Scale-free networks

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Outline

Scale-free networks
Main story
Model details
Analysis
A more plausible mechanism
Robustness
Krapivsky & Redner's model
Generalized model
Analysis
Universality?
Sublinear attachment kernels
Superlinear attachment kernels
Nutshell
References


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE DOORS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


THE NETWORKS FREE OF SCALE

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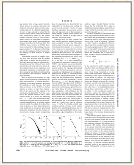
- One of the seminal works in complex networks:



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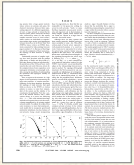

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- Somewhat misleading nomenclature...

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- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are ‘scale-free’ or not…

Some real data (we are feeling brave):

From Barabási and Albert's original paper [2]:

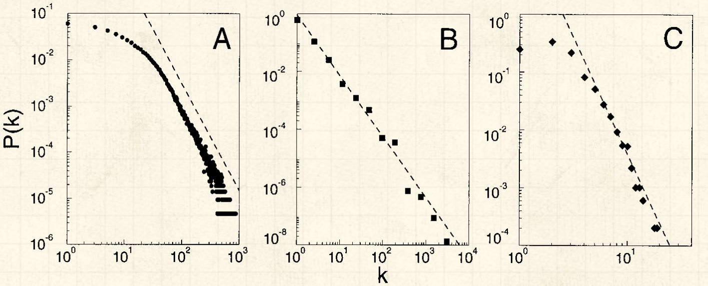
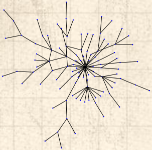
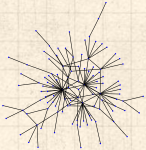


Fig. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\text{actor}} = 2.3$, (B) $\gamma_{\text{www}} = 2.1$ and (C) $\gamma_{\text{power}} = 4$.

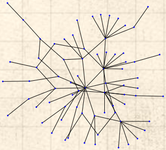
Random networks: largest components



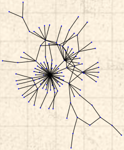
γ = 2.5
$\langle k \rangle$ = 1.8



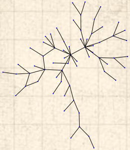
γ = 2.5
$\langle k \rangle$ = 2.05333



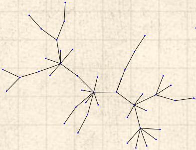
γ = 2.5
$\langle k \rangle$ = 1.66667



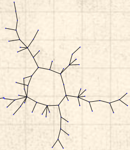
γ = 2.5
$\langle k \rangle$ = 1.92



γ = 2.5
$\langle k \rangle$ = 1.6



γ = 2.5
$\langle k \rangle$ = 1.50667



γ = 2.5
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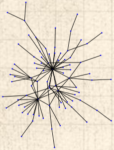

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A big deal for scale-free networks:

- How does the exponent γ depend on the mechanism?
- Do the mechanism details matter?

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- In essence, we have a rich-gets-richer scheme.
- Yes, we've seen this all before in Simon's model.

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- Main story
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- Analysis
- A more plausible mechanism
- Robustness
- Krapivsky & Redner’s model
- Generalized model
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- Universality?
- Sublinear attachment kernels
- Superlinear attachment kernels
- Nutshell

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where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

Approximate analysis

- When $(N+1)$th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

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where $t = N(t) - m_0$.

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- Next find c_i ...

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- Clearly, a Ponzi scheme.

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- Degree of node i is the size of the ith ranked node:

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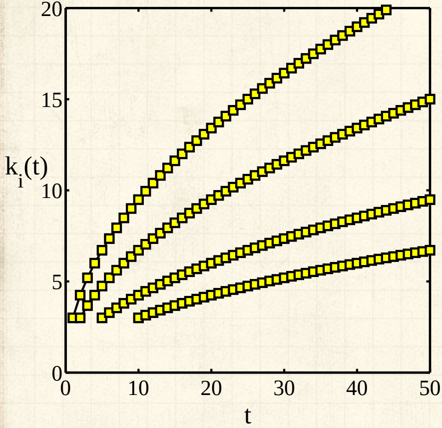
- Our connection $\alpha = 1/(\gamma - 1)$ or $\gamma = 1 + 1/\alpha$ then gives

$$\boxed{\gamma = 1 + 1/(1/2) = 3.}$$



- $m = 3$
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Back to that real data:

From Barabási and Albert's original paper [2]:

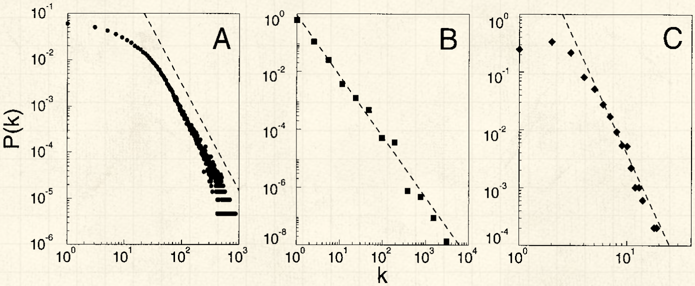

Fig. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\text{actor}} = 2.3$, (B) $\gamma_{\text{www}} = 2.1$ and (C) $\gamma_{\text{power}} = 4$.

Examples

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The Internets is a different business...

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Outline

Scale-free networks

Main story

Model details

Analysis

A more plausible mechanism

Robustness

Krapivsky & Redner's model

Generalized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachment kernels

Nutshell

References

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- So rich-gets-richer scheme can now be seen to work in a natural way.

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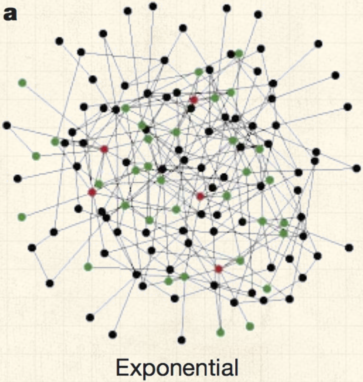
Robustness

- Albert et al., Nature, 2000:
  "Error and attack tolerance of complex networks" [1]
- Standard random networks (Erdős-Rényi) versus Scale-free networks:



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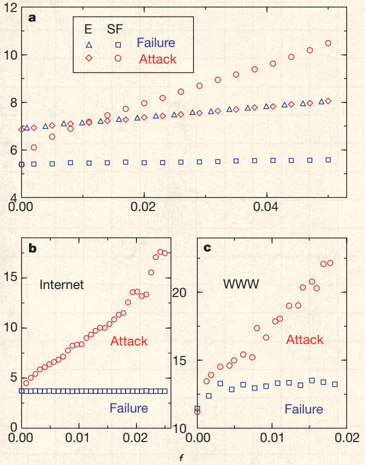

from Albert et al., 2000

- Plots of network diameter as a function of fraction of nodes removed
- Erdős-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

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- Need to explore cost of various targeting schemes.

Robustness

Not a robust paper:

"The "Robust yet Fragile" nature of the Internet"
Doyle et al.,
Proc. Natl. Acad. Sci., **2005**, 14497–14502, 2005. [3]

- HOT networks versus scale-free networks
- Same degree distributions, different arrangements.
- Doyle *et al.* take a look at the actual Internet.

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- KR model will be fully studied in CoNKS.

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- Detail: we are ignoring initial seed network's edges.

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$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution: $N_k = n_k t$.
- We replace $\mathrm{d}N_k/\mathrm{d}t$ with $\mathrm{d}n_k t/\mathrm{d}t = n_k$.
- We arrive at a difference equation:

$$n_k = \frac{1}{2\cancel{t}}\left[(k-1)n_{k-1}\cancel{t} - kn_k\cancel{t}\right] + \delta_{k1}$$

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- Keep A_k linear in k but tweak details.
- Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

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- As before, also assume $N_k(t) = n_k t$.

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- We can solve for μ in some cases.
- Our assumption that $A = \mu t$ looks to be not too horrible.

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- Distribution of degree is universal providing $\nu < 1$.

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- And for $1/(r+1) < \nu < 1/r$, we have r pieces in exponential.

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Scale-free networks

- Main story
- Model details
- Analysis
- A more plausible mechanism
- Robustness
- Krapivsky & Redner's model
- Generalized model
- Analysis
- Universality?
- Sublinear attachment kernels
- Superlinear attachment kernels
- Nutshell

References

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- For $\nu > 2$, all but a finite # of nodes connect to one node.

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Overview Key Points for Models of Networks:

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Neural reboot (NR):

Turning the corner:

https://www.youtube.com/watch?v=axrTxEVQqN4?rel=0

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