Mixed, correlated random networks

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Outline

Random directed networks:

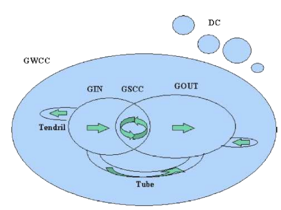
- So far, we've largely studied networks with undirected, unweighted edges.
- Now consider directed, unweighted edges.
- Nodes have $k_{\rm i}$ and $k_{\rm o}$ incoming and outgoing edges, otherwise random.
- Network defined by joint in- and out-degree distribution: $P_{k_{\rm i},k_{\rm o}}$
- Normalization: $\sum_{k_{\rm i}=0}^{\infty}\sum_{k_{\rm o}=0}^{\infty} P_{k_{\rm i},k_{\rm o}} = 1$
- Marginal in-degree and out-degree distributions:

$$P_{k_{\rm i}} = \sum_{k_{\rm o}=0}^{\infty} P_{k_{\rm i},k_{\rm o}} \text{ and } P_{k_{\rm o}} = \sum_{k_{\rm i}=0}^{\infty} P_{k_{\rm i},k_{\rm o}}$$

- Required balance:

$$\langle k_{\rm i}\rangle = \sum_{k_{\rm i}=0}^{\infty}\sum_{k_{\rm o}=0}^{\infty} k_{\rm i} P_{k_{\rm i},k_{\rm o}} = \sum_{k_{\rm i}=0}^{\infty}\sum_{k_{\rm o}=0}^{\infty} k_{\rm o} P_{k_{\rm i},k_{\rm o}} = \langle k_{\rm o}\rangle$$

Directed network structure:

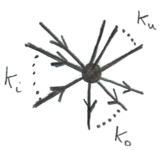


From Boguñá and Serano.[1]

- GWCC = Giant Weakly Connected Component (directions removed);
- GIN = Giant In-Component;
- GOUT = Giant Out-Component;
- GSCC = Giant Strongly Connected Component;
- DC = Disconnected Components (finite).
- When moving through a family of increasingly connected directed random networks, GWCC usually appears before GIN, GOUT, and GSCC which tend to appear together.[4, 1]

Important observation:

- Directed and undirected random networks are separate families ...
- ...and analyses are also disjoint.
- Need to examine a larger family of random networks with mixed directed and undirected edges.



- Consider nodes with three types of edges:
  1. k_{u} undirected edges,
  2. k_{i} incoming directed edges,
  3. k_{o} outgoing directed edges.
- Define a node by generalized degree:

$$\vec{k} = [\, k_{\mathrm{u}} \;\; k_{\mathrm{i}} \;\; k_{\mathrm{o}} \,]^{\mathsf{T}}.$$

- Joint degree distribution:

$$P_{\vec{k}} \text{ where } \vec{k} = [\, k_{\mathrm{u}} \;\; k_{\mathrm{i}} \;\; k_{\mathrm{o}} \,]^{\mathsf{T}}.$$

- As for directed networks, require in- and out-degree averages to match up:

$$\langle k_{\mathrm{i}} \rangle = \sum_{k_{\mathrm{u}}=0}^{\infty} \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{i}} P_{\vec{k}} = \sum_{k_{\mathrm{u}}=0}^{\infty} \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{o}} P_{\vec{k}} = \langle k_{\mathrm{o}} \rangle$$

- Otherwise, no other restrictions and connections are random.
- Directed and undirected random networks are disjoint subfamilies:

$$\text{Undirected: } P_{\vec{k}} = P_{k_{\mathrm{u}}} \delta_{k_{\mathrm{i}},0} \delta_{k_{\mathrm{o}},0},$$

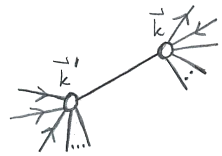
$$\text{Directed: } P_{\vec{k}} = \delta_{k_{\mathrm{u}},0} P_{k_{\mathrm{i}},k_{\mathrm{o}}}.$$

Correlations:

- Now add correlations (two point or Markovian) ☐:
  1. $P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')$ = probability that an undirected edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node.
  2. $P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an in-directed edge relative to the destination node.
  3. $P^{(\mathrm{o})}(\vec{k}\,|\,\vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an out-directed edge relative to the destination node.
- Now require more refined (detailed) balance.
- Conditional probabilities cannot be arbitrary.
  1. $P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')$ must be related to $P^{(\mathrm{u})}(\vec{k}'\,|\,\vec{k})$.
  2. $P^{(\mathrm{o})}(\vec{k}\,|\,\vec{k}')$ and $P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}')$ must be connected.

Correlations—Undirected edge balance:

- Randomly choose an edge, and randomly choose one end.
- Say we find a degree $\vec{k}$ node at this end, and a degree $\vec{k}'$ node at the other end.
- Define probability this happens as $P^{(\mathrm{u})}(\vec{k}, \vec{k}')$.
- Observe we must have $P^{(\mathrm{u})}(\vec{k}, \vec{k}') = P^{(\mathrm{u})}(\vec{k}', \vec{k})$.



- Conditional probability connection:

$$P^{(\mathrm{u})}(\vec{k}, \vec{k}') = P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}') \frac{k'_{\mathrm{u}} P(\vec{k}')}{\langle k'_{\mathrm{u}} \rangle}$$

$$P^{(\mathrm{u})}(\vec{k}', \vec{k}) = P^{(\mathrm{u})}(\vec{k}'\,|\,\vec{k}) \frac{k_{\mathrm{u}} P(\vec{k})}{\langle k_{\mathrm{u}} \rangle}.$$

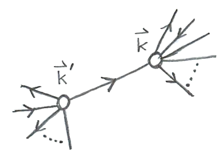
Correlations—Directed edge balance:

- The quantities

$$\frac{k_{\mathrm{o}} P(\vec{k})}{\langle k_{\mathrm{o}} \rangle} \text{ and } \frac{k_{\mathrm{i}} P(\vec{k})}{\langle k_{\mathrm{i}} \rangle}$$

give the probabilities that in starting at a random end of a randomly selected edge, we begin at a degree $\vec{k}$ node and then find ourselves travelling:

1. along an outgoing edge, or
2. against the direction of an incoming edge.



- We therefore have

$$P^{(\mathrm{dir})}(\vec{k}, \vec{k}') = P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}') \frac{k'_{\mathrm{o}} P(\vec{k}')}{\langle k'_{\mathrm{o}} \rangle} = P^{(\mathrm{o})}(\vec{k}'\,|\,\vec{k}) \frac{k_{\mathrm{i}} P(\vec{k})}{\langle k_{\mathrm{i}} \rangle}.$$

- Note that $P^{(\mathrm{dir})}(\vec{k}, \vec{k}')$ and $P^{(\mathrm{dir})}(\vec{k}', \vec{k})$ are in general not related if $\vec{k} \neq \vec{k}'$.

Global spreading condition: [2]

When are cascades possible?:

- Consider uncorrelated mixed networks first.
- Recall our first result for undirected random networks, that edge gain ratio must exceed 1:

$$\mathbf{R} = \sum_{k_{\mathrm{u}}=0}^{\infty} \frac{k_{\mathrm{u}} P_{k_{\mathrm{u}}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}},1} > 1.$$

- Similar form for purely directed networks:

$$\mathbf{R} = \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} \frac{k_{\mathrm{i}} P_{k_{\mathrm{i}},k_{\mathrm{o}}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{o}} \bullet B_{k_{\mathrm{i}},1} > 1.$$

- Both are composed of (1) probability of connection to a node of a given type; (2) number of newly infected edges if successful; and (3) probability of infection.

Global spreading condition:

Local growth equation:

- Define number of infected edges leading to nodes a distance d away from the original seed as $f(d)$.
- Infected edge growth equation:

$$f(d+1) = \mathbf{R} f(d).$$

- Applies for discrete time and continuous time contagion processes.
- Now see $B_{k_{\mathrm{u}},1}$ is the probability that an infected edge eventually infects a node.
- Also allows for recovery of nodes (SIR).

Global spreading condition:

Mixed, uncorrelated random netwoks:

- Now have two types of edges spreading infection: directed and undirected.
- Gain ratio now more complicated:
  1. Infected directed edges can lead to infected directed or undirected edges.
  2. Infected undirected edges can lead to infected directed or undirected edges.
- Define $f^{(\mathrm{u})}(d)$ and $f^{(\mathrm{o})}(d)$ as the expected number of infected undirected and directed edges leading to nodes a distance d from seed.

- Gain ratio now has a matrix form:

$$\begin{bmatrix} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{bmatrix}$$

- Two separate gain equations:

$$f^{(\mathrm{u})}(d+1) = \sum_{\vec{k}} \left[\frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{u})}(d) + \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{u}} \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{o})}(d) \right]$$

$$f^{(\mathrm{o})}(d+1) = \sum_{\vec{k}} \left[\frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet k_{\mathrm{o}} B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{u})}(d) + \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{o}} \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{o})}(d) \right]$$

- Gain ratio matrix:

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) & \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{u}} \\ \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet k_{\mathrm{o}} & \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1}$$

- Spreading condition: max eigenvalue of $\mathbf{R} > 1$.

Global spreading condition:

- Useful change of notation for making results more general: write $P^{(\mathrm{u})}(\vec{k} \,|\, *) = \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle}$ and $P^{(\mathrm{i})}(\vec{k} \,|\, *) = \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle}$ where $*$ indicates the starting node's degree is irrelevant (no correlations).
- Also write $B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$ to indicate a more general infection probability, but one that does not depend on the edge's origin.
- Now have, for the example of mixed, uncorrelated random networks:

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathrm{u})}(\vec{k} \,|\, *) \bullet (k_{\mathrm{u}} - 1) & P^{(\mathrm{i})}(\vec{k} \,|\, *) \bullet k_{\mathrm{u}} \\ P^{(\mathrm{u})}(\vec{k} \,|\, *) \bullet k_{\mathrm{o}} & P^{(\mathrm{i})}(\vec{k} \,|\, *) \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$$

Summary of contagion conditions for uncorrelated networks:

- I. Undirected, Uncorrelated—$f(d+1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_{\mathrm{u}}} P^{(\mathrm{u})}(k_{\mathrm{u}} \,|\, *) \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}},*}$$

- II. Directed, Uncorrelated—$f(d+1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_{\mathrm{i}}, k_{\mathrm{o}}} P^{(\mathrm{i})}(k_{\mathrm{i}}, k_{\mathrm{o}} \,|\, *) \bullet k_{\mathrm{o}} \bullet B_{k_{\mathrm{i}},*}$$

- III. Mixed Directed and Undirected, Uncorrelated—

$$\begin{bmatrix} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{bmatrix}$$

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathrm{u})}(\vec{k} \,|\, *) \bullet (k_{\mathrm{u}} - 1) & P^{(\mathrm{i})}(\vec{k} \,|\, *) \bullet k_{\mathrm{u}} \\ P^{(\mathrm{u})}(\vec{k} \,|\, *) \bullet k_{\mathrm{o}} & P^{(\mathrm{i})}(\vec{k} \,|\, *) \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$$

Correlated version:

- Now have to think of transfer of infection from edges emanating from degree $\vec{k}'$ nodes to edges emanating from degree $\vec{k}$ nodes.
- Replace $P^{(\mathrm{i})}(\vec{k} \,|\, *)$ with $P^{(\mathrm{i})}(\vec{k} \,|\, \vec{k}')$ and so on.
- Edge types are now more diverse beyond directed and undirected as originating node type matters.
- Sums are now over $\vec{k}'$.

Summary of contagion conditions for correlated networks:

- IV. Undirected, Correlated—$f_{k_u}(d+1) = \sum_{k'_u} R_{k_u k'_u} f_{k'_u}(d)$

$$R_{k_u k'_u} = P^{(u)}(k_u \,|\, k'_u) \bullet (k_u - 1) \bullet B_{k_u k'_u}$$

- V. Directed, Correlated—$f_{k_i k_o}(d+1) = \sum_{k'_i, k'_o} R_{k_i k_o k'_i k'_o} f_{k'_i k'_o}(d)$

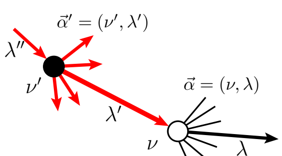
$$R_{k_i k_o k'_i k'_o} = P^{(i)}(k_i, k_o \,|\, k'_i, k'_o) \bullet k_o \bullet B_{k_i k_o k'_i k'_o}$$

- VI. Mixed Directed and Undirected, Correlated—

$$\begin{bmatrix} f^{(u)}_{\vec{k}}(d+1) \\ f^{(o)}_{\vec{k}}(d+1) \end{bmatrix} = \sum_{\vec{k}'} \mathbf{R}_{\vec{k}\vec{k}'} \begin{bmatrix} f^{(u)}_{\vec{k}'}(d) \\ f^{(o)}_{\vec{k}'}(d) \end{bmatrix}$$

$$\mathbf{R}_{\vec{k}\vec{k}'} = \begin{bmatrix} P^{(u)}(\vec{k} \,|\, \vec{k}') \bullet (k_u - 1) & P^{(i)}(\vec{k} \,|\, \vec{k}') \bullet k_u \\ P^{(u)}(\vec{k} \,|\, \vec{k}') \bullet k_o & P^{(i)}(\vec{k} \,|\, \vec{k}') \bullet k_o \end{bmatrix} \bullet B_{\vec{k}\vec{k}'}$$

Full generalization:



$$f_{\vec{\alpha}}(d+1) = \sum_{\vec{\alpha}'} R_{\vec{\alpha}\vec{\alpha}'} f_{\vec{\alpha}'}(d)$$

$R_{\vec{\alpha}\vec{\alpha}'}$ is the gain ratio matrix and has the form:

$$R_{\vec{\alpha}\vec{\alpha}'} = P_{\vec{\alpha}\vec{\alpha}'} \bullet k_{\vec{\alpha}\vec{\alpha}'} \bullet B_{\vec{\alpha}\vec{\alpha}'}.$$

- $P_{\vec{\alpha}\vec{\alpha}'}$ = conditional probability that a type λ' edge emanating from a type ν' node leads to a type ν node.
- $k_{\vec{\alpha}\vec{\alpha}'}$ = potential number of newly infected edges of type λ emanating from nodes of type ν.
- $B_{\vec{\alpha}\vec{\alpha}'}$ = probability that a type ν node is eventually infected by a single infected type λ' link arriving from a neighboring node of type ν'.
- Generalized contagion condition:

$$\max|\mu| : \mu \in \sigma(\mathbf{R}) > 1$$

- As we saw earlier, the triggering probability for simple contagion on random networks can be determined with a straightforward physical argument.
- Two good things:

$$Q_{\text{trig}} = \sum_{k=0}^{\infty} \frac{k P_k}{\langle k \rangle} \bullet B_{k1} \bullet \left[1 - \left(1 - Q_{\text{trig}}\right)^{k-1}\right],$$

$$P_{\text{trig}} = S_{\text{trig}} = \sum_k P_k \bullet \left[1 - (1 - Q_{\text{trig}})^k\right].$$

- Equivalent to result found via the eldritch route of generating functions.
- Generating functions arguably make some kinds of calculations easier (but perhaps we don't care about component sizes that much).
- On the other hand, a plainspoken physical argument helps us generalize to correlated networks more easily.

Summary of triggering probabilities for uncorrelated networks:[3]

- I. Undirected, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_u} P^{(u)}(k'_u \,|\, \cdot) B_{k'_u 1} \left[1 - (1 - Q_{\text{trig}})^{k'_u - 1}\right]$$

$$P_{\text{trig}} = S_{\text{trig}} = \sum_{k'_u} P(k'_u) \left[1 - (1 - Q_{\text{trig}})^{k'_u}\right]$$

- II. Directed, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_i, k'_o} P^{(u)}(k'_i, k'_o | \cdot) B_{k'_i 1} \left[1 - (1 - Q_{\text{trig}})^{k'_o}\right]$$

$$S_{\text{trig}} = \sum_{k'_i, k'_o} P(k'_i, k'_o) \left[1 - (1 - Q_{\text{trig}})^{k'_o}\right]$$

Summary of triggering probabilities for uncorrelated networks:

- III. Mixed Directed and Undirected, Uncorrelated—

$$Q^{(u)}_{\text{trig}} = \sum_{\vec{k}'} P^{(u)}(\vec{k}' | \cdot) B_{\vec{k}' 1} \left[1 - (1 - Q^{(u)}_{\text{trig}})^{k'_u - 1} (1 - Q^{(o)}_{\text{trig}})^{k'_o}\right]$$

$$Q^{(o)}_{\text{trig}} = \sum_{\vec{k}'} P^{(i)}(\vec{k}' | \cdot) B_{\vec{k}' 1} \left[1 - (1 - Q^{(u)}_{\text{trig}})^{k'_u} (1 - Q^{(o)}_{\text{trig}})^{k'_o}\right]$$

$$S_{\text{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q^{(u)}_{\text{trig}})^{k'_u} (1 - Q^{(o)}_{\text{trig}})^{k'_o}\right]$$

Summary of triggering probabilities for correlated networks:

- IV. Undirected, Correlated— $Q_{\text{trig}}(k_u) = \sum_{k'_u} P^{(u)}(k'_u \,|\, k_u) B_{k'_u 1} \left[1 - (1 - Q_{\text{trig}}(k'_u))^{k'_u - 1}\right]$

$$S_{\text{trig}} = \sum_{k'_u} P(k'_u) \left[1 - (1 - Q_{\text{trig}}(k'_u))^{k'_u}\right]$$

- V. Directed, Correlated— $Q_{\text{trig}}(k_i, k_o) = \sum_{k'_i, k'_o} P^{(u)}(k'_i, k'_o | \, k_i, k_o) B_{k'_i 1} \left[1 - (1 - Q_{\text{trig}}(k'_i, k'_o))^{k'_o}\right]$

$$S_{\text{trig}} = \sum_{k'_i, k'_o} P(k'_i, k'_o) \left[1 - (1 - Q_{\text{trig}}(k'_i, k'_o))^{k'_o}\right]$$

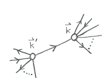
Summary of triggering probabilities for correlated networks:

- VI. Mixed Directed and Undirected, Correlated—

$$Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}-1}(1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

$$Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}}(1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}}(1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$



Nutshell:

- Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.
- Spreading conditions and triggering probabilities of contagion processes can be determined using a direct, physical approach.
- These conditions can be generalized to arbitrary random networks with arbitrary node and edge types.
- More generalizations: bipartite affiliation graphs and multilayer networks.

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Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
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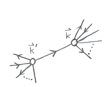

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