

Random walks and diffusion on networks

Last updated: 2019/01/14, 23:14:28

Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2019

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont



These slides are brought to you by:

CocoNuTS
@networksvox

Diffusion

Random walks on
networks

Sealie & Lambie
Productions



CocoNuTS
Complex Networks
@networksvox
Everything is connected



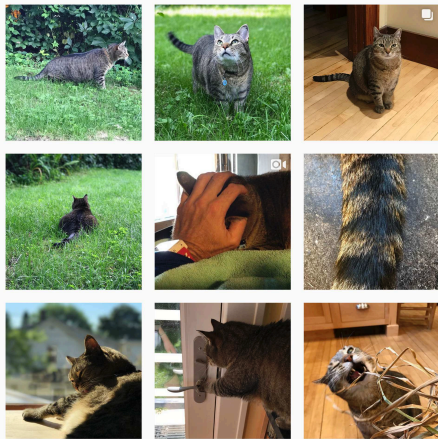
These slides are also brought to you by:



COCO NuTS
@networksvox

Diffusion

Special Guest Executive Producer

Random walks on
networks



 On Instagram at [pratchett_the_cat](https://www.instagram.com/pratchett_the_cat) 



Outline

CocoNuTS
@networksvox

Diffusion

Random walks on
networks

Random walks on networks



Random walks on networks—basics:

- Imagine a single random walker moving around on a network.
- At $t = 0$, start walker at node j and take time to be discrete.
- Q:** What's the long term probability distribution for where the walker will be?
- Define $p_i(t)$ as the probability that at time step t , our walker is at node i .
- We want to characterize the evolution of $\vec{p}(t)$.
- First task: connect $\vec{p}(t + 1)$ to $\vec{p}(t)$.
- Let's call our walker **Barry**.
- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Worse still: Barry is **texting**.



Where is Barry?

- Consider simple undirected, ergodic (strongly connected) networks.
- As usual, represent network by **adjacency matrix** A where


$$a_{ij} = 1 \text{ if } i \text{ has an edge leading to } j, \\ a_{ij} = 0 \text{ otherwise.}$$


- Barry is at node j at time t with probability $p_j(t)$.
- In the next time step, he **randomly lurches** toward one of j 's neighbors.
- Barry arrives at node i from node j with probability $\frac{1}{k_j}$ if an edge connects j to i .
- Equation-wise:

$$p_i(t+1) = \sum_{j=1}^n \frac{1}{k_j} a_{ji} p_j(t).$$

where k_j is j 's degree. Note: $k_i = \sum_{j=1}^n a_{ij}$.





 **Excellent observation:** The same equation applies for stuff moving around a network, such that at each time step all material at node i is sent to its neighbors.

 $x_i(t)$ = amount of stuff at node i at time t .




$$x_i(t+1) = \sum_{j=1}^n \frac{1}{k_j} a_{ji} x_j(t).$$

 Random walking is equivalent to diffusion .





Where is Barry?


-  Linear algebra-based excitement:
 $p_i(t+1) = \sum_{j=1}^n a_{ji} \frac{1}{k_j} p_j(t)$ is more usefully viewed as


$$\vec{p}(t+1) = A^T K^{-1} \vec{p}(t)$$

where $[K_{ij}] = [\delta_{ij} k_i]$ has node degrees on the main diagonal and zeros everywhere else.

-  So... we need to find the **dominant eigenvalue** of $A^T K^{-1}$.


-  Expect this eigenvalue will be 1 (doesn't make sense for total probability to change).

-  The corresponding eigenvector will be the limiting probability distribution (or invariant measure).

-  Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.








Where is Barry?

 By inspection, we see that

$$\vec{p}(\infty) = \frac{1}{\sum_{i=1}^n k_i} \vec{k}$$

satisfies $\vec{p}(\infty) = A^T K^{-1} \vec{p}(\infty)$ with eigenvalue 1.

-  We will find Barry at node i with probability proportional to its degree k_i .
-  Beautiful implication: probability of finding Barry travelling along any edge is **uniform**.
-  Diffusion in real space smooths things out.
-  On networks, uniformity occurs on edges.
-  So in fact, diffusion in real space is **about the edges too** but we just don't see that.

Other pieces:

Goodness: $A^T K^{-1}$ is similar to a real symmetric matrix if $A = A^T$.

Consider the transformation $M = K^{-1/2}$:

$$K^{-1/2} A^T K^{-1} K^{1/2} = K^{-1/2} A^T K^{-1/2}.$$

Since $A^T = A$, we have

$$(K^{-1/2} A K^{-1/2})^T = K^{-1/2} A K^{-1/2}.$$

Upshot: $A^T K^{-1} = A K^{-1}$ has real eigenvalues and a complete set of orthogonal eigenvectors.

Can also show that maximum eigenvalue magnitude is indeed 1.

