# Generating Functions and Networks

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## Complex Networks | @networksvox
## CSYS/MATH 303, Spring, 2019


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Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont










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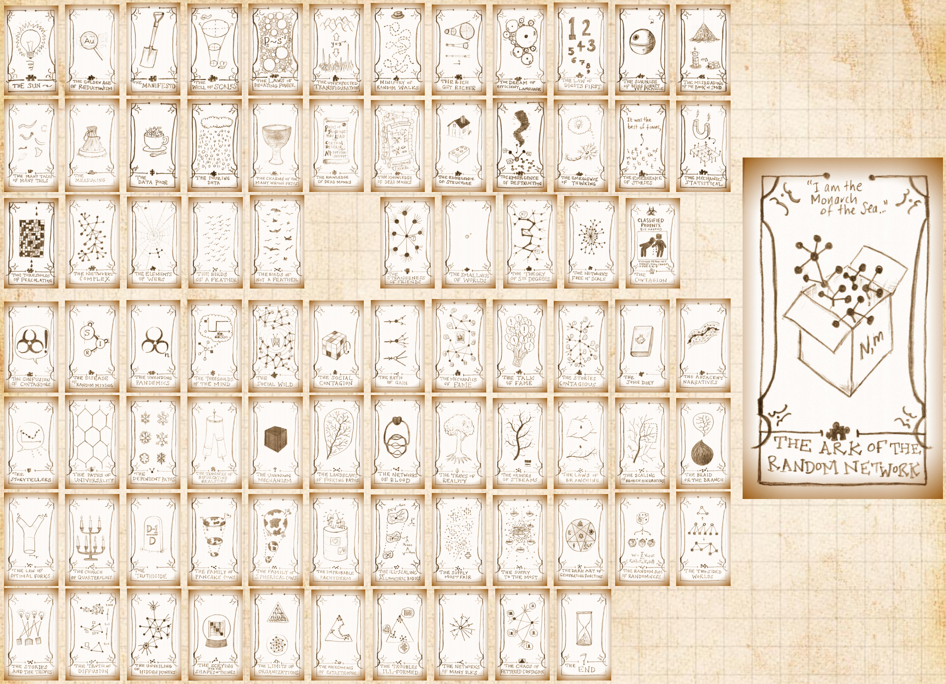
# Outline

Generating Functions
- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- A few examples
- Average Component Size

References




THE SUN
THE GOLDEN AGE OF REDUCTIONISM
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THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
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THE EMERGENCE OF STRUCTURE
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THE NETWORKS COMPLEX
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THE THEORY OF SIX DEGREES
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"I am the Monarch of the Sea."
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# Outline





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- Idea: Given a sequence $a_0, a_1, a_2, \dots,$ associate each element with a distinct function or other mathematical object.





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- Roughly: transforms a vector in $R^\infty$ into a function defined on $R^1$.
- Related to Fourier, Laplace, Mellin, ...





# Simple examples:

## Rolling dice and flipping coins:

$p_k^{(\epsdice{3})} = \mathbf{Pr}(\text{throwing a } k) = 1/6$ where $k = 1, 2, \ldots, 6$.

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- We'll come back to these simple examples as we derive various delicious properties of generating functions.



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- $k$th element of sequence (general):

$$P_k = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) \bigg|_{x=0}$$





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  3. Add a coin flip to one die roll.

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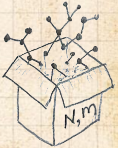

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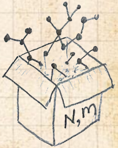

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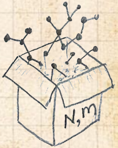

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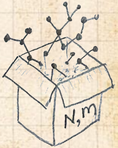

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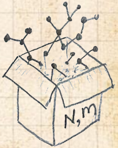

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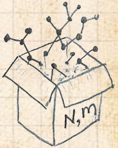

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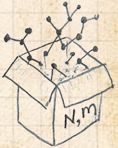

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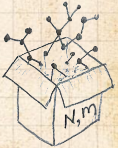

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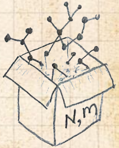
- Now find how $F_R$ is related to $F_P$ ...







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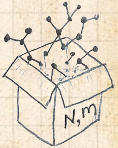

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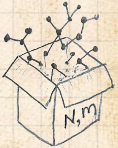

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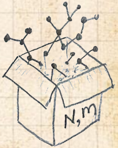

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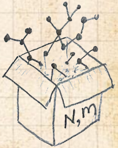

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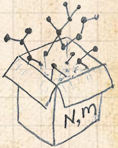

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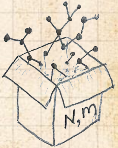

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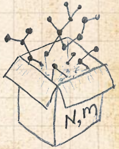

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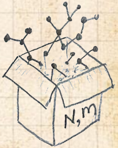

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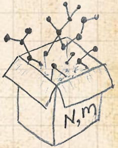
Finally, since $\langle k \rangle = F_P'(1)$,

$$\boxed{F_R(x) = \frac{F_P'(x)}{F_P'(1)}}$$







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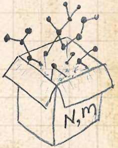
- Recall giant component condition is $\langle k \rangle_R = F_R'(1) > 1$.







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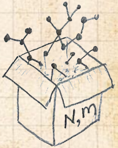
- Recall giant component condition is $\langle k \rangle_R = F'_R(1) > 1$.
- Since we have $F_R(x) = F'_P(x)/F'_P(1)$,







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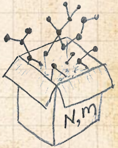

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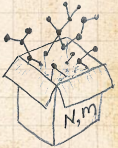
- Setting $x = 1$, our condition becomes

$$\boxed{\frac{F''_P(1)}{F'_P(1)} > 1}$$



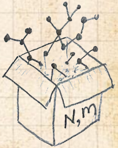




# Outline





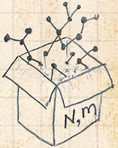


# Size distributions

To figure out the size of the largest component ($S_1$), we need more resolution on component sizes.







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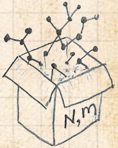

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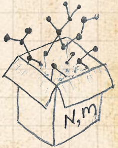

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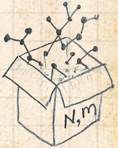
## Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

neighbors $\Leftrightarrow$ components







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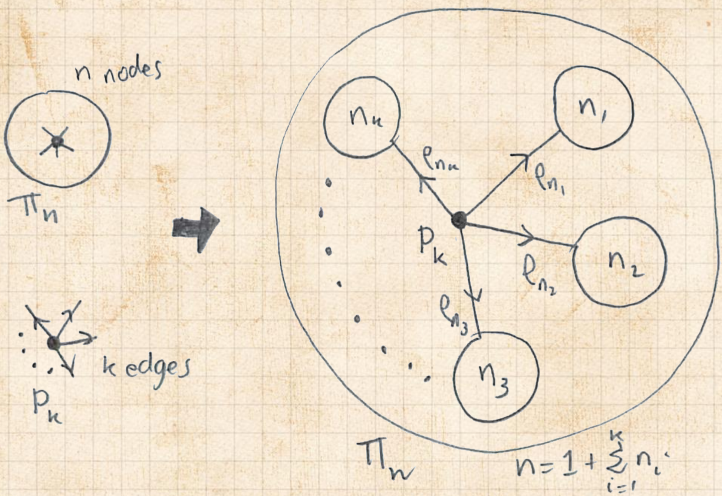
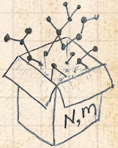



- Markov property of random networks connects $\pi_n$, $\rho_n$, and $P_k$.







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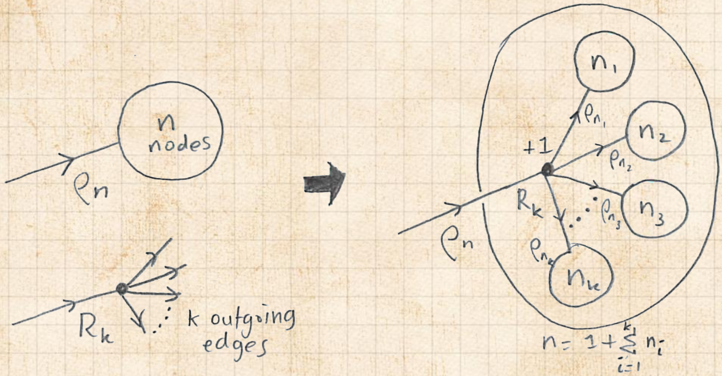


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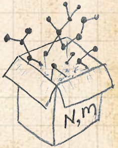

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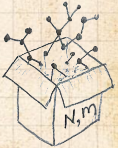

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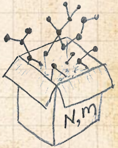

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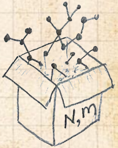

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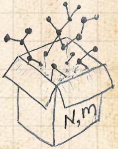
# Our mission, which we accept:

- Determine and connect the four generating functions

$$F_P, F_R, F_\pi, \text{ and } F_\rho.$$







# Outline





# Useful results we'll need for g.f.'s

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THE SUN
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THE UNEXPECTED TRANSFIGURATION
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THE MISREADING OF THE BOOK OF SAND
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It was the best of times
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THE STORYTELLERS
THE PATHS OF UNIVERSALITY
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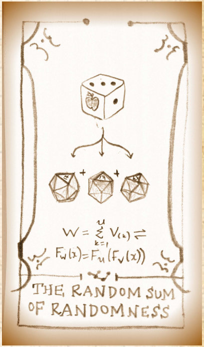
THE CHAOS OF FETTERED CONTAGION
THE END


$W = \sum_{k=1}^{U} V_{(k)}$
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- Alternate, groovier proof in the accompanying assignment.





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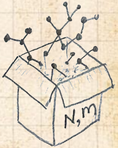
# Outline

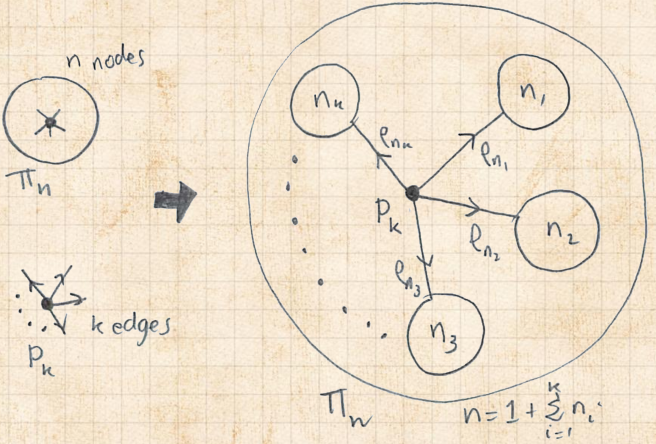






# Connecting generating functions:

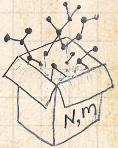
- Goal: figure out forms of the component generating functions, $F_\pi$ and $F_\rho$.




- Relate $\pi_n$ to $P_k$ and $\rho_n$ through one step of recursion.







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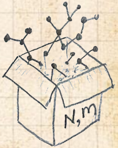

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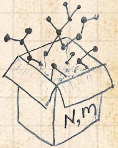

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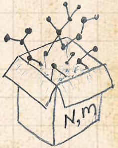

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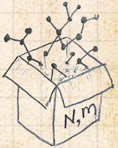

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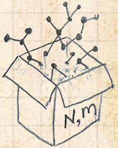

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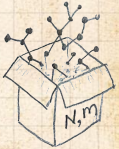

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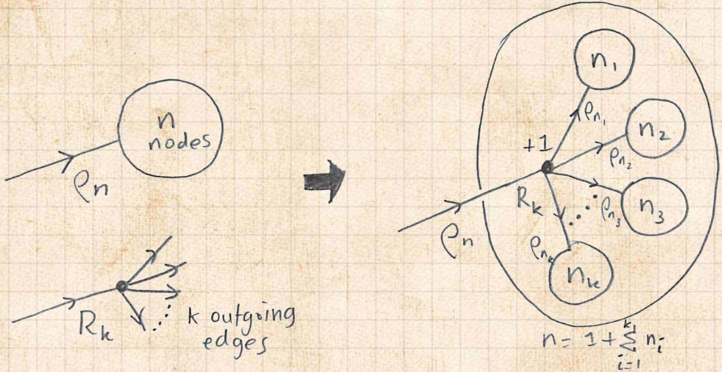


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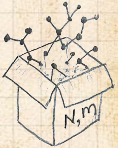

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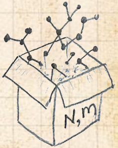

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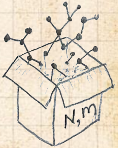

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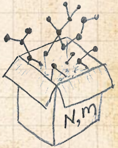

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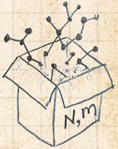

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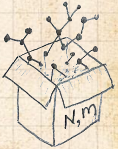

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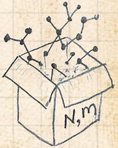

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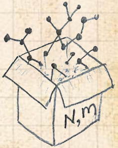

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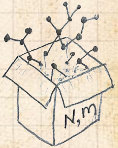

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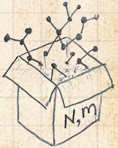

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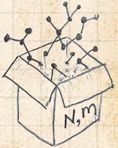

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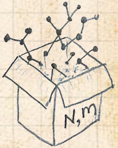
- The first equation then immediately gives us $F_\pi$ in terms of $F_\rho$ and $F_R$.





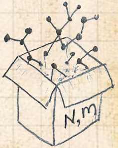


# Component sizes

- Remembering vaguely what we are doing:







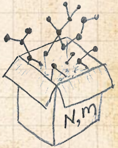
# Component sizes





- Remembering vaguely what we are doing:

  Finding $F_{\pi}$ to obtain the fractional size of the largest component $S_1 = 1 - F_{\pi}(1)$.



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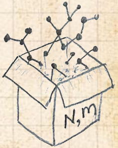

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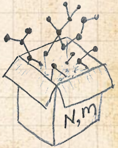

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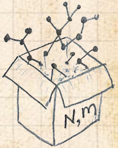

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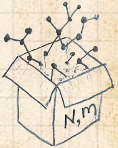

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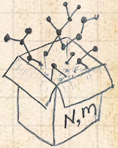
**Example**: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$







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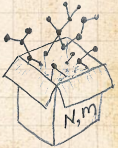

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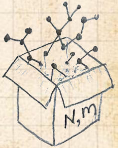

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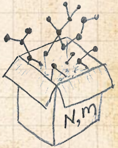

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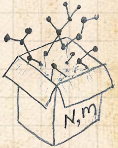

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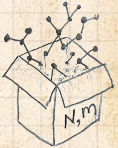

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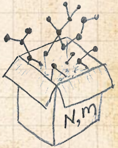

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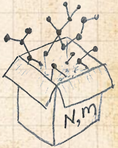

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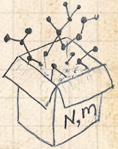
- $\pi_n = \rho_n$ just as $P_k = R_k$.





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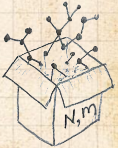

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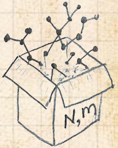

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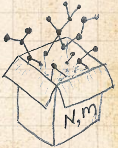

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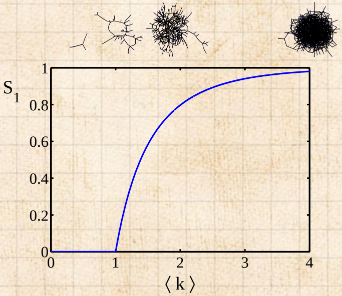






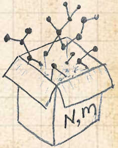

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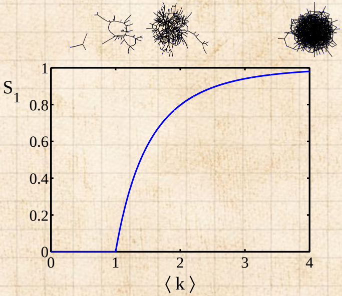


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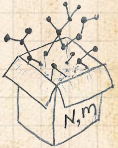

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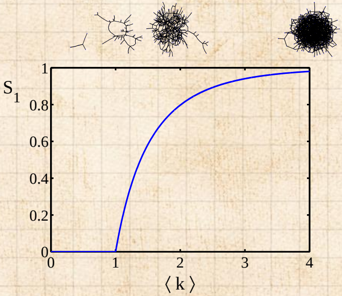
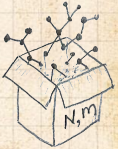



- Just as we found with our dirty trick ...
- Again, we (usually) have to resort to numerics ...



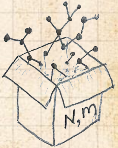


# Outline



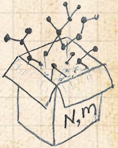






# A few simple random networks to contemplate and play around with:





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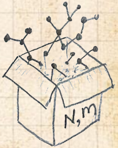

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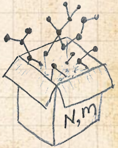

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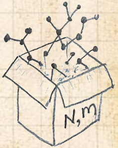

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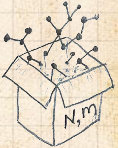

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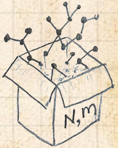

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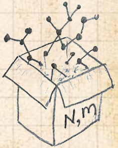

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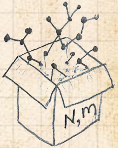

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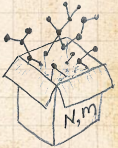

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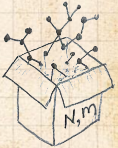

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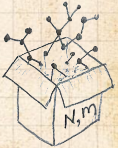
$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

- We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}.$







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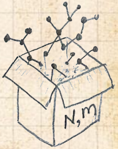

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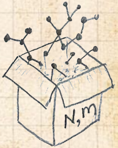

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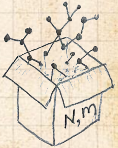
$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2$$

- Check for goodness:







## A joyful example:

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

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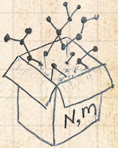
  - $F_R(x) = F_P'(x)/F_P'(1)$ and $F_P(1) = F_R(1) = 1$.







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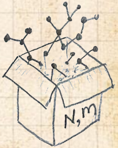

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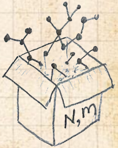
  - $F'_P(1) = \langle k \rangle_P = 2$ and $F'_R(1) = \langle k \rangle_R = \frac{3}{2}$.
- Things to figure out: Component size generating functions for $\pi_n$ and $\rho_n$, and the size of the giant component.



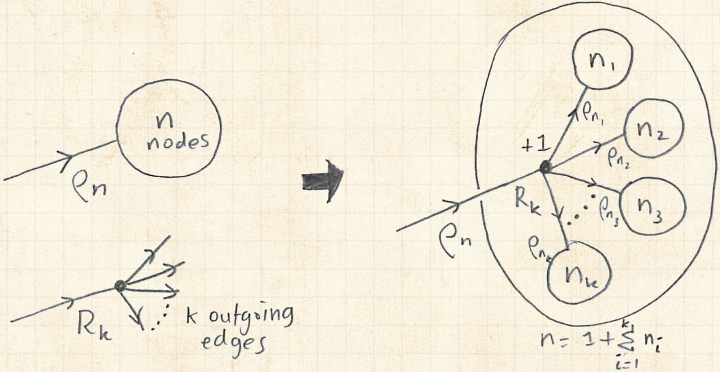




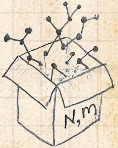
## Find $F_\rho(x)$ first:

- We know:

$$F_\rho(x) = xF_R\left(F_\rho(x)\right).$$





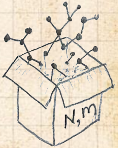



## Sticking things in things, we have:

$$F_\rho(x) = x\left(\frac{1}{4} + \frac{3}{4}\left[F_\rho(x)\right]^2\right).$$







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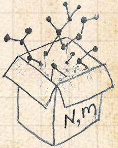
Rearranging:

$$3x\left[F_\rho(x)\right]^2 - 4F_\rho(x) + x = 0.$$







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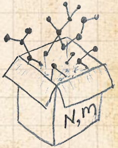
$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right)$$





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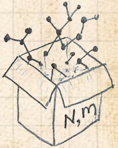

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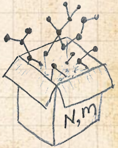
- The promise: non-negative powers of $x$ with non-negative coefficients.







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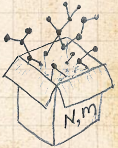
- Time for a Taylor series expansion.
- The promise: non-negative powers of $x$ with non-negative coefficients.
- First: which sign do we take?



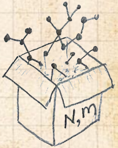




Because $\rho_n$ is a probability distribution, we know $F_\rho(1) \leq 1$ and $F_\rho(x) \leq 1$ for $0 \leq x \leq 1$.







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- Thinking about the limit $x \to 0$ in

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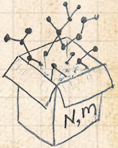
we see that the positive sign solution blows to smithereens, and the negative one is okay.







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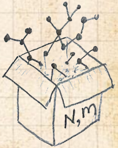

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- So we must have:

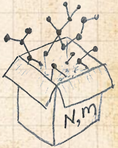
$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right),$$

- We can now deploy the Taylor expansion:

$$(1+z)^\theta = \binom{\theta}{0}z^0 + \binom{\theta}{1}z^1 + \binom{\theta}{2}z^2 + \binom{\theta}{3}z^3 + \ldots$$





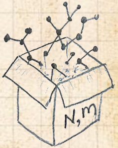


Let's define a binomial for arbitrary $\theta$ and $k = 0, 1, 2, \ldots$:

$$\binom{\theta}{k} = \frac{\Gamma(\theta+1)}{\Gamma(k+1)\Gamma(\theta-k+1)}$$







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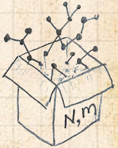

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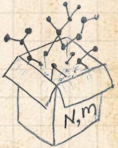

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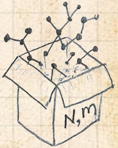
$$= 1 + \frac{1}{2} z - \frac{1}{8} z^2 + \frac{1}{16} z^3 - \ldots$$

where we've used $\Gamma(x+1) = x\Gamma(x)$ and noted that $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.



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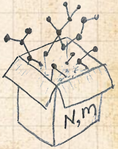
- Note: $(1+z)^\theta \sim 1 + \theta z$ always.







Totally psyched, we go back to here:

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- Giving:

$$F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n =$$

$$\frac{1}{4}x + \frac{3}{64}x^3 + \frac{9}{512}x^5 + \ldots + \frac{2}{3}\left(\frac{3}{4}\right)^k \frac{(-1)^{k+1}\Gamma(\frac{3}{2})}{\Gamma(k+1)\Gamma(\frac{3}{2}-k)}x^{2k-1} + \ldots$$

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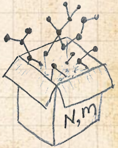
- Do odd powers make sense?

We can now find $F_\pi(x)$ with:

$$F_\pi(x) = xF_P\left(F_\rho(x)\right)$$





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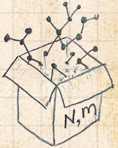

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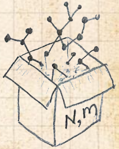

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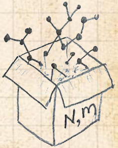
- Delicious.





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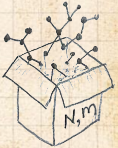
- Delicious.
- In principle, we can now extract all the $\pi_n$.





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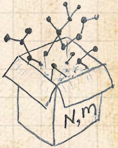
- Delicious.
- In principle, we can now extract all the $\pi_n$.
- But let's just find the size of the giant component.



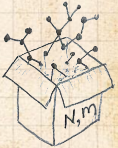


## First, we need $F_\rho(1)$:

$$F_\rho(x)\big|_{x=1} = \frac{2}{3\cdot 1}\left(1-\sqrt{1-\frac{3}{4}1^2}\right) = \frac{1}{3}.$$







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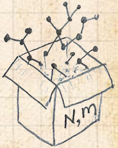

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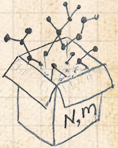

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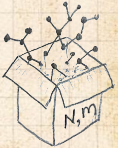

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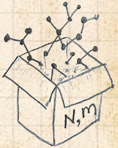

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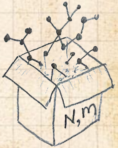

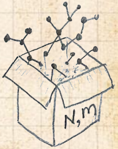


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- This is the probability that a random chosen node belongs to a finite component.

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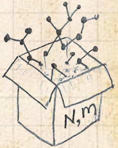
- This is the probability that a random chosen node belongs to a finite component.
- Finally, we have

$$S_1 = 1 - F_\pi(1) = 1 - \frac{5}{27} = \frac{22}{27}.$$



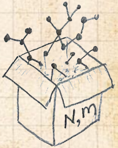


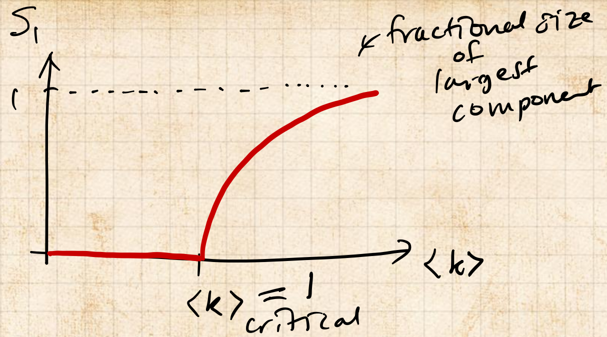


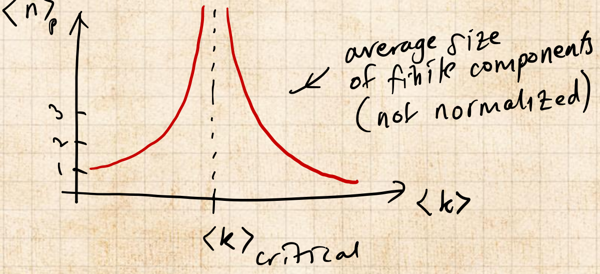
# Outline








$S_1$
1
fractional size of largest component
$\langle k \rangle$
$\langle k \rangle_{critical} = 1$

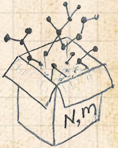

$\langle n \rangle_p$
3
2
1
average size of finite components (not normalized)
$\langle k \rangle$
$\langle k \rangle_{critical}$

# Average component size

- Next: find average size of finite components $\langle n \rangle$.





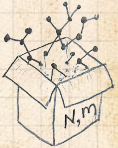


# Average component size

- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.







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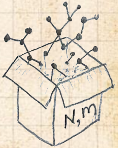
- Try to avoid finding $F_\pi(x)$ ...







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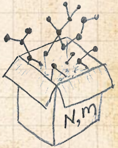
$$F'_\pi(x) = F_P\left(F_\rho(x)\right) + xF'_\rho(x)F'_P\left(F_\rho(x)\right)$$







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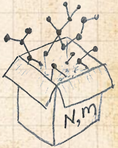
- While $F_\rho(x) = x F_R\left(F_\rho(x)\right)$ gives

$$F'_\rho(x) = F_R\left(F_\rho(x)\right) + x F'_\rho(x) F'_R\left(F_\rho(x)\right)$$





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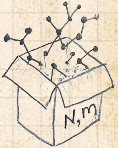
- Now set $x = 1$ in both equations.





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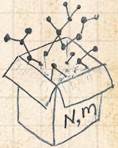

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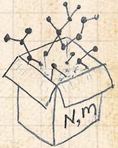
- Plug $F'_\rho(1)$ and $F_\rho(1)$ into first equation to find $F'_\pi(1)$.





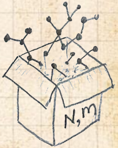


# Average component size

**Example**: Standard random graphs.







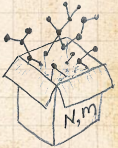
# Average component size

Example: Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.







# Average component size

Example: Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- Two differentiated equations reduce to only one:

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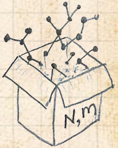

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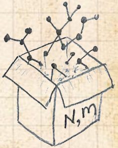

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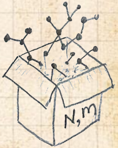
- Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$







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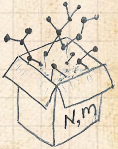

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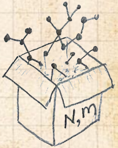
- Set $x = 1$ and replace $F_\pi(1)$ with $1 - S_1$.







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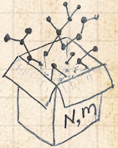
- Replace $F_P(F_\pi(x))$ using $F_\pi(x) = xF_P(F_\pi(x))$.
- Set $x = 1$ and replace $F_\pi(1)$ with $1 - S_1$.

End result: $$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$







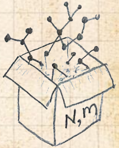
# Average component size

- Our result for standard random networks:

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$







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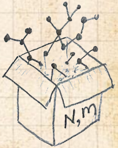

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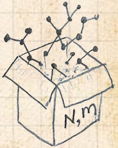
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.







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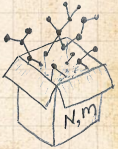
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- We have $S_1 = 0$ for all $\langle k \rangle < 1$







# Average component size

- Our result for standard random networks:

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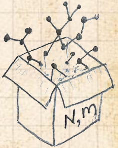
- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- This blows up as $\langle k \rangle \to 1$.







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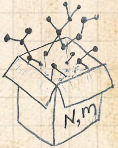
- This blows up as $\langle k \rangle \rightarrow 1$.
- Reason: we have a power law distribution of component sizes at $\langle k \rangle = 1$.







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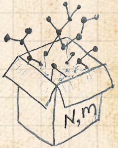
- We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- This blows up as $\langle k \rangle \to 1$.
- Reason: we have a power law distribution of component sizes at $\langle k \rangle = 1$.
- Typical critical point behavior ...







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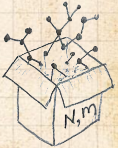

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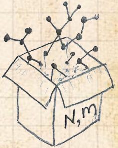

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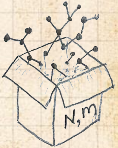

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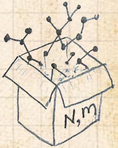

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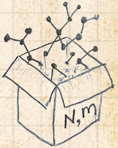

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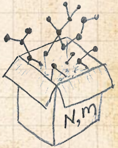

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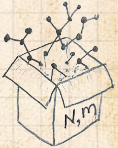
- For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}/N$.
- For $\langle k \rangle < 1$, $S_1 \sim (\log N)/N$.



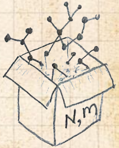




Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.







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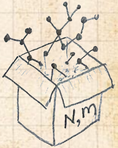

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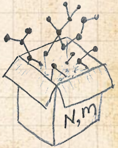

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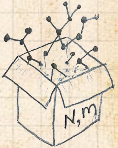
- Place stick between teeth, and recall that we have:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2.$$





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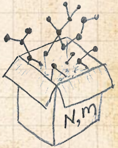
- Differentiation gives us:

$$F'_P(x) = \frac{1}{2} + \frac{3}{2}x^2 \text{ and } F'_R(x) = \frac{3}{2}x.$$







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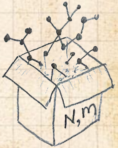

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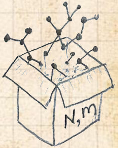

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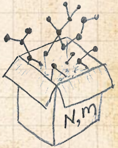

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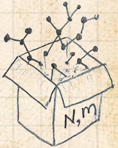

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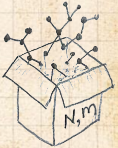

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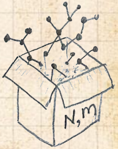

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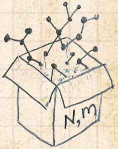

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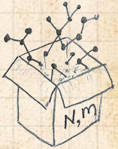

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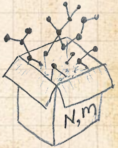
- So, kinda small.





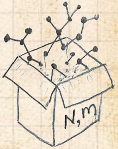


# Nutshell

- Generating functions allow us to strangely calculate features of random networks.







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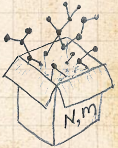

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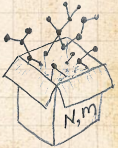
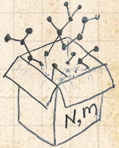


# Nutshell





- Generating functions allow us to strangely calculate features of random networks.
- They're a bit scary and magical.
- We'll find generating functions useful for contagion.
- But we'll also see that more direct, physics-bearing calculations are possible.



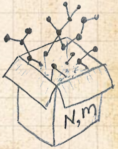
# Neural reboot (NR):

## Elevation:

https://www.youtube.com/watch?v=bGBoZbT7cR8?rel=0







# References I



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