Mixed, correlated random networks

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Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2019

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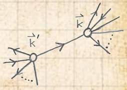

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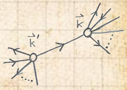

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Outline

Directed random networks

Mixed random networks
- Definition
- Correlations

Mixed Random Network Contagion
- Spreading condition
- Full generalization
- Triggering probabilities

Nutshell

References


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
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THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
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THE RATIO OF GAIN
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THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


"I am the Monarch of the Sea."
N,m
THE ARK OF THE RANDOM NETWORK


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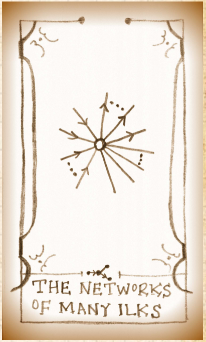
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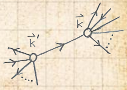
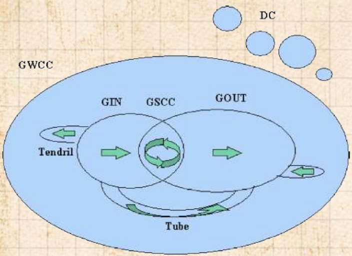
- Required balance:

$$\langle k_\mathrm{i} \rangle = \sum_{k_\mathrm{i}=0}^{\infty}\sum_{k_\mathrm{o}=0}^{\infty} k_\mathrm{i} P_{k_\mathrm{i},k_\mathrm{o}} = \sum_{k_\mathrm{i}=0}^{\infty}\sum_{k_\mathrm{o}=0}^{\infty} k_\mathrm{o} P_{k_\mathrm{i},k_\mathrm{o}} = \langle k_\mathrm{o} \rangle$$



Directed network structure:



From Boguñá and Serano.[1]

- GWCC = Giant Weakly Connected Component (directions removed);
- GIN = Giant In-Component;
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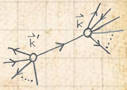
- GSCC = Giant Strongly Connected Component;
- DC = Disconnected Components (finite).



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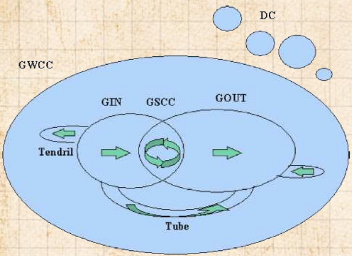

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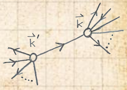
- When moving through a family of increasingly connected directed random networks, GWCC usually appears before GIN, GOUT, and GSCC which tend to appear together.[4, 1]

Outline



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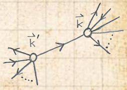
- Directed and undirected random networks are separate families ...



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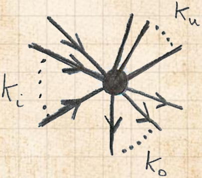

- Consider nodes with three types of edges:
  1. $k_{\rm u}$ undirected edges,
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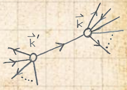

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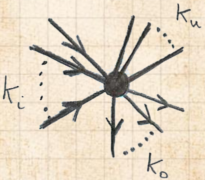

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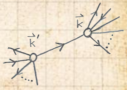
- Define a node by generalized degree:

$$\vec{k} = [\, k_{\mathrm{u}} \;\; k_{\mathrm{i}} \;\; k_{\mathrm{o}} \,]^{\mathsf{T}}.$$



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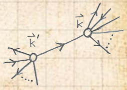

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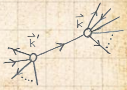

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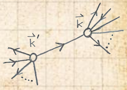

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- Otherwise, no other restrictions and connections are random.
- Directed and undirected random networks are disjoint subfamilies:

$$\text{Undirected: } P_{\vec{k}} = P_{k_{\mathsf{u}}} \delta_{k_{\mathsf{i}},0} \delta_{k_{\mathsf{o}},0},$$

$$\text{Directed: } P_{\vec{k}} = \delta_{k_{\mathsf{u}},0} P_{k_{\mathsf{i}},k_{\mathsf{o}}}.$$

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Mixed Random Network Contagion

- Spreading condition
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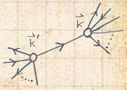

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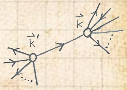

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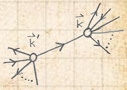

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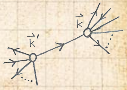
- Now require more refined (detailed) balance.



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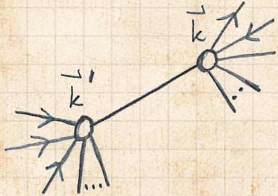
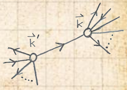
Correlations—Undirected edge balance:

- Randomly choose an edge, and randomly choose one end.





Correlations—Undirected edge balance:

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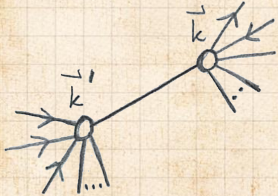

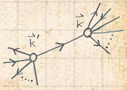

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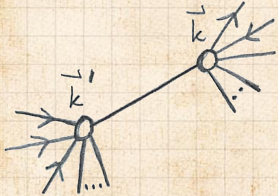

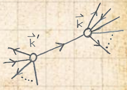

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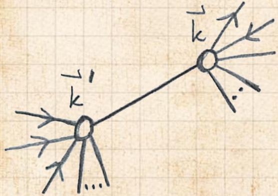

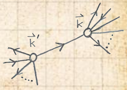

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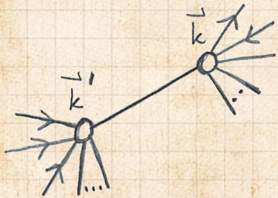

- Conditional probability connection:

$$P^{(\mathrm{u})}(\vec{k}, \vec{k}') = P^{(\mathrm{u})}(\vec{k} \,|\, \vec{k}') \frac{k'_{\mathrm{u}} P(\vec{k}')}{\langle k'_{\mathrm{u}} \rangle}$$

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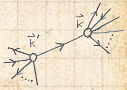

Correlations—Directed edge balance:

- The quantities

$$\frac{k_{\mathrm{o}} P(\vec{k})}{\langle k_{\mathrm{o}} \rangle} \text{ and } \frac{k_{\mathrm{i}} P(\vec{k})}{\langle k_{\mathrm{i}} \rangle}$$

give the probabilities that in starting at a random end of a randomly selected edge, we begin at a degree $\vec{k}$ node and then find ourselves travelling:

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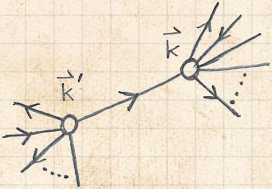

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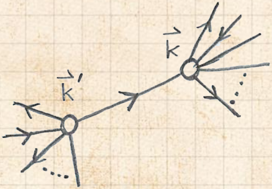

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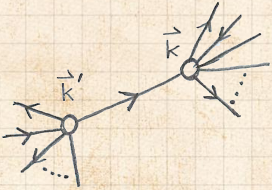

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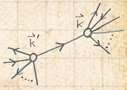


Outline

Directed random networks

Mixed random networks
- Definition
- Correlations

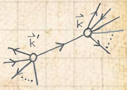
Mixed Random Network Contagion
- Spreading condition
- Full generalization
- Triggering probabilities

Nutshell

References

Global spreading condition: [2]

When are cascades possible?:



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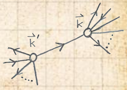

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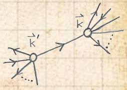

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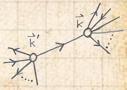

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- Both are composed of (1) probability of connection to a node of a given type; (2) number of newly infected edges if successful; and (3) probability of infection.

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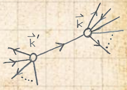
Local growth equation:

- Define number of infected edges leading to nodes a distance d away from the original seed as $f(d)$.



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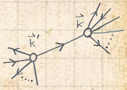

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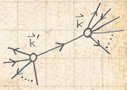

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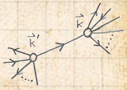

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Mixed, uncorrelated random netwoks:

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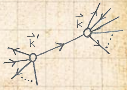

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  1. Infected directed edges can lead to infected directed or undirected edges.
  2. Infected undirected edges can lead to infected directed or undirected edges.
- Define $f^{(\mathrm{u})}(d)$ and $f^{(\mathrm{o})}(d)$ as the expected number of infected undirected and directed edges leading to nodes a distance d from seed.

Gain ratio now has a matrix form:

$$\begin{bmatrix} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{bmatrix}$$

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$$\left[\begin{array}{c} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{array}\right] = \mathbf{R} \left[\begin{array}{c} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{array}\right]$$

- Two separate gain equations:

$$f^{(\mathrm{u})}(d+1) = \sum_{\vec{k}} \left[\frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{u})}(d) + \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{u}} \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{o})}(\right.$$

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$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} \frac{k_{\mathsf{u}} P_{\vec{k}}}{\langle k_{\mathsf{u}} \rangle} \bullet (k_{\mathsf{u}} - 1) & \frac{k_{\mathsf{i}} P_{\vec{k}}}{\langle k_{\mathsf{i}} \rangle} \bullet k_{\mathsf{u}} \\ \frac{k_{\mathsf{u}} P_{\vec{k}}}{\langle k_{\mathsf{u}} \rangle} \bullet k_{\mathsf{o}} & \frac{k_{\mathsf{i}} P_{\vec{k}}}{\langle k_{\mathsf{i}} \rangle} \bullet k_{\mathsf{o}} \end{bmatrix} \bullet B_{k_{\mathsf{u}}+k_{\mathsf{i}},1}$$

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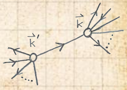
- Spreading condition: max eigenvalue of $\mathbf{R} > 1$.

Global spreading condition:

- Useful change of notation for making results more general: write $P^{(\mathrm{u})}(\vec{k} \,|\, *) = \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle}$ and $P^{(\mathrm{i})}(\vec{k} \,|\, *) = \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle}$ where $*$ indicates the starting node's degree is irrelevant (no correlations).



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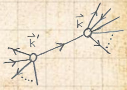

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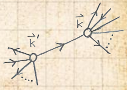
- Also write $B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$ to indicate a more general infection probability, but one that does not depend on the edge's origin.
- Now have, for the example of mixed, uncorrelated random networks:

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet (k_{\mathrm{u}} - 1) & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{u}} \\ P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$$



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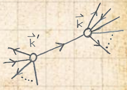

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- III. Mixed Directed and Undirected, Uncorrelated—

$$\begin{bmatrix} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{bmatrix}$$

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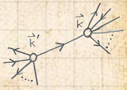

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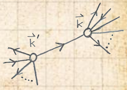

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Summary of contagion conditions for correlated networks:

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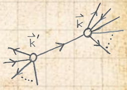
Outline



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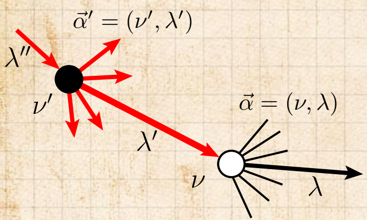

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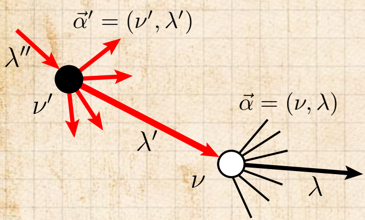

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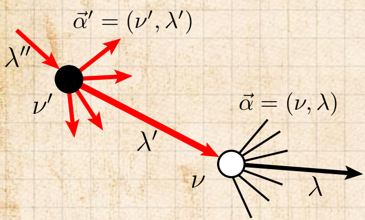

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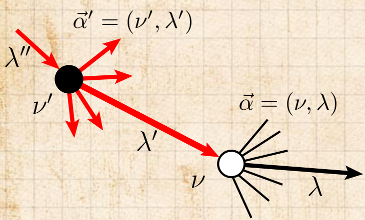

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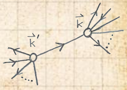

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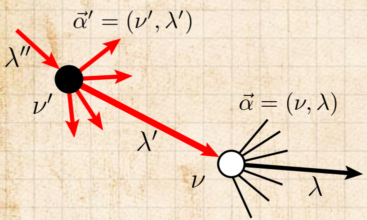

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- Generalized contagion condition:

$$\max|\mu| : \mu \in \sigma(\mathbf{R}) > 1$$

Outline

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- Generating functions arguably make some kinds of calculations easier (but perhaps we don't care about component sizes that much).
- On the other hand, a plainspoken physical argument helps us generalize to correlated networks more easily.

Summary of triggering probabilities for uncorrelated networks: [3]

- I. Undirected, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_{\text{u}}} P^{(\text{u})}(k'_{\text{u}} \,|\, \cdot) B_{k'_{\text{u}}1} \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{u}}-1}\right]$$

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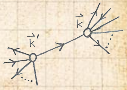

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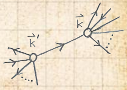

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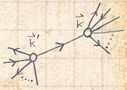
Nutshell:

- Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.



Nutshell:

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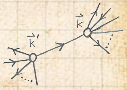
- Spreading conditions and triggering probabilities of contagion processes can be determined using a direct, physical approach.



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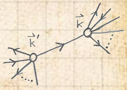

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- More generalizations: bipartite affiliation graphs and multilayer networks.

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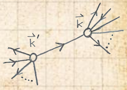

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