### Optimal Supply Networks III: Redistribution

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Complex Networks | @networksvox CSYS/MATH 303, Spring, 2019

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#### **Distributed Sources**

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Size-density law Cartograms A reasonable derivation Global redistribution Public versus Private

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### Many sources, many sinks

#### How do we distribute sources?

- A Focus on 2-d (results generalize to higher dimensions).
- Sources = hospitals, post offices, pubs, ...
- & Key problem: How do we cope with uneven population densities?
- & Obvious: if density is uniform then sources are best distributed uniformly.
- Which lattice is optimal? The hexagonal lattice
- Q2: Given population density is uneven, what do we do?
- & We'll follow work by Stephan (1977, 1984) [4, 5], Gastner and Newman (2006) [2], Um et al. (2009) [6], and work cited by them.





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# Optimal source allocation

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### Solidifying the basic problem

- & Given a region with some population distribution  $\rho$ , most likely uneven.
- $\mathcal{L}$  Given resources to build and maintain N facilities.
- $\bigcirc$  O: How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?





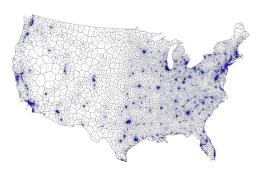
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On Instagram at pratchett the cat



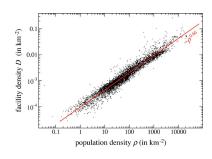
"Optimal design of spatial distribution networks"

Gastner and Newman, Phys. Rev. E, **74**, 016117, 2006. [2]



- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.
- Simulated annealing + Voronoi tessellation.

## Optimal source allocation



- $lap{8}$  Optimal facility density  $ho_{
  m fac}$  vs. population density
- $\mbox{\&}$  Fit is  $\rho_{\rm fac} \propto \rho_{\rm pop}^{0.66}$  with  $r^2=0.94$ .
- & Looking good for a 2/3 power ...

# Optimal source allocation

#### Size-density law:



 $\rho_{\rm fac} \propto \rho_{\rm pop}^{2/3}$ 

- ♣ Why?
- Again: Different story to branching networks where there was either one source or one sink.
- Now sources & sinks are distributed throughout region.

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## Optimal source allocation



"Territorial Division: The Least-Time Constraint Behind the Formation of Subnational Boundaries"

G. Edward Stephan, Science, **196**, 523-524, 1977. [4]

- We first examine Stephan's treatment (1977) [4, 5]
- A "Territorial Division: The Least-Time Constraint Behind the Formation of Subnational Boundaries" (Science, 1977)
- Zipf-like approach: invokes principle of minimal
- 💫 Also known as the Homer Simpson principle.

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access every day.

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- on the length scale of the region which is  $\sim A^{1/2}$
- Average time expended per person in accessing facility is therefore

& Assume isometry: average travel distance  $\bar{d}$  will be

 Consider a region of area A and population P with a single functional center that everyone needs to

Build up a general cost function based on time

 $\clubsuit$  Write average travel distance to center as  $\bar{d}$  and assume average speed of travel is  $\bar{v}$ .

expended to access and maintain center.

$$\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$$

where c is an unimportant shape factor.





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If burden of mainenance is shared then average cost per person is  $\tau/P$  where P = population.

Next assume facility requires regular maintenance

- & Replace P by  $\rho_{\mathsf{pop}} A$  where  $\rho_{\mathsf{pop}}$  is density.
- Important assumption: uniform density.
- Total average time cost per person:

Optimal source allocation

(person-hours per day).

& Call this quantity  $\tau$ .

$$T = \bar{d}/\bar{v} + \tau/(\rho_{\mathsf{pop}}A) = cA^{1/2}/\bar{v} + \tau/(\rho_{\mathsf{pop}}A).$$

 $\aleph$  Now Minimize with respect to  $A \dots$ 







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## Optimal source allocation

Differentiating ...

$$\begin{split} \frac{\partial T}{\partial A} &= \frac{\partial}{\partial A} \left( c A^{1/2}/\bar{v} + \tau/(\rho_{\mathsf{pop}} A) \right) \\ &= \frac{c}{2\bar{v} A^{1/2}} - \frac{\tau}{\rho_{\mathsf{pop}} A^2} = 0 \end{split}$$

Rearrange:

$$A = \left(\frac{2\bar{v}\tau}{c\rho_{\mathsf{pop}}}\right)^{2/3} \, \propto \rho_{\mathsf{pop}}^{-2/3}$$

 $\clubsuit$  # facilities per unit area  $\rho_{\text{fac}}$ :

$$ho_{
m fac} \propto A^{-1} \propto 
ho_{
m pop}^{2/3}$$

🚳 Groovy ...

## Optimal source allocation

#### An issue:

Cartograms

Standard world map:

- Maintenance  $(\tau)$  is assumed to be independent of population and area (P and A)
- Stephan's online book
  "The Division of Territory in Society" is here ☑.
- 🙈 (It used to be here 🗹.)
- ♣ The Readme is well worth reading (1995).





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Cartogram of countries 'rescaled' by population:



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### Cartograms

#### Diffusion-based cartograms:

- & Idea of cartograms is to distort areas to more accurately represent some local density  $\rho_{\rm pop}$  (e.g. population).
- Many methods put forward—typically involve some kind of physical analogy to spreading or repulsion.
- Algorithm due to Gastner and Newman (2004) [1] is based on standard diffusion:

$$\nabla^2 \rho_{\rm pop} - \frac{\partial \rho_{\rm pop}}{\partial t} = 0. \label{eq:pop_pop}$$

- Allow density to diffuse and trace the movement of individual elements and boundaries.

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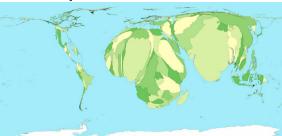
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Cartograms

# Child mortality:



## Cartograms

### Energy consumption:



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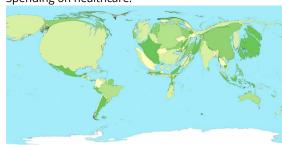
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### Spending on healthcare:

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# Cartograms

#### Gross domestic product:



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### Greenhouse gas emissions:

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# Cartograms

- A larger collection can be found at worldmapper.org ♂.

WSRLDMAPPER The world as you've never seen it be





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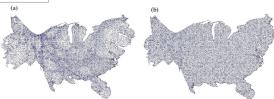
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### Size-density law



"Optimal design of spatial distribution networks"  $\square$ 

Gastner and Newman, Phys. Rev. E, **74**, 016117, 2006. [2]



- Left: population density-equalized cartogram.
- Right: (population density) $^{2/3}$ -equalized cartogram.
- $\mbox{\&}$  Facility density is uniform for  $ho_{
  m pop}^{2/3}$  cartogram.

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# Approximations:

- & For a given set of source placements  $\{\vec{x}_1,\dots,\vec{x}_n\}$ , the region  $\Omega$  is divided up into Voronoi cells  $\mathbb{Z}$ , one per source.
- $\Leftrightarrow$  Define  $A(\vec{x})$  as the area of the Voronoi cell containing  $\vec{x}$ .
- $\Leftrightarrow$  As per Stephan's calculation, estimate typical distance from  $\vec{x}$  to the nearest source (say i) as

$$c_i A(\vec{x})^{1/2}$$

where  $c_i$  is a shape factor for the ith Voronoi cell.

 $\clubsuit$  Approximate  $c_i$  as a constant c.



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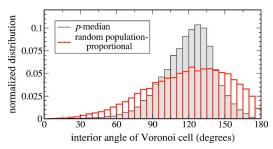
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# Size-density law



From Gastner and Newman (2006)<sup>[2]</sup>

Cartogram's Voronoi cells are somewhat hexagonal.

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# Size-density law

### Carrying on:

The cost function is now

$$F = c \int_{\Omega} \rho_{\rm pop}(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} \,. \label{eq:F_pop}$$

- We also have that the constraint that Voronoi cells divide up the overall area of  $\Omega$ :  $\sum_{i=1}^n A(\vec{x}_i) = A_{\Omega}$ .
- Sneakily turn this into an integral constraint:

$$\int_{\Omega} \frac{\mathrm{d}\vec{x}}{A(\vec{x})} = n.$$

- $\Re$  Within each cell,  $A(\vec{x})$  is constant.
- & So ...integral over each of the n cells equals 1.



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# Size-density law

#### Deriving the optimal source distribution:

- & Basic idea: Minimize the average distance from a random individual to the nearest facility. [2]
- & Assume given a fixed population density  $ρ_{pop}$  defined on a spatial region Ω.

$$F(\{\vec{x}_1,\ldots,\vec{x}_n\}) = \int_{\Omega} \rho_{\mathsf{pop}}(\vec{x}) \, \mathsf{min}_i ||\vec{x} - \vec{x}_i|| \mathrm{d}\vec{x} \,.$$

- Also known as the p-median problem.
- Not easy ...in fact this one is an NP-hard problem. [2]
- Approximate solution originally due to Gusein-Zade [3].

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# Now a Lagrange multiplier story:

 $\Re$  By varying  $\{\vec{x}_1, \dots, \vec{x}_n\}$ , minimize

$$G(A) = c \int_{\Omega} \rho_{\mathsf{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} - \lambda \left( n - \int_{\Omega} \left[ A(\vec{x}) \right]^{-1} \mathrm{d}\vec{x} \right) \underset{\text{Size density law and the properties}}{\operatorname{Size density law}} \int_{\mathbb{R}^{2}} \mathrm{d}\vec{x} d\vec{x} d\vec{x} d\vec{x} d\vec{x} = 0$$

♣ I Can Haz Calculus of Variations 
☐?

- & Compute  $\delta G/\delta A$ , the functional derivative  $\mathbb{Z}$  of the functional G(A).
- This gives

$$\int_{\Omega} \left[ \frac{c}{2} \rho_{\mathsf{pop}}(\vec{x}) A(\vec{x})^{-1/2} - \lambda \left[ A(\vec{x}) \right]^{-2} \right] \mathrm{d}\vec{x} \, = 0.$$

Setting the integrand to be zilch, we have:

$$\rho_{\rm pop}(\vec{x}) = 2\lambda c^{-1} A(\vec{x})^{-3/2}.$$





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### Size-density law

#### Now a Lagrange multiplier story:

Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho_{\text{pop}}^{-2/3}.$$

- $\Longrightarrow$  Finally, we indentify  $1/A(\vec{x})$  as  $\rho_{fac}(\vec{x})$ , an approximation of the local source density.
- Substituting  $\rho_{\mathsf{fac}} = 1/A$ , we have

$$\rho_{\rm fac}(\vec{x}) = \left(\frac{c}{2\lambda}\rho_{\rm pop}\right)^{2/3}. \label{eq:rhofac}$$

 $\aleph$  Normalizing (or solving for  $\lambda$ ):

$$\rho_{\rm fac}(\vec{x}) = n \frac{[\rho_{\rm pop}(\vec{x})]^{2/3}}{\int_{\Omega} [\rho_{\rm pop}(\vec{x})]^{2/3} {\rm d}\vec{x}} \propto [\rho_{\rm pop}(\vec{x})]^{2/3}.$$

#### COcoNuTS Public versus private facilities Optimal Supply Networks III

#### & "Scaling laws between population and facility densities" by Um et al., Proc. Natl. Acad. Sci., 2009. [6]

A Um et al. find empirically and argue theoretically that the connection between facility and population density

$$ho_{\mathsf{fac}} \propto 
ho_{\mathsf{pop}}^{lpha}$$

does not universally hold with  $\alpha = 2/3$ .

Two idealized limiting classes:

Beyond minimizing distances:

- 1. For-profit, commercial facilities:  $\alpha = 1$ ;
- 2. Pro-social, public facilities:  $\alpha = 2/3$ .
- Um et al. investigate facility locations in the United States and South Korea.

# Public versus Private

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## Global redistribution networks

#### One more thing:

- How do we supply these facilities?
- How do we best redistribute mail? People?
- How do we get beer to the pubs?
- Gastner and Newman model: cost is a function of basic maintenance and travel time:

$$C_{\text{maint}} + \gamma C_{\text{travel}}$$
.

Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance  $\ell_{ij}$  and number of legs to journey:

$$(1-\delta)\ell_{ij} + \delta(\#\mathsf{hops}).$$

& When  $\delta = 1$ , only number of hops matters.

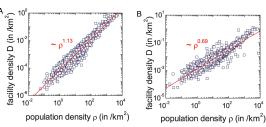
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# Public versus private facilities: evidence



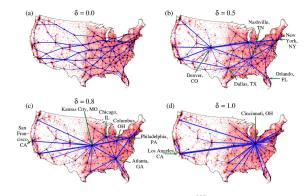
- COcoNuTS Optimal Supply Networks III
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- & Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.
- Note: break in scaling for public schools. Transition from  $\alpha \simeq 2/3$  to  $\alpha = 1$  around  $\rho_{\mathsf{pop}} \simeq 100$ .





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# Global redistribution networks



From Gastner and Newman (2006) [2]

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# Public versus private facilities: evidence

Beauty care         1.08(1)         0.86           Laundry         1.05(1)         0.90           Automotive repair         0.99(1)         0.92           Private school         0.95(1)         0.82           Restaurant         0.93(1)         0.70           Accommodation         0.89(1)         0.70           Bank         0.88(1)         0.89           Gas station         0.86(1)         0.94           Death care         0.79(1)         0.80           * Fire station         0.78(3)         0.93           * Police station         0.71(6)         0.75           Public school         0.69(1)         0.87           * SK facility         α (5E)         R²           Bank         1.18(2)         0.96           Parking place         1.13(2)         0.91           * Primary clinic         1.09(2)         1.00           * Hospital         0.96(5)         0.97           * Liniversity/college         0.93(9)         0.89           Market place         0.87(2)         0.90           * Secondary school         0.77(3)         0.98           * Primary school         0.77(3)         0.98 <t< th=""><th>US facility</th><th>α (SE)</th><th>R<sup>2</sup></th></t<>	US facility	α (SE)	R <sup>2</sup>
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Automotive repair 0.99(1) 0.92 Private school 0.95(1) 0.82 Restaurant 0.93(1) 0.89 Accommodation 0.89(1) 0.70 Bank 0.88(1) 0.89 Gas station 0.86(1) 0.94 Death care 0.79(1) 0.80 * Fire station 0.78(3) 0.93 * Police station 0.71(6) 0.75 Public school 0.69(1) 0.87 SK facility a (SE) R² Bank 1.18(2) 0.96 Parking place 1.13(2) 0.91 * Primary clinic 1.09(2) 1.00 * Primary clinic 1.09(2) 1.00 * Hospital 0.96(5) 0.97 * University/college 0.93(9) 0.89 Market place 0.87(2) 0.90 * Secondary school 0.77(3) 0.98 * Primary school 0.77(3) 0.97 Social welfare org. 0.75(2) 0.84 * Police station 0.71(5) 0.94 * Police station 0.71(5) 0.94 * Police station 0.70(1) 0.93 * Police station 0.00(4) 0.93	Beauty care	1.08(1)	0.86
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Gas station         0.86(1)         0.94           Death care         0.79(1)         0.80           * Fire station         0.78(3)         0.93           * Police station         0.71(6)         0.75           Public station         0.69(1)         0.87           SK facility         α (SE)         R²           Bank         1.18(2)         0.96           Parking place         1.13(2)         0.91           * Primary clinic         1.09(2)         1.00           * Hospital         0.96(5)         0.97           * University/college         0.93(9)         0.89           Market place         0.87(2)         0.90           * Secondary school         0.77(3)         0.98           * Primary school         0.77(3)         0.97           Social welfare org.         0.75(2)         0.84           * Police station         0.71(5)         0.94           Government office         0.70(1)         0.93           * Fire station         0.60(4)         0.93	Accommodation	0.89(1)	0.70
Death care 0.79(1) 0.80 * Fire station 0.78(3) 0.93 * Police station 0.71(6) 0.75 Public school 0.69(1) 0.87 SK facility α(5E) R² Sank 1.18(2) 0.96 Parking place 1.13(2) 0.91 * Primary clinic 1.09(2) 1.00 * Police station 0.96(5) 0.97 * University/college 0.93(9) 0.89 Market place 0.87(2) 0.90 Market place 0.87(2) 0.90 * University/college 0.93(9) 0.89 * Primary school 0.77(3) 0.98 * Primary school 0.77(3) 0.97 * Cocial welfare org. 0.75(2) 0.84 * Police station 0.71(5) 0.94 * Police station 0.70(1) 0.93 * Overnment office 0.70(1) 0.93 * Fire station 0.60(4) 0.93	Bank	0.88(1)	0.89
Fire station 0.78(3) 0.93  * Police station 0.71(6) 0.75  Public station 0.91(1) 0.87  SK facility α(SE) R²  Bank 1.18(2) 0.96  Parking place 1.13(2) 0.91  * Primary clinic 1.09(2) 1.00  * Hospital 0.96(5) 0.97  * University/college 0.93(9) 0.89  Market place 0.87(2) 0.90  * Secondary school 0.77(3) 0.98  * Primary school 0.77(3) 0.97  Social welfare org. 0.75(2) 0.84  * Police station 0.71(5) 0.94  Solice station 0.70(1) 0.93  * Fire station 0.60(4) 0.93	Gas station	0.86(1)	0.94
* Police station 0.71(6) 0.75 Public school 0.69(1) 0.87  SK facility α(SE) R²  Bank 1.18(2) 0.96 Parking place 1.13(2) 0.91 * Primary clinic 1.09(2) 1.00 * Hospital 0.96(5) 0.97 * University/college 0.93(9) 0.89 Market place 0.87(2) 0.90 * Secondary school 0.77(3) 0.98 * Primary school 0.77(3) 0.97 Social welfare org. 0.75(2) 0.84 * Police station 0.71(5) 0.94 Sovernment office 0.70(1) 0.93 * Fire station 0.60(4) 0.93	Death care	0.79(1)	0.80
Public school         0.69(1)         0.87           SK facility         α (SE)         R²           Sank         1.18(2)         0.96           Parking place         1.13(2)         0.91           P Primary clinic         1.09(2)         1.00           Hospital         0.96(5)         0.97           University/college         0.93(9)         0.89           Market place         0.87(2)         0.90           * Pecondary school         0.77(3)         0.98           * Primary school         0.77(3)         0.97           Social welfare org.         0.75(2)         0.84           * Police station         0.71(5)         0.94           Sovernment office         0.70(1)         0.93           * Fire station         0.60(4)         0.93	* Fire station	0.78(3)	0.93
SK facility α (SE) R²  Bank 1.18(2) 0.96  Parking place 1.13(2) 0.91  Primary clinic 1.09(2) 1.00  Hospital 0.96(5) 0.97  *University/college 0.93(9) 0.89  Market place 0.87(2) 0.90  *Secondary school 0.77(3) 0.98  *Primary school 0.77(3) 0.97  Social welfare org. 0.75(2) 0.84  *Police station 0.71(5) 0.94  Sovernment office 0.70(1) 0.93  Fire striction 0.60(4) 0.93	* Police station	0.71(6)	0.75
Sank 1.18(2) 0.96 Parking place 1.13(2) 0.91 Primary clinic 1.09(2) 1.00 Hospital 0.96(5) 0.97 University/college 0.93(9) 0.89 Market place 0.87(2) 0.90 Secondary school 0.77(3) 0.98 Primary school 0.77(3) 0.97 Social welfare org. 0.75(2) 0.84 Police station 0.71(5) 0.94 Sovernment office 0.70(1) 0.93 Fire tribution 0.60(4) 0.93	Public school	0.69(1)	0.87
Parking place 1.13(2) 0.91  P Primary clinic 1.09(2) 1.00  Hospital 0.96(5) 0.97  * University/college 0.93(9) 0.89  Market place 0.87(2) 0.90  * Secondary school 0.77(3) 0.98  * Primary school 0.77(3) 0.97  Social welfare org. 0.75(2) 0.84  * Police station 0.71(5) 0.94  Sovernment office 0.70(1) 0.93  Fire trie station 0.60(4) 0.93	SK facility	α (SE)	R <sup>2</sup>
№ Primary clinic         1.09(2)         1.00           № Hospital         0.96(5)         0.97           № University/college         0.93(9)         0.89           Market place         0.87(2)         0.90           № Fecondary school         0.77(3)         0.98           № Primary school         0.77(3)         0.97           Social welfare org.         0.75(2)         0.84           № Police station         0.71(5)         0.94           Government office         0.70(1)         0.93           F Fire station         0.60(4)         0.93	Bank	1.18(2)	0.96
* Hospital 0.96(5) 0.97 * University/college 0.93(9) 0.89 Market place 0.87(2) 0.90 * Secondary school 0.77(3) 0.98 * Primary school 0.77(3) 0.97 Social welfare org. 0.75(2) 0.84 * Police station 0.71(5) 0.94 Government office 0.70(1) 0.93 * Fire station 0.60(4) 0.93	Parking place	1.13(2)	0.91
* University/college         0.3(9)         0.89           Market place         0.87(2)         0.90           * Secondary school         0.77(3)         0.98           * Primary school         0.77(3)         0.97           Social welfare org.         0.75(2)         0.84           * Police station         0.71(5)         0.94           Government office         0.70(1)         0.93           * Fire station         0.60(4)         0.93	* Primary clinic	1.09(2)	1.00
Market place         0.87(2)         0.90           * Secondary school         0.77(3)         0.98           * Primary school         0.77(3)         0.97           Social welfare org.         0.75(2)         0.84           * Police station         0.71(5)         0.94           Government office         0.70(1)         0.93           * Fire station         0.60(4)         0.93	* Hospital	0.96(5)	0.97
* Secondary school 0.77(3) 0.98 * Primary school 0.77(3) 0.97 Social welfare org. 0.75(2) 0.84 * Police station 0.71(5) 0.94 Sovernment office 0.70(1) 0.93 Fire station 0.60(4) 0.93	* University/college	0.93(9)	0.89
P Frimary school         0.77(3)         0.97           Social welfare org.         0.75(2)         0.84           P Police station         0.71(5)         0.94           Government office         0.70(1)         0.93           F Fire station         0.60(4)         0.93	Market place	0.87(2)	0.90
Social welfare org.         0.75(2)         0.84           * Police station         0.71(5)         0.94           Government office         0.70(1)         0.93           * Fire station         0.60(4)         0.93	* Secondary school	0.77(3)	0.00
* Police station 0.71(5) 0.94 Government office 0.70(1) 0.93 * Fire station 0.60(4) 0.93		0.77(3)	0.98
Government office 0.70(1) 0.93 * Fire station 0.60(4) 0.93	* Primary school		
* Fire station 0.60(4) 0.93		0.77(3)	0.97
	Social welfare org.	0.77(3) 0.75(2)	0.97 0.84
	* Primary school Social welfare org. * Police station Government office	0.77(3) 0.75(2) 0.71(5)	0.97 0.84 0.94
	Social welfare org. * Police station	0.77(3) 0.75(2) 0.71(5) 0.70(1)	0.97 0.84 0.94 0.93

Rough transition between public and private at  $\alpha \simeq 0.8$ .

Note: \* indicates analysis is at state/province level; otherwise county level.

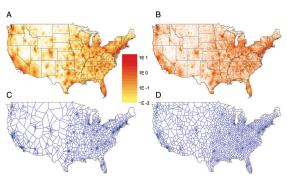
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## Public versus private facilities: evidence



A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.

# Public versus private facilities: the story So what's going on?

- Social institutions seek to minimize distance of travel.
- Commercial institutions seek to maximize the number of visitors.
- & Defns: For the ith facility and its Voronoi cell  $V_i$ , define
  - $n_i$  = population of the *i*th cell;
  - $\langle r_i \rangle$  = the average travel distance to the *i*th facility.
  - $A_i$  = area of *i*th cell ( $s_i$  in Um *et al.* [6])
- Objective function to maximize for a facility (highly constructed):

$$v_i = n_i \langle r_i \rangle^{\beta}$$
 with  $0 \le \beta \le 1$ .

- & Limits:
  - $\beta = 0$ : purely commercial.
  - $\beta = 1$ : purely social.

# Public versus private facilities: the story

Either proceeding as per the Gastner-Newman-Gusein-Zade calculation or, as Um et al. do, observing that the cost for each cell should be the same, we have:

$$\rho_{\rm fac}(\vec{x}) = n \frac{[\rho_{\rm pop}(\vec{x})]^{2/(\beta+2)}}{\int_{\Omega} [\rho_{\rm pop}(\vec{x})]^{2/(\beta+2)} \mathrm{d}\vec{x}} \propto [\rho_{\rm pop}(\vec{x})]^{2/(\beta+2)}. \label{eq:rhofac}$$

- $\Re$  For  $\beta = 0$ ,  $\alpha = 1$ : commercial scaling is linear.
- $\Re$  For  $\beta = 1$ ,  $\alpha = 2/3$ : social scaling is sublinear.
- You can try this too:
  Insert question from assignment 4

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