimal design of spatial distribution networks"
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]



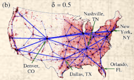


- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.

Optimal source allocation



- Optimal facility density ρ_{fac} vs. population density ρ_{pop}.



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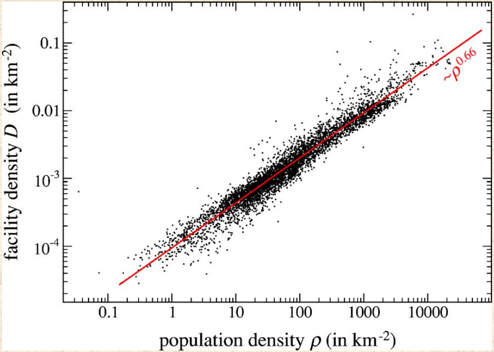

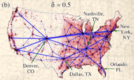

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- Fit is $\rho_{\text{fac}} \propto \rho_{\text{pop}}^{0.66}$ with $r^2 = 0.94$.

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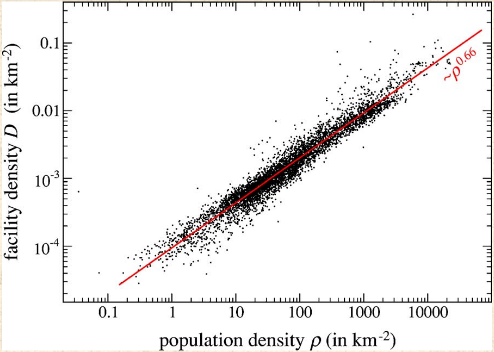

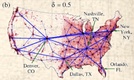

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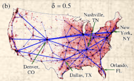
- Fit is $\rho_{\text{fac}} \propto \rho_{\text{pop}}^{0.66}$ with $r^2 = 0.94$.
- Looking good for a 2/3 power ...

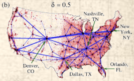
Outline



Optimal source allocation

Size-density law:

$$\boxed{\rho_{\text{fac}} \propto \rho_{\text{pop}}^{2/3}}$$



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Size-density law:

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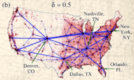
- Why?



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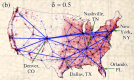

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- Why?
- Again: Different story to branching networks where there was either one source or one sink.

Optimal source allocation

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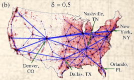
- Why?
- Again: Different story to branching networks where there was either one source or one sink.
- Now sources & sinks are distributed throughout region.



Optimal source allocation

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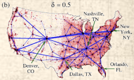
- We first examine Stephan's treatment (1977) [4, 5]



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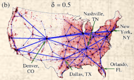

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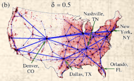
- Also known as the Homer Simpson principle.



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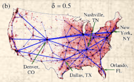
- Consider a region of area A and population P with a single functional center that everyone needs to access every day.



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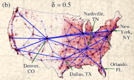

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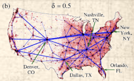

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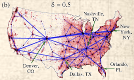

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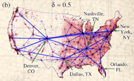
- Assume isometry: average travel distance $\bar{d}$ will be on the length scale of the region which is $\sim A^{1/2}$
- Average time expended per person in accessing facility is therefore

$$\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$$

where c is an unimportant shape factor.



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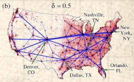

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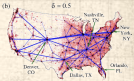
- Call this quantity τ.



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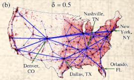
- If burden of mainenance is shared then average cost per person is τ/P where P = population.



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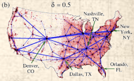

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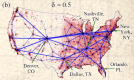

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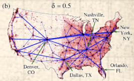
$$T = \bar{d}/\bar{v} + \tau/(\rho_{\text{pop}}A)$$



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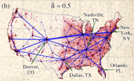

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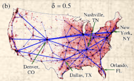
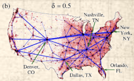
- Now Minimize with respect to A ...



Optimal source allocation

- Differentiating ...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\bar{v} + \tau/(\rho_{\text{pop}}A)\right)$$



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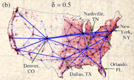

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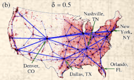

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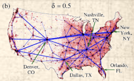

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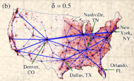

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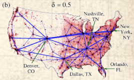

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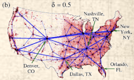

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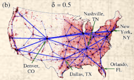

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Optimal source allocation

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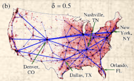
- Stephan's online book "The Division of Territory in Society" is here.
- (It used to be here.)
- The Readme is well worth reading (1995).



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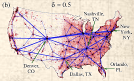
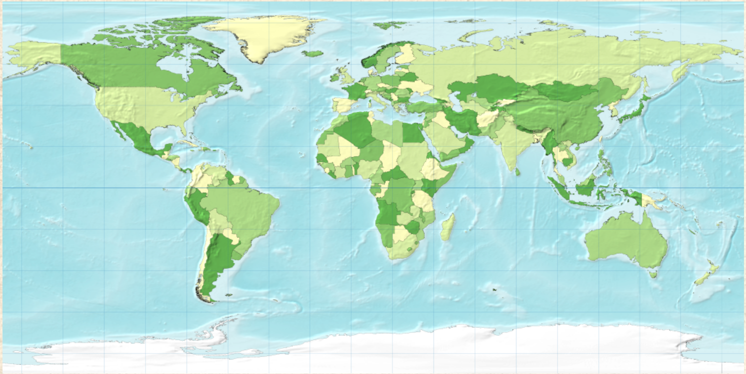
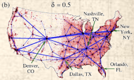

(b)
$\delta = 0.5$
Nashville, TN
New York, NY
Denver, CO
Dallas, TX
Orlando, FL

Cartograms

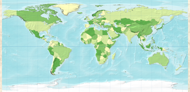
Standard world map:

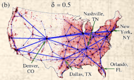




Cartograms

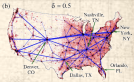
Cartogram of countries 'rescaled' by population:





Cartograms

Diffusion-based cartograms:



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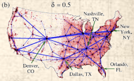

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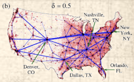

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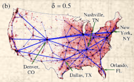

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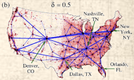

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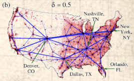
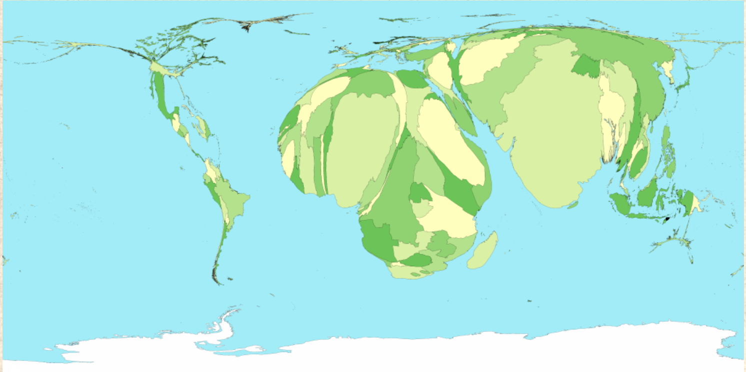
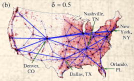


Cartograms

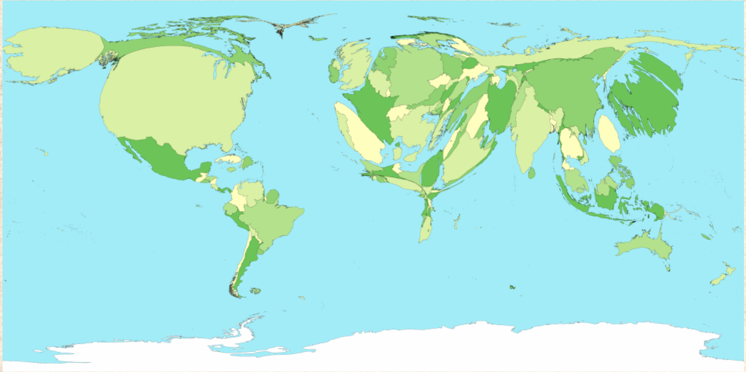
Child mortality:

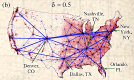




Cartograms

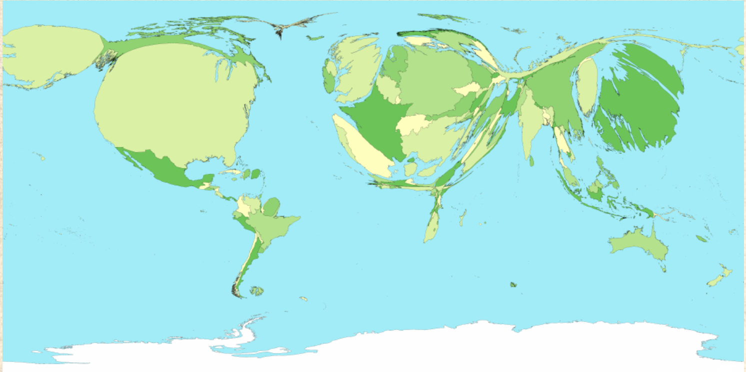
Energy consumption:

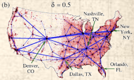




Cartograms

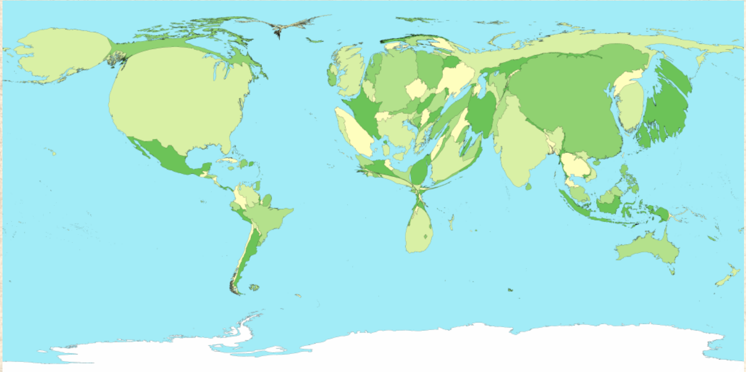
Gross domestic product:

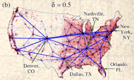




Cartograms

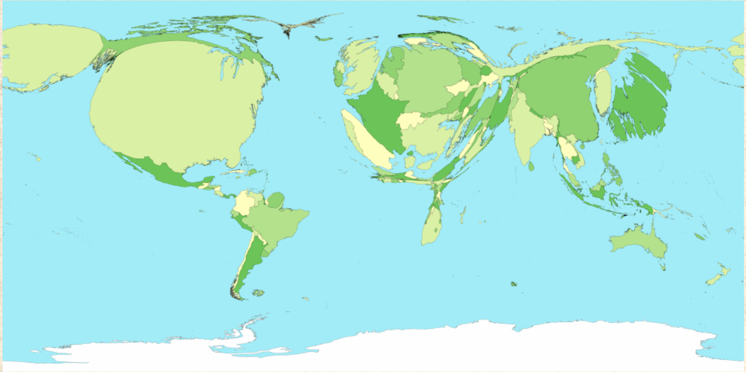
Greenhouse gas emissions:

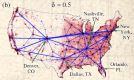




Cartograms

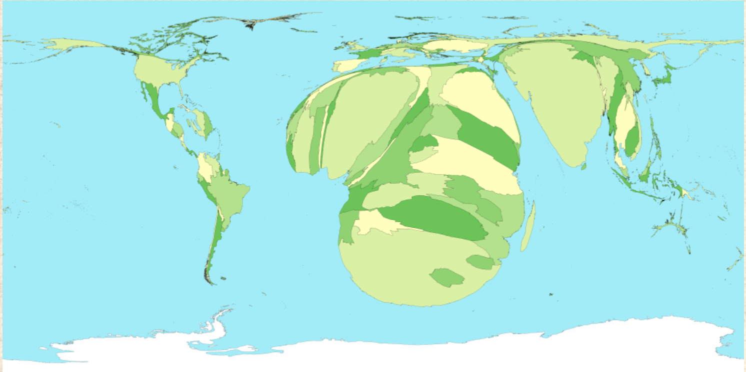
Spending on healthcare:

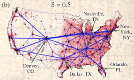




Cartograms

People living with HIV:

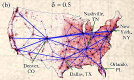




Cartograms

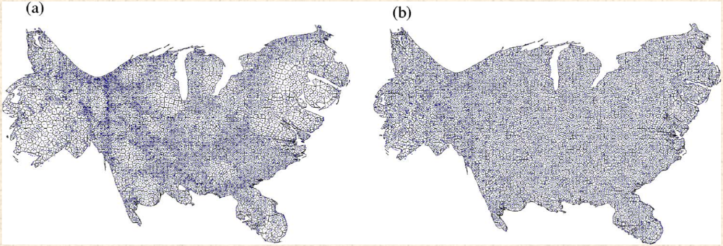
- The preceding sampling of Gastner & Newman's cartograms lives here.
- A larger collection can be found at worldmapper.org.

WORLDMAPPER The world as you've never seen it before

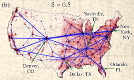


Size-density law

"Optimal design of spatial distribution networks"
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]



- Left: population density-equalized cartogram.



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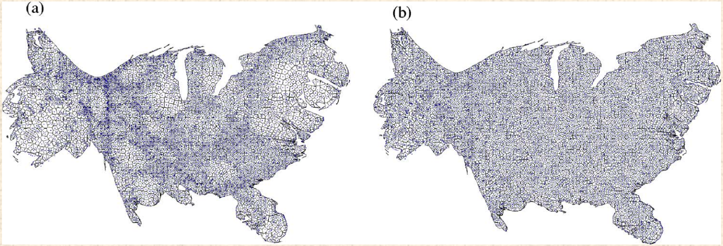

- Left: population density-equalized cartogram.
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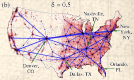

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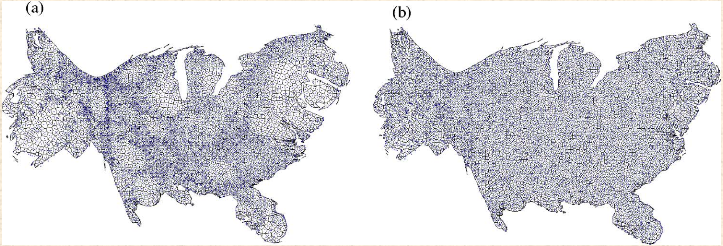

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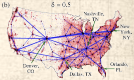
- Facility density is uniform for $\rho_{\text{pop}}^{2/3}$ cartogram.



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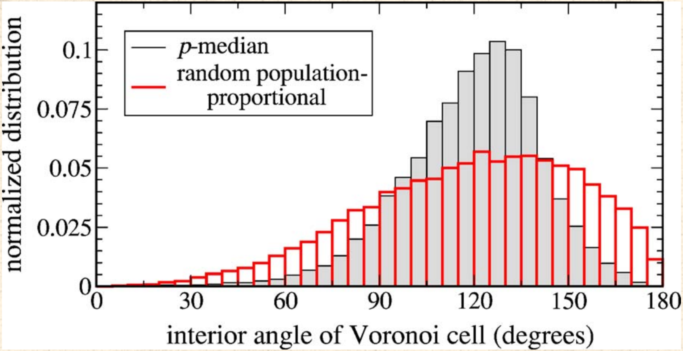
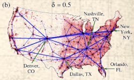


From Gastner and Newman (2006) [2]

- Cartogram's Voronoi cells are somewhat hexagonal.



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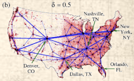

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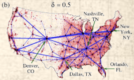

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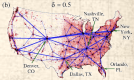

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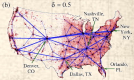

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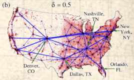

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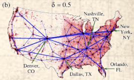
- Also known as the p-median problem.



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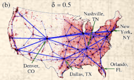

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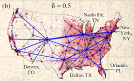

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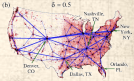
Approximations:

- For a given set of source placements $\{\vec{x}_1, \ldots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells, one per source.



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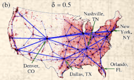
- Define $A(\vec{x})$ as the area of the Voronoi cell containing $\vec{x}$.



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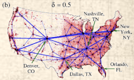

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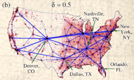
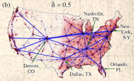


Size-density law

Carrying on:

- The cost function is now

$$F = c \int_{\Omega} \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x}\,.$$



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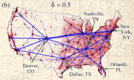

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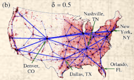

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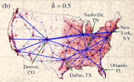

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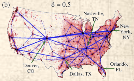
- So ...integral over each of the n cells equals 1.



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- By varying $\{\vec{x}_1, \dots, \vec{x}_n\}$, minimize

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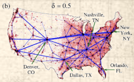

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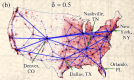

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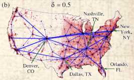

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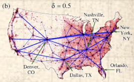

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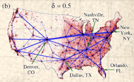
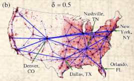


Size-density law

Now a Lagrange multiplier story:

- Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho_{\text{pop}}^{-2/3}.$$



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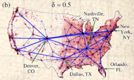

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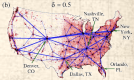

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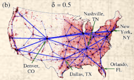
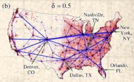
- Normalizing (or solving for λ):

$$\rho_{\text{fac}}(\vec{x}) = n \frac{[\rho_{\text{pop}}(\vec{x})]^{2/3}}{\int_{\Omega} [\rho_{\text{pop}}(\vec{x})]^{2/3} \mathrm{d}\vec{x}} \propto [\rho_{\text{pop}}(\vec{x})]^{2/3}.$$



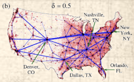
Outline



Global redistribution networks

One more thing:

- How do we supply these facilities?



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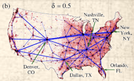

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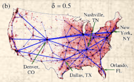

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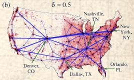

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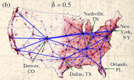

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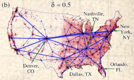

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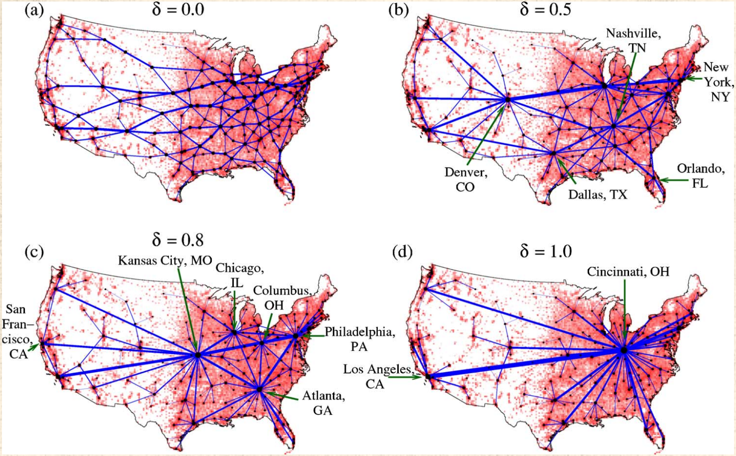
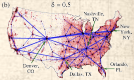
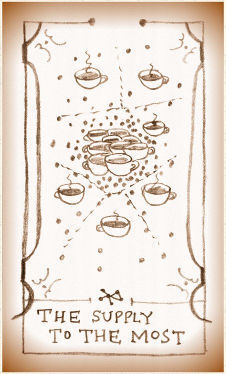
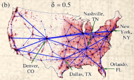


From Gastner and Newman (2006) [2]

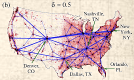



THE JOHN DORY
THE SUN
THE NETWORKS COMPLEX
"I am the Monarch of the Sea."
THE ARK OF THE RANDOM NETWORK
THE NETWORKS FREE OF SCALE
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE UNKNOWN MECHANISM
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
D-1
D
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR


THE SUPPLY
TO THE MOST

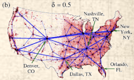

(b)
δ = 0.5
Nashville, TN
New York, NY
Orlando, FL
Denver, CO
Dallas, TX

Outline



Public versus private facilities

Beyond minimizing distances:



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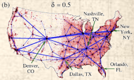

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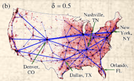

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- Two idealized limiting classes:
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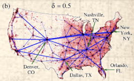

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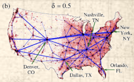
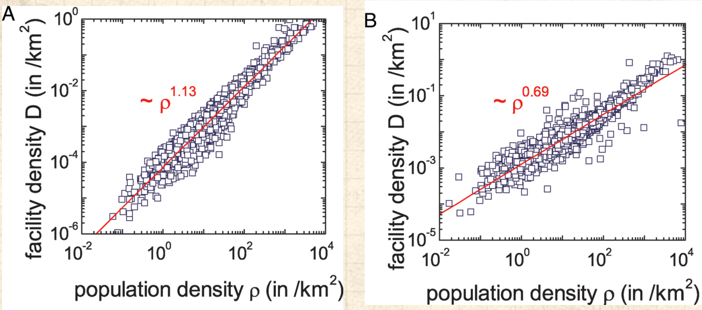
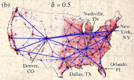
- Two idealized limiting classes:
  1. For-profit, commercial facilities: $\alpha = 1$;
  2. Pro-social, public facilities: $\alpha = 2/3$.
- Um *et al.* investigate facility locations in the United States and South Korea.



Public versus private facilities: evidence



- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.



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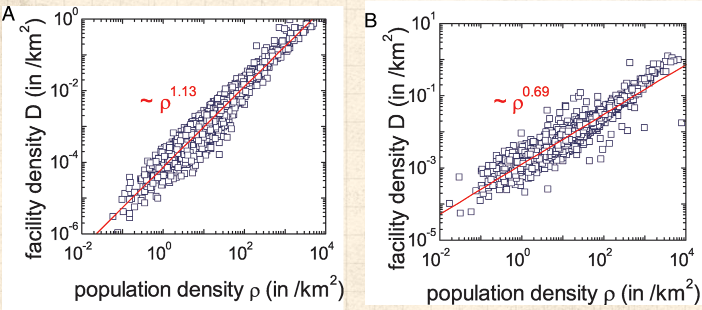
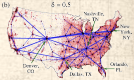


- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.
- Note: break in scaling for public schools. Transition from $\alpha \simeq 2/3$ to $\alpha = 1$ around $\rho_{\text{pop}} \simeq 100$.



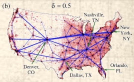
Public versus private facilities: evidence

US facility	α (SE)	R^2
Ambulatory hospital	1.13(1)	0.93
Beauty care	1.08(1)	0.86
Laundry	1.05(1)	0.90
Automotive repair	0.99(1)	0.92
Private school	0.95(1)	0.82
Restaurant	0.93(1)	0.89
Accommodation	0.89(1)	0.70
Bank	0.88(1)	0.89
Gas station	0.86(1)	0.94
Death care	0.79(1)	0.80
* Fire station	0.78(3)	0.93
* Police station	0.71(6)	0.75
Public school	0.69(1)	0.87

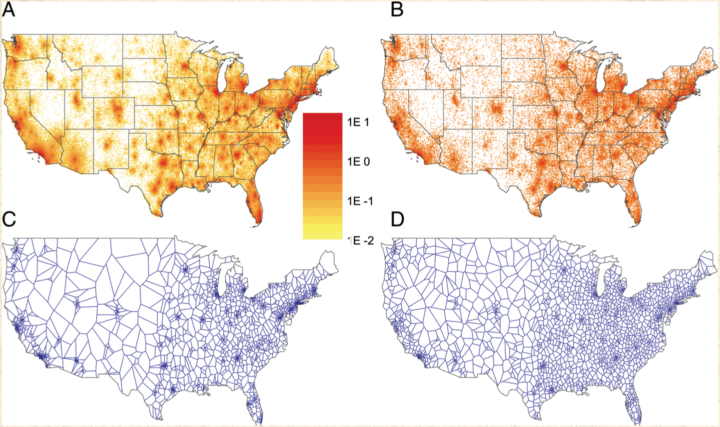
SK facility	α (SE)	R^2
Bank	1.18(2)	0.96
Parking place	1.13(2)	0.91
* Primary clinic	1.09(2)	1.00
* Hospital	0.96(5)	0.97
* University/college	0.93(9)	0.89
Market place	0.87(2)	0.90
* Secondary school	0.77(3)	0.98
* Primary school	0.77(3)	0.97
Social welfare org.	0.75(2)	0.84
* Police station	0.71(5)	0.94
Government office	0.70(1)	0.93
* Fire station	0.60(4)	0.93
* Public health center	0.09(5)	0.19

Rough transition between public and private at $\alpha \simeq 0.8$.

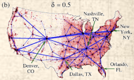
Note: * indicates analysis is at state/province level; otherwise county level.



Public versus private facilities: evidence



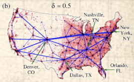
A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.



Public versus private facilities: the story

So what's going on?

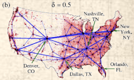
- Social institutions seek to minimize distance of travel.



Public versus private facilities: the story

So what's going on?

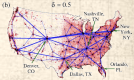
- Social institutions seek to minimize distance of travel.
- Commercial institutions seek to maximize the number of visitors.



Public versus private facilities: the story

So what's going on?

- Social institutions seek to minimize distance of travel.
- Commercial institutions seek to maximize the number of visitors.
- Defns: For the ith facility and its Voronoi cell V_i, define
  - n_i = population of the ith cell;
  - $\langle r_i \rangle$ = the average travel distance to the ith facility.
  - A_i = area of ith cell (s_i in Um *et al.* [6])



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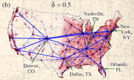
$$v_i = n_i \langle r_i \rangle^{\beta} \text{ with } 0 \leq \beta \leq 1.$$



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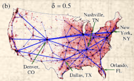
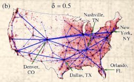
- Limits:
  - $\beta = 0$: purely commercial.
  - $\beta = 1$: purely social.



Public versus private facilities: the story

- Either proceeding as per the Gastner-Newman-Gusein-Zade calculation or, as Um *et al.* do, observing that the cost for each cell should be the same, we have:

$$\rho_{\mathsf{fac}}(\vec{x}) = n \frac{[\rho_{\mathsf{pop}}(\vec{x})]^{2/(\beta+2)}}{\int_{\Omega} [\rho_{\mathsf{pop}}(\vec{x})]^{2/(\beta+2)} \mathsf{d}\vec{x}} \propto [\rho_{\mathsf{pop}}(\vec{x})]^{2/(\beta+2)}.$$



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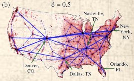
- For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.



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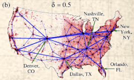
- For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.
- For $\beta = 1$, $\alpha = 2/3$: social scaling is sublinear.



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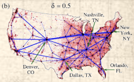

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