Branching Networks II

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Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2019

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

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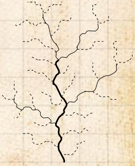
Everything is connected



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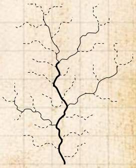

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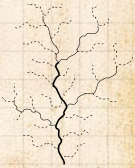
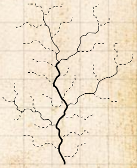
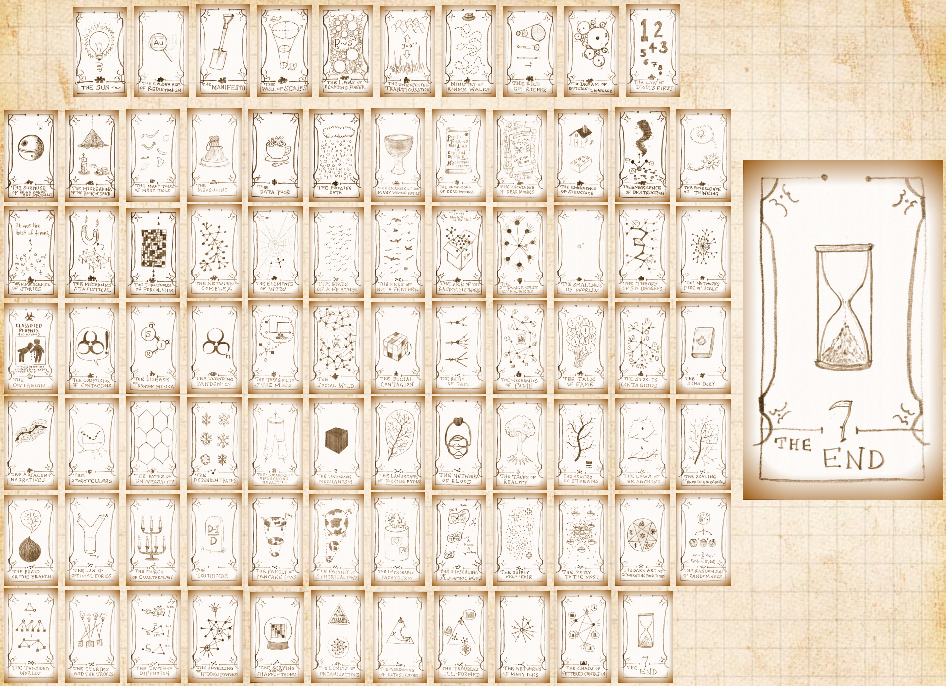


Outline

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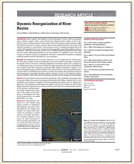
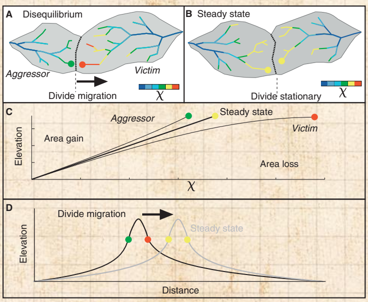
Piracy on the high χ's:



"Dynamic Reorganization of River Basins"
Willett et al.,
Science Magazine, **343**, 1248765, 2014. [21]



$$\frac{\partial z(x,t)}{\partial t} = U - K A^m \left| \frac{\partial z(x,t)}{\partial x} \right|^n$$

$$z(x) = z_\mathrm{b} + \left(\frac{U}{K A_0^m} \right)^{1/n} \chi$$

$$\chi = \int_{x_\mathrm{b}}^{x} \left(\frac{A_0}{A(x')} \right)^{m/n} \mathrm{d}x'$$

Piracy on the high χ's:

http://www.youtube.com/watch?v=FnroL1_-l2c?rel=0

More: How river networks move across a landscape (Science Daily)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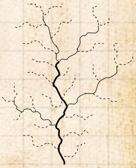
Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:



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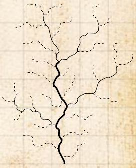

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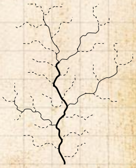
- Oddly, Horton's laws have four parameters and Tokunaga has two parameters.



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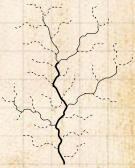
  Insert question from assignment 1



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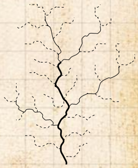
- To make a connection, clearest approach is to start with Tokunaga's law ...



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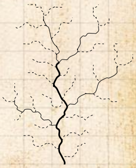
- Known result: Tokunaga $\rightarrow$ Horton [18, 19, 20, 9, 2]



Let us make them happy

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Space-fillingness

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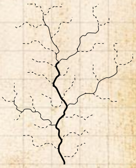
- A network is space-filling if the average distance between adjacent streams is roughly constant.



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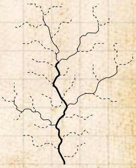
- Reasonable for river and cardiovascular networks



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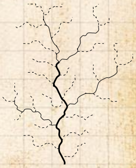
- For river networks:
  Drainage density ρ_{dd} = inverse of typical distance between channels in a landscape.



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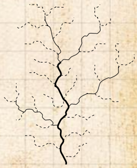
- In terms of basin characteristics:

$$\rho_{\mathrm{dd}} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}}$$



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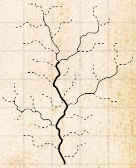
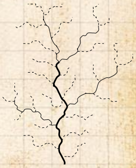
$$\rho_{\mathrm{dd}} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \frac{\sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega}{a_\Omega}$$



More with the happy-making thing

Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$



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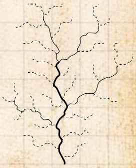

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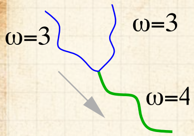

1. Running into another stream of order ω and generating a stream of order $\omega + 1$ ...

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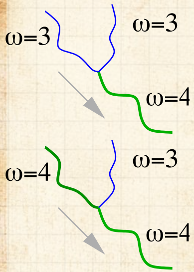

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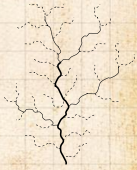
2. Running into and being absorbed by a stream of higher order $\omega' > \omega$ ...



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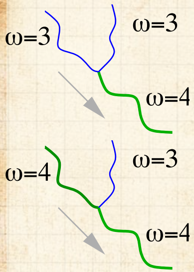

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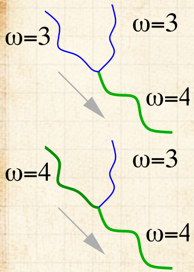

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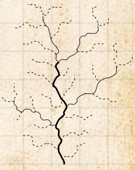
   - $n_{\omega'} T_{\omega'-\omega}$ streams of order ω do this



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Putting things together:

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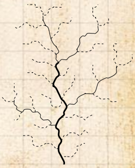
- Insert question from assignment 1



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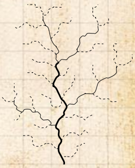
- Insert question from assignment 1
- Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

(The larger value is the one we want.)



Finding other Horton ratios

Connect Tokunaga to R_s

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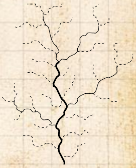

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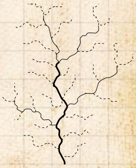

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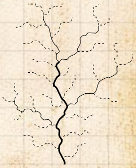

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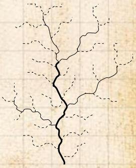

Horton and Tokunaga are happy

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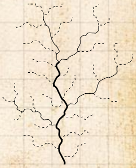

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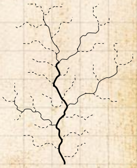

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- And from before:

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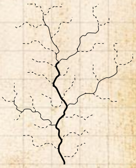
Some observations:

- R_n and R_ℓ depend on T_1 and R_T.



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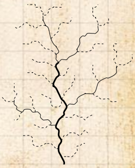

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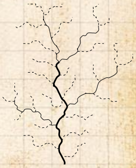

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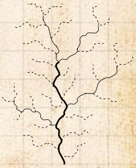

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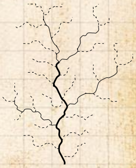

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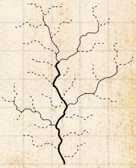
- We'll in fact see that $R_a = R_n$.
- Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between non-trivial statistical distributions. [3, 4]

Horton and Tokunaga are happy

The other way round

- Note: We can invert the expresssions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.



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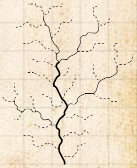

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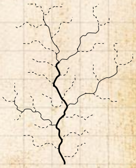
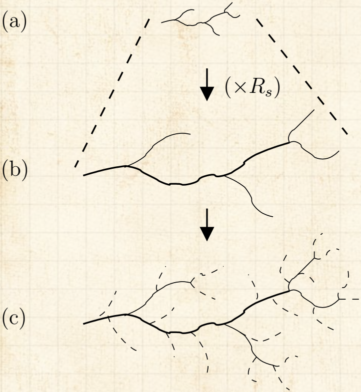
- Suggests we should be able to argue that Horton's laws imply Tokunaga's laws (if drainage density is uniform) ...



Horton and Tokunaga are friends

From Horton to Tokunaga [2]



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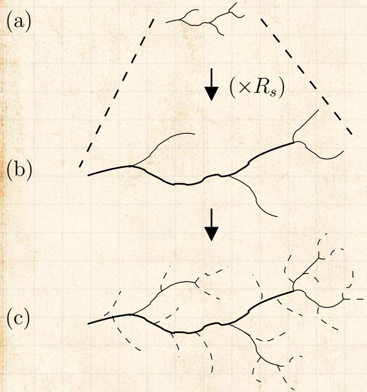

- Assume Horton's laws hold for number and length

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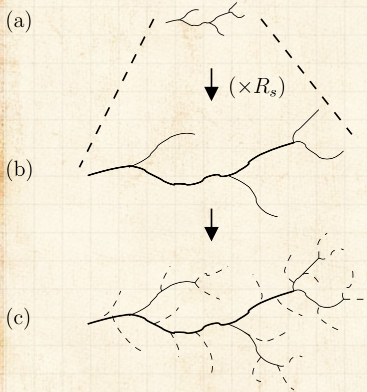

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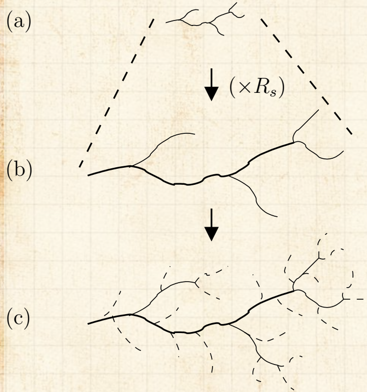

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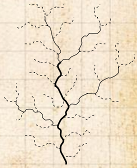

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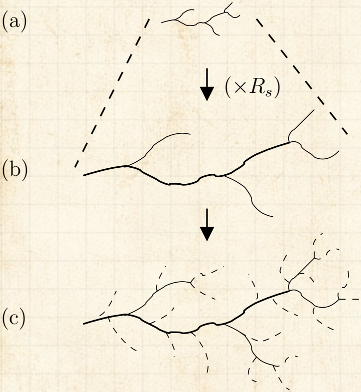

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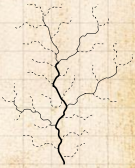
- Scale up by a factor of R_ℓ, orders increment to $\omega+1$ and ω.
- Maintain drainage density by adding new order $\omega-1$ streams



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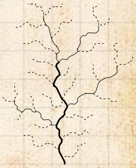

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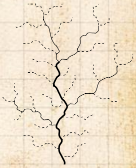

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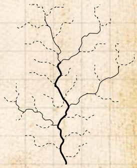

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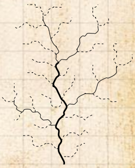
- For large ω, Tokunaga's law is the solution—let's check ...



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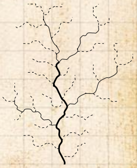

Just checking:

- Substitute Tokunaga's law $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$ into

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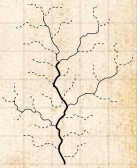

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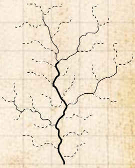

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$$\simeq (R_\ell - 1) T_1 \frac{R_\ell^{k-1}}{R_\ell - 1}$$

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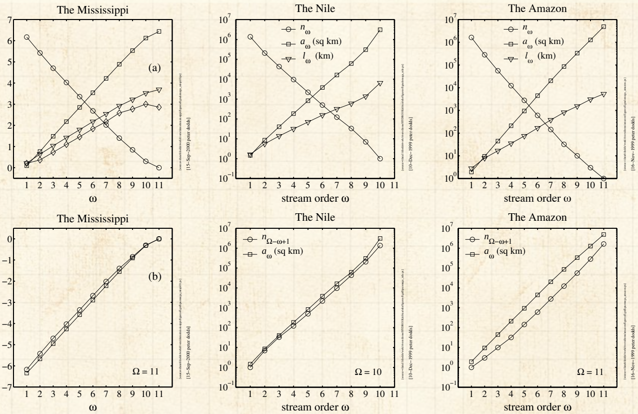
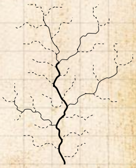
$$\simeq (R_\ell - 1) T_1 \frac{R_\ell^{\,k-1}}{R_\ell - 1} = T_1 R_\ell^{k-1}$$ ...yep.

Horton's laws of area and number:

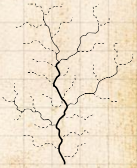


- In bottom plots, stream number graph has been flipped vertically.
- Highly suggestive that $R_n \equiv R_a$ ...



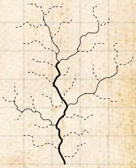
Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?



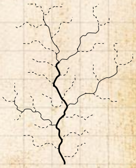
Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?
- Rule of thumb: discard data for two smallest and two largest orders.



Mississippi:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
$[2,3]$	5.27	5.26	2.48	2.30	1.00
$[2,5]$	4.86	4.96	2.42	2.31	1.02
$[2,7]$	4.77	4.88	2.40	2.31	1.02
$[3,4]$	4.72	4.91	2.41	2.34	1.04
$[3,6]$	4.70	4.83	2.40	2.35	1.03
$[3,8]$	4.60	4.79	2.38	2.34	1.04
$[4,6]$	4.69	4.81	2.40	2.36	1.02
$[4,8]$	4.57	4.77	2.38	2.34	1.05
$[5,7]$	4.68	4.83	2.36	2.29	1.03
$[6,7]$	4.63	4.76	2.30	2.16	1.03
$[7,8]$	4.16	4.67	2.41	2.56	1.12
mean μ	4.69	4.85	2.40	2.33	1.04
std dev σ	0.21	0.13	0.04	0.07	0.03
σ/μ	0.045	0.027	0.015	0.031	0.024



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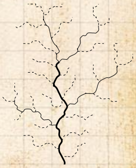
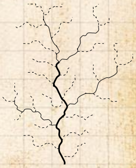
ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
$[2,3]$	4.78	4.71	2.47	2.08	0.99
$[2,5]$	4.55	4.58	2.32	2.12	1.01
$[2,7]$	4.42	4.53	2.24	2.10	1.02
$[3,5]$	4.45	4.52	2.26	2.14	1.01
$[3,7]$	4.35	4.49	2.20	2.10	1.03
$[4,6]$	4.38	4.54	2.22	2.18	1.03
$[5,6]$	4.38	4.62	2.22	2.21	1.06
$[6,7]$	4.08	4.27	2.05	1.83	1.05
mean μ	4.42	4.53	2.25	2.10	1.02
std dev σ	0.17	0.10	0.10	0.09	0.02
σ/μ	0.038	0.023	0.045	0.042	0.019



Reducing Horton's laws:

Rough first effort to show $R_n \equiv R_a$:



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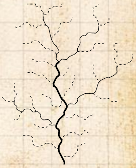

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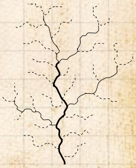
- So:

$$a_\Omega \simeq \sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega / \rho_{\text{dd}}$$



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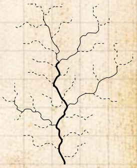

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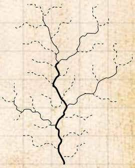

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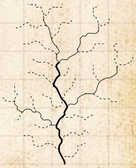

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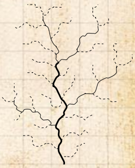

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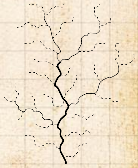

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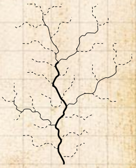

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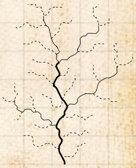

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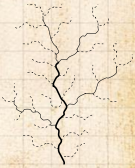
- So, a_Ω is growing like R_n^Ω and therefore:

$$\boxed{R_n \equiv R_a}$$



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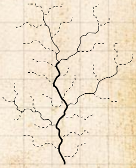

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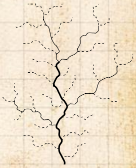

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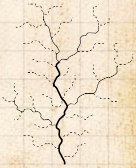
- Insert question from assignment 2



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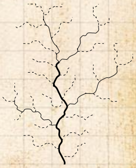
Intriguing division of area:

- Observe: Combined area of basins of order ω independent of ω.



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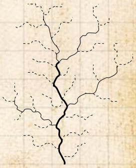

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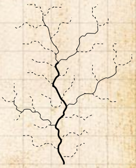

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- Reason:

$$n_\omega \propto (R_n)^{-\omega}$$

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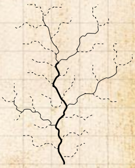
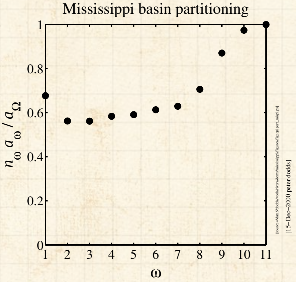
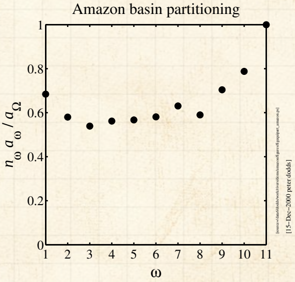
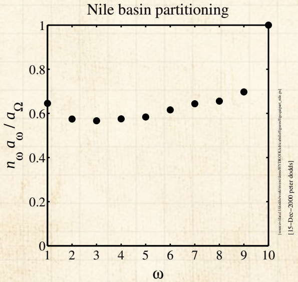


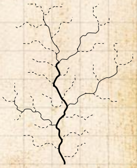
Equipartitioning:

Some examples:









Neural Reboot: Fwoompf

http://www.youtube.com/watch?v=5mUs70SqD4o?rel=0

Scaling laws

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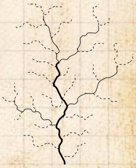

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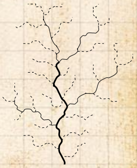

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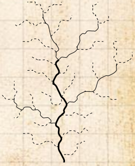

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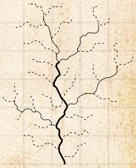

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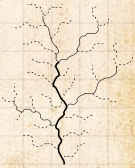

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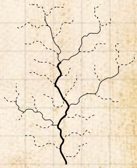

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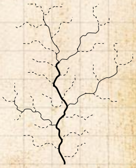

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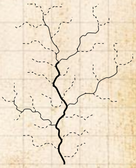

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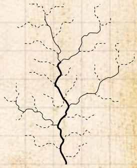

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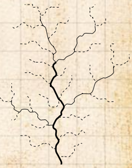

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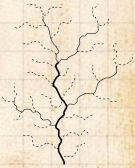

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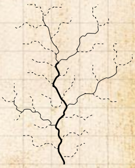

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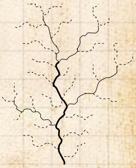

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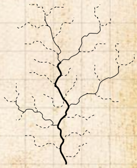

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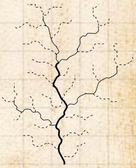

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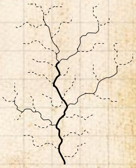

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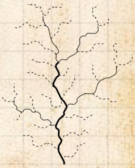

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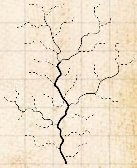

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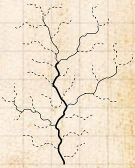

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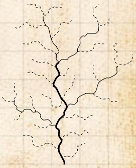

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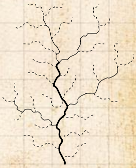

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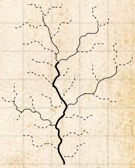

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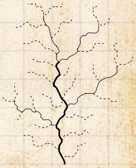

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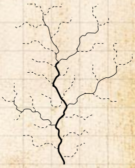

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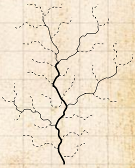

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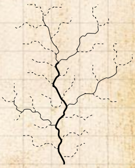

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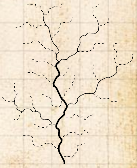

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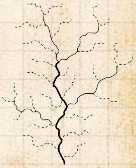

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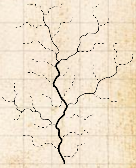

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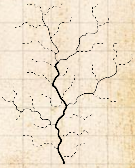

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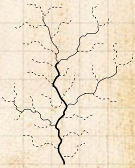
- Also known as the exceedance probability.



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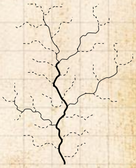

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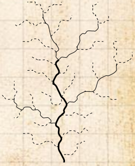

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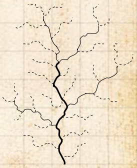

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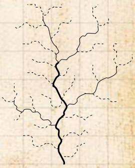

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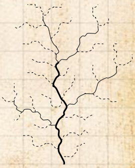

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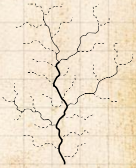

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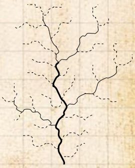

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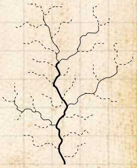

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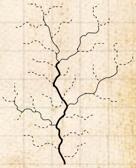

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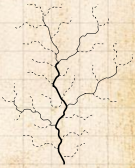

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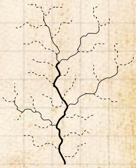

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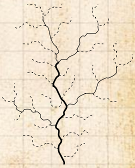

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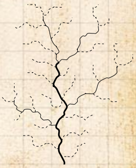

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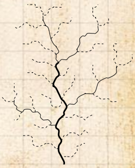

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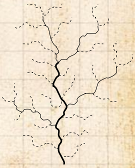

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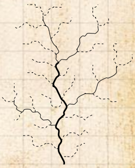

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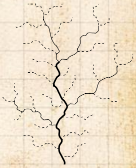

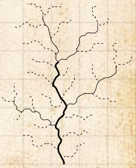

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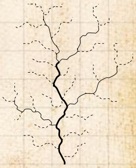

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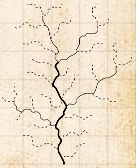

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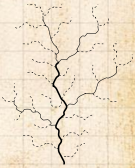

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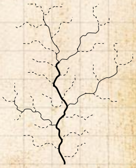

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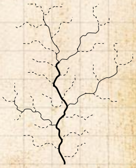

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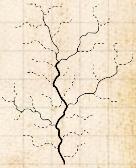

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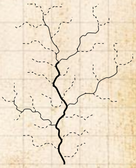

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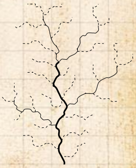

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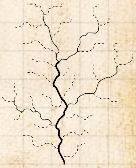

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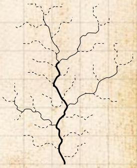

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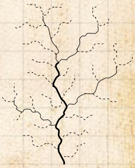

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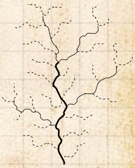
Insert question from assignment 2



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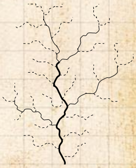

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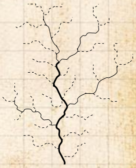
Insert question from assignment 2

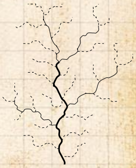
- Such connections between exponents are called scaling relations
- Let's connect to one last relationship: Hack's law



Scaling laws

Hack's law: [6]

$$\ell \propto a^h$$



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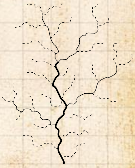

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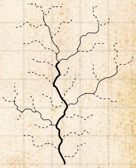

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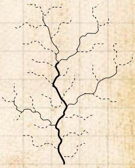

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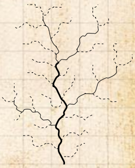

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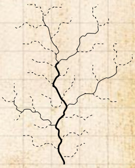


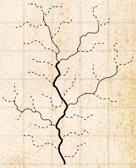
We mentioned there were a good number of 'laws': [2]

Relation:	Name or description:
$T_k = T_1 (R_T)^{k-1}$	Tokunaga's law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega / n_{\omega+1} = R_n$	Horton's law of stream numbers
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	Horton's law of main stream lengths
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	Horton's law of basin areas
$\bar{s}_{\omega+1} / \bar{s}_\omega = R_s$	Horton's law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack's law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein's law
$\lambda \sim L^\varphi$	variation of Langbein's law

Connecting exponents

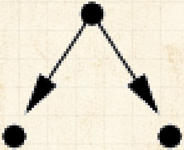
Only 3 parameters are independent:
e.g., take d, R_n, and R_s

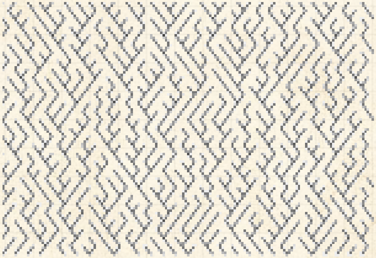
relation:	**scaling relation/parameter:** [2]
$\ell \sim L^d$	d
$T_k = T_1 (R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_\omega / n_{\omega+1} = R_n$	R_n
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	$R_a = R_n$
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \ln R_s / \ln R_n$
$a \sim L^D$	$D = d/h$
$L_\perp \sim L^H$	$H = d/h - 1$
$P(a) \sim a^{-\tau}$	$\tau = 2 - h$
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim a^\beta$	$\beta = 1 + h$
$\lambda \sim L^\varphi$	$\varphi = d$



Scheidegger's model

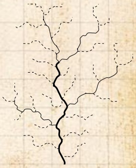
Directed random networks [11, 12]





$$P(\searrow) = P(\swarrow) = 1/2$$

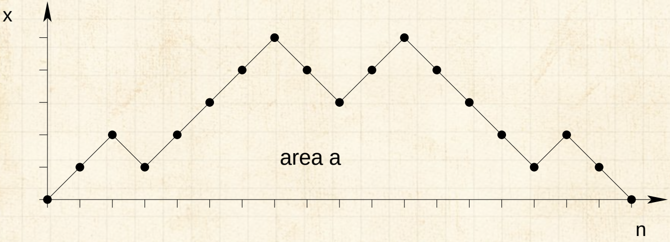
- Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]
- Useful and interesting test case

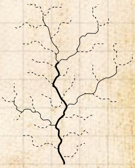


A toy model—Scheidegger's model

Random walk basins:

- Boundaries of basins are random walks





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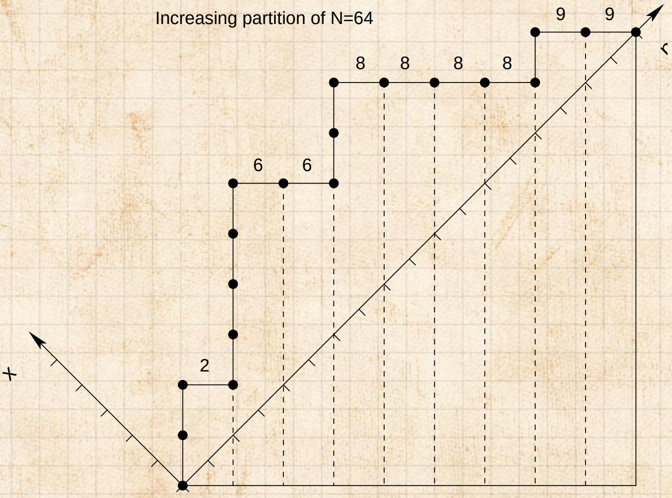

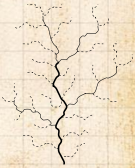

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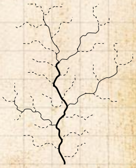

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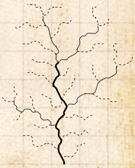

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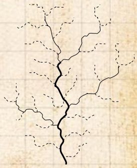
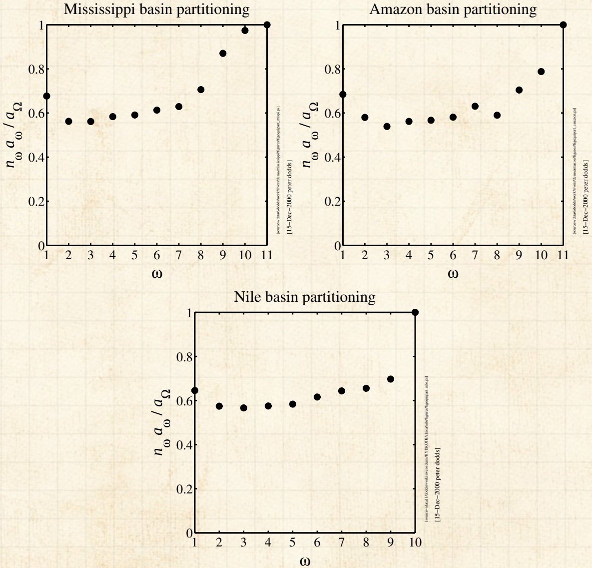
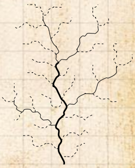


Equipartitioning reexamined:

Recall this story:





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- What about

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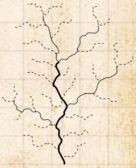

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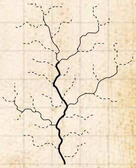

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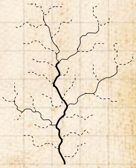

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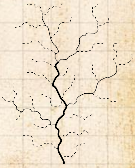
- while stream ordering separates basins ...

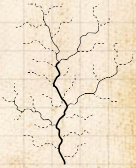


Hard neural reboot (sound matters):

https://twitter.com/round_boys/status/951873765964681216

Fluctuations

Moving beyond the mean:



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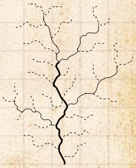

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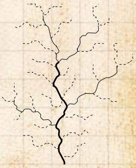

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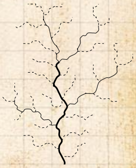
- Natural generalization to consider relationships between probability distributions
- Yields rich and full description of branching network structure



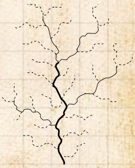
Fluctuations

Moving beyond the mean:

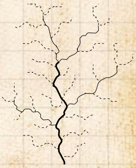
- Both Horton's laws and Tokunaga's law relate average properties, e.g.,

$$\bar{s}_{\omega}/\bar{s}_{\omega-1} = R_s$$

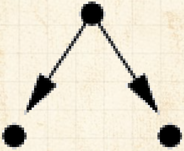
- Natural generalization to consider relationships between probability distributions
- Yields rich and full description of branching network structure
- See into the heart of randomness ...

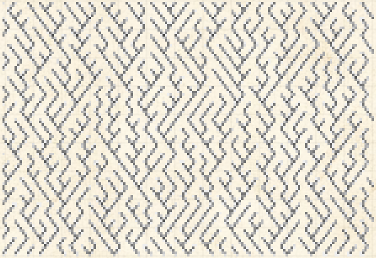


A toy model—Scheidegger's model



Directed random networks [11, 12]

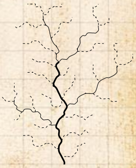




- $$P(\searrow) = P(\swarrow) = 1/2$$
- Flow is directed downwards

Generalizing Horton's laws

- $\bar{\ell}_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell|\omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell/R_\ell^\omega)$



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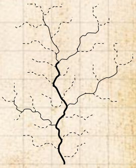

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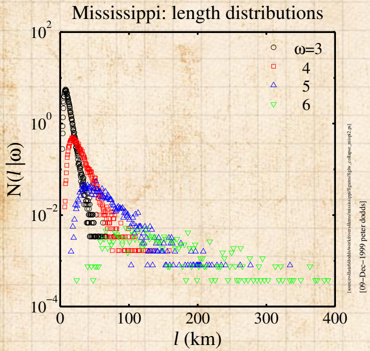

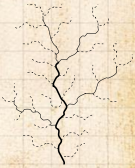

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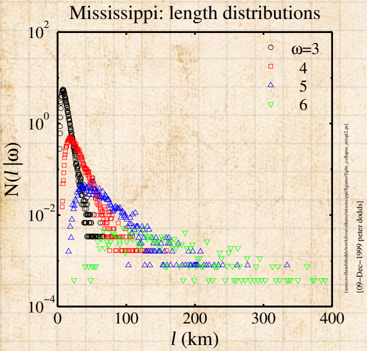
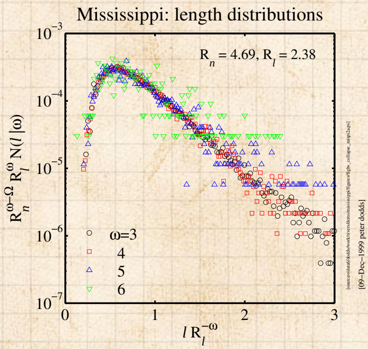
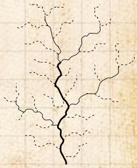




- Scaling collapse works well for intermediate orders



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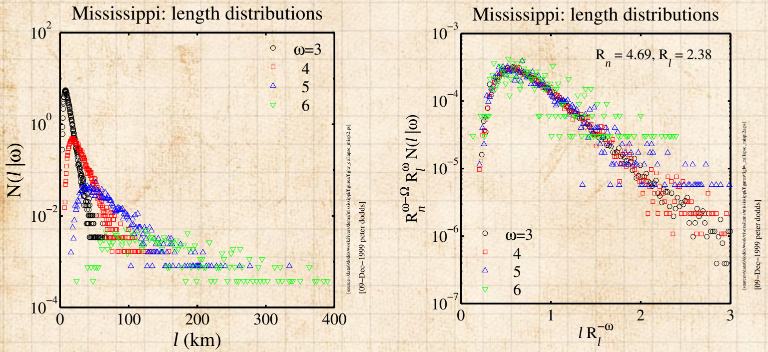
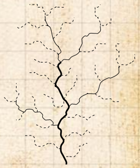
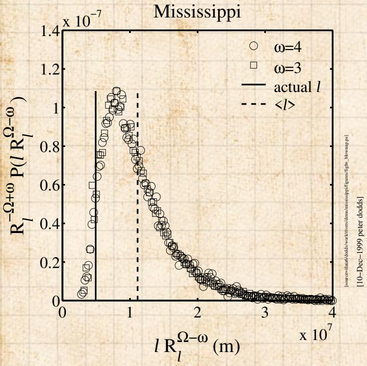


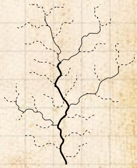


- Scaling collapse works well for intermediate orders
- All moments grow exponentially with order

Generalizing Horton's laws

- How well does overall basin fit internal pattern?





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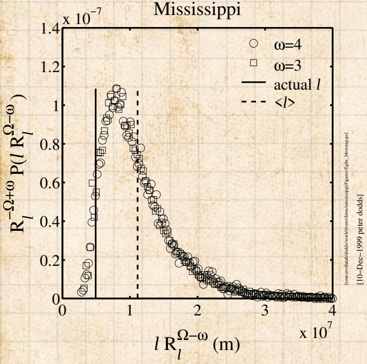
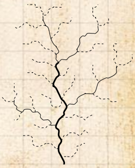


- Actual length = 4920 km (at 1 km res)



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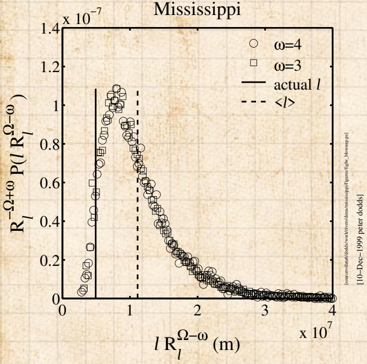
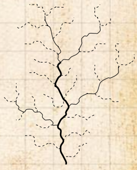


- Actual length = 4920 km (at 1 km res)
- Predicted Mean length = 11100 km



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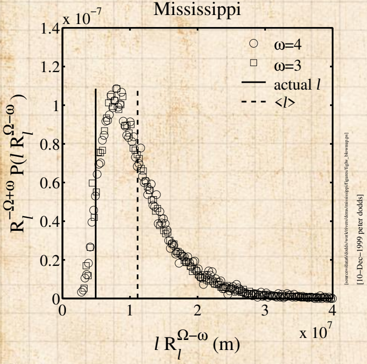
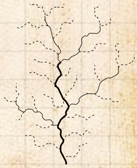


- Actual length = 4920 km (at 1 km res)
- Predicted Mean length = 11100 km
- Predicted Std dev = 5600 km



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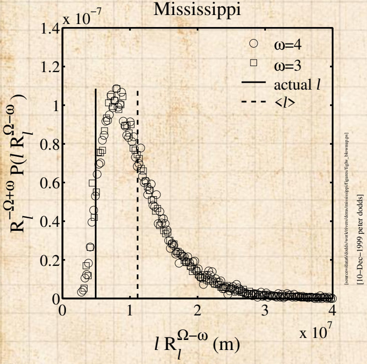

- Actual length = 4920 km (at 1 km res)
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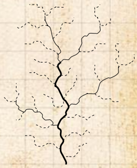
- Actual length/Mean length = 44 %



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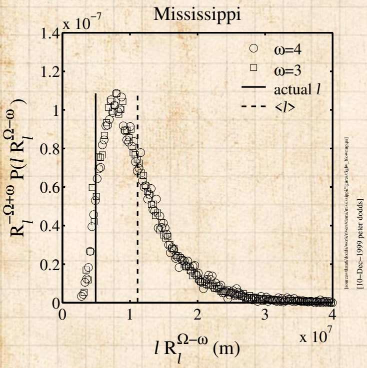

- Actual length = 4920 km (at 1 km res)
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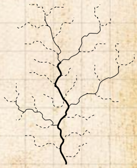
- Predicted Std dev = 5600 km
- Actual length/Mean length = 44 %
- Okay.

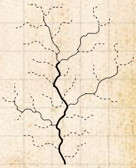


Generalizing Horton's laws

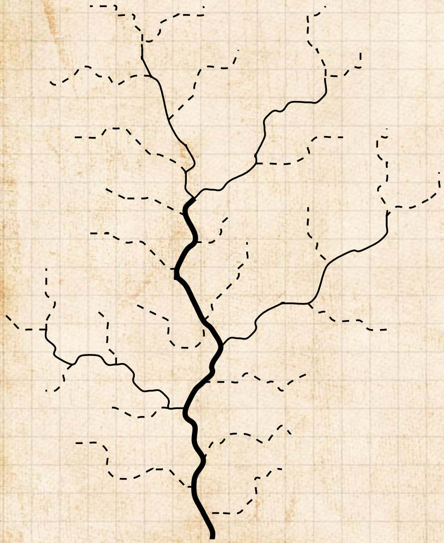
Comparison of predicted versus measured main stream lengths for large scale river networks (in 10^3 km):

basin:	ℓ_Ω	$\bar{\ell}_\Omega$	σ_ℓ	$\ell_\Omega/\bar{\ell}_\Omega$	$\sigma_\ell/\bar{\ell}_\Omega$
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73

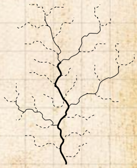
	a_Ω	$\bar{a}_\Omega$	σ_a	$a_\Omega/\bar{a}_\Omega$	$\sigma_a/\bar{a}_\Omega$
Mississippi	2.74	7.55	5.58	0.36	0.74
Amazon	5.40	9.07	8.04	0.60	0.89
Nile	3.08	0.96	0.79	3.19	0.82
Congo	3.70	10.09	8.28	0.37	0.82
Kansas	0.14	0.49	0.42	0.28	0.86



Combining stream segments distributions:



- Stream segments sum to give main stream lengths
- $$\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$$



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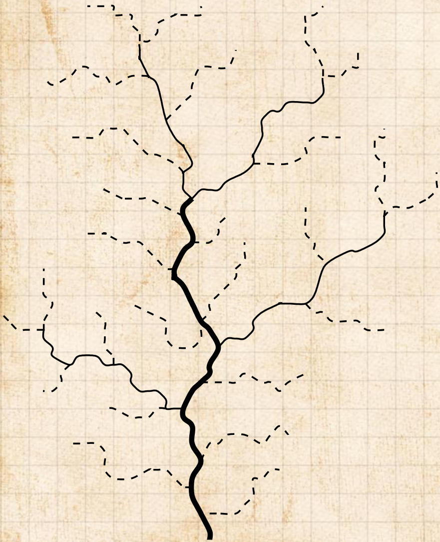
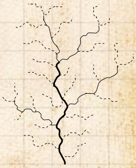


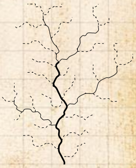
- Stream segments sum to give main stream lengths
- $$\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$$
- $P(\ell_\omega)$ is a convolution of distributions for the s_ω



Generalizing Horton's laws

- Sum of variables $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$ leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \cdots * N(s|\omega)$$



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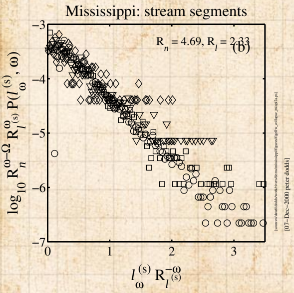
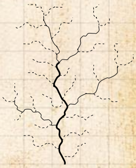


$$N(s|\omega) = \frac{1}{R_n^\omega R_\ell^\omega} F\left(s/R_\ell^\omega\right)$$

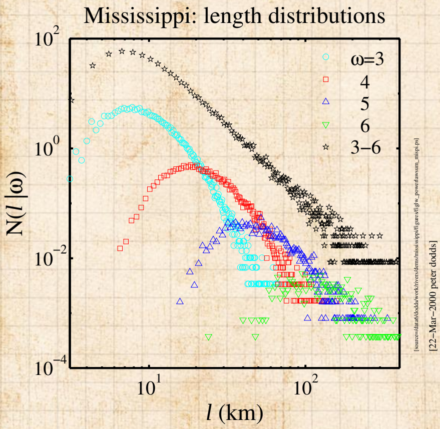
$$F(x) = e^{-x/\xi}$$

Mississippi: $\xi \simeq 900$ m.

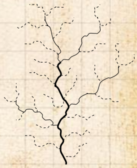


Generalizing Horton's laws

- Next level up: Main stream length distributions must combine to give overall distribution for stream length



- $P(\ell) \sim \ell^{-\gamma}$



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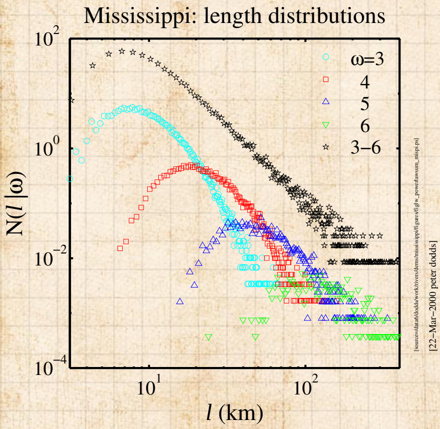
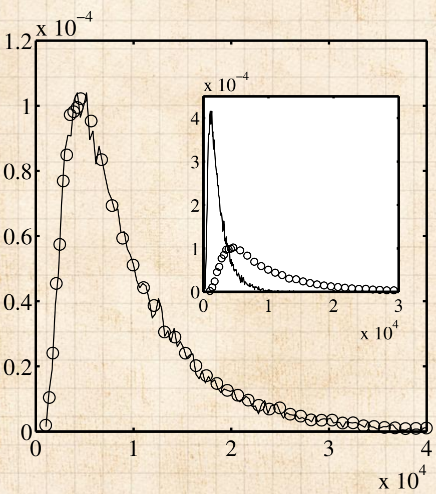
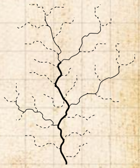


- $P(\ell) \sim \ell^{-\gamma}$
- Another round of convolutions [3]
- Interesting ...

Generalizing Horton's laws

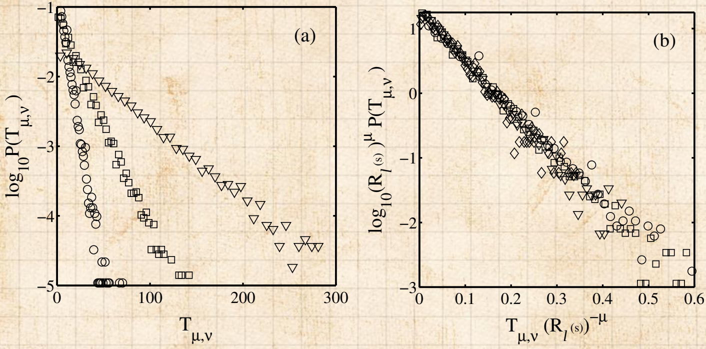
- Number and area distributions for the Scheidegger model [3]
- $P(n_{1,6})$ versus $P(a_6)$ for a randomly selected $\omega = 6$ basin.

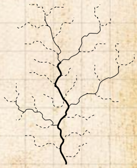




Generalizing Tokunaga's law

Scheidegger:

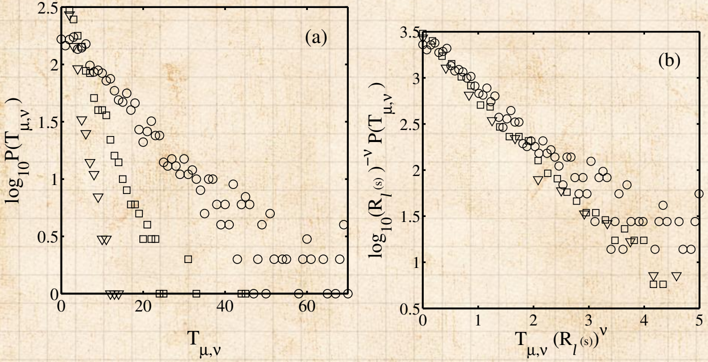




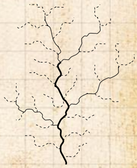
- Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using R_s

Generalizing Tokunaga's law

Mississippi:



- Same data collapse for Mississippi ...



Generalizing Tokunaga's law

So

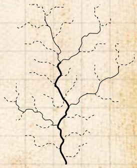
$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t \left[T_{\mu,\nu}/(R_s)^{\mu-\nu-1}\right]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

$$\boxed{P(s_\mu) \Leftrightarrow P(T_{\mu,\nu})}$$

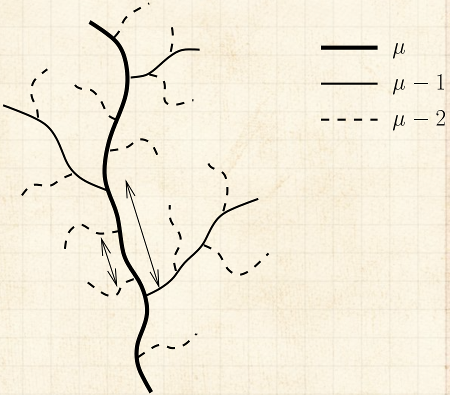
- Exponentials arise from randomness.
- Look at joint probability $P(s_\mu, T_{\mu,\nu})$.

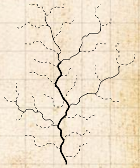


Generalizing Tokunaga's law

Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments





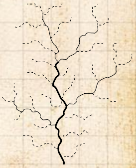
Generalizing Tokunaga's law

- Follow streams segments down stream from their beginning

Generalizing Tokunaga's law

- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is constant:

$$\tilde{p}_\mu \simeq 1/(R_s)^{\mu-1}\xi_s$$



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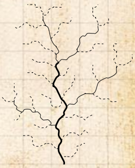
- Inter-tributary lengths exponentially distributed



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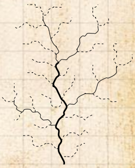
- Probability decays exponentially with stream order
- Inter-tributary lengths exponentially distributed
- $\Rightarrow$ random spatial distribution of stream segments

Generalizing Tokunaga's law

- Joint distribution for generalized version of Tokunaga's law:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

  where
  - p_ν = probability of absorbing an order ν side stream



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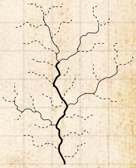
where

- $p_\nu =$ probability of absorbing an order ν side stream
- $\tilde{p}_\mu =$ probability of an order μ stream terminating



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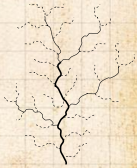
  where
  - p_ν = probability of absorbing an order ν side stream
  - $\tilde{p}_\mu$ = probability of an order μ stream terminating
- Approximation: depends on distance units of s_μ
- In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.



Generalizing Tokunaga's law

- Now deal with this thing:

$$P(s_{\mu}, T_{\mu,\nu}) = \tilde{p}_{\mu} \binom{s_{\mu} - 1}{T_{\mu,\nu}} p_{\nu}^{T_{\mu,\nu}} (1 - p_{\nu} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu,\nu} - 1}$$

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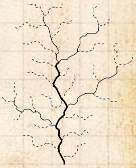

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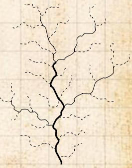
- Obtain

$$P(x, y) = N x^{-1/2} \left[F(y/x)\right]^x$$

where

$$F(v) = \left(\frac{1-v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$



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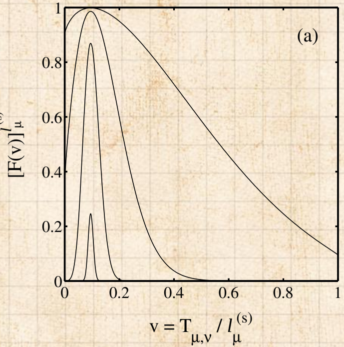

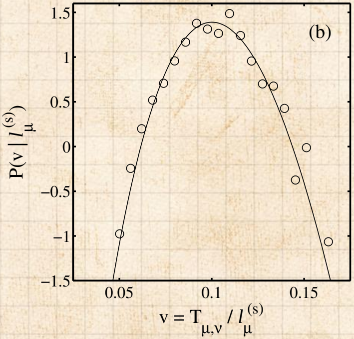

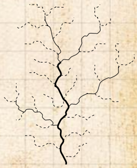

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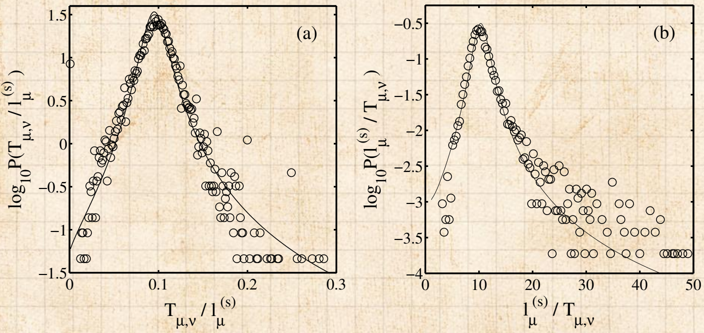

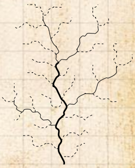

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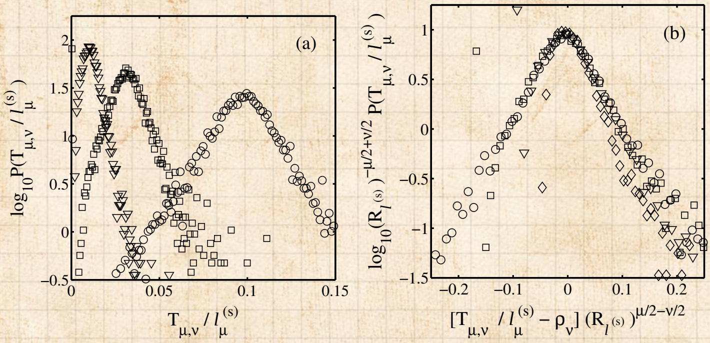

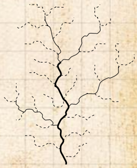

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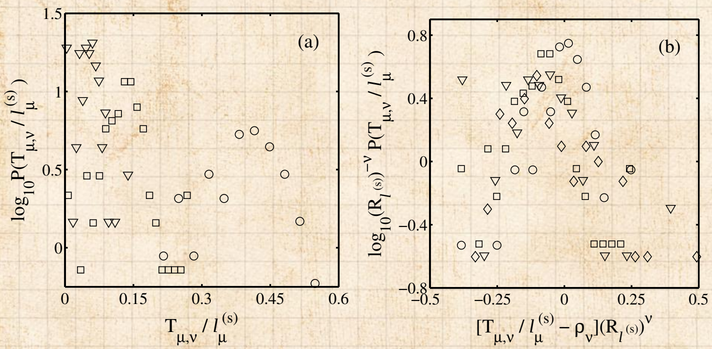

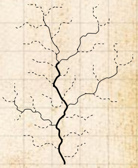

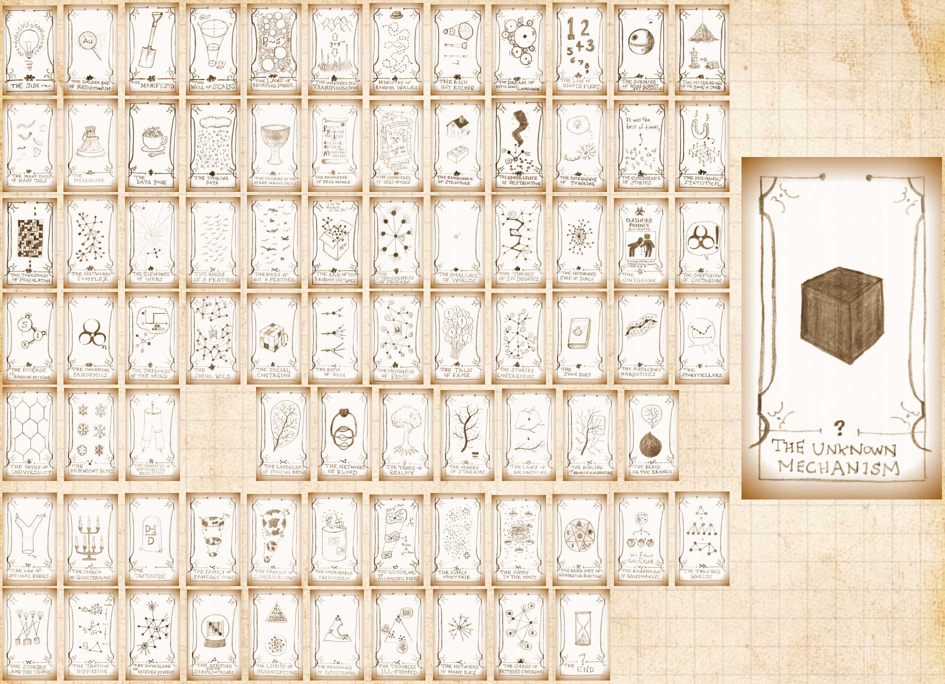
THE SUN
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It was the best of times
THE EMERGENCE OF STORIES
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THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
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THE ARKS OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
THE CONTAGION
THE CONFUSION OF CONTAGIONS
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THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
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THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE PATHS OF UNIVERSALITY
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THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
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THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
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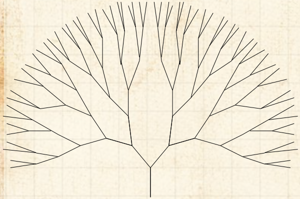
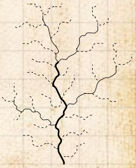
THE END
?
THE UNKNOWN MECHANISM

Random subnetworks on a Bethe lattice [13]





Models

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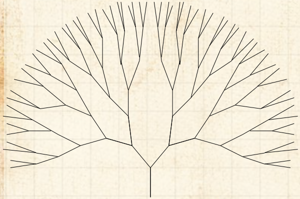

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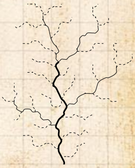

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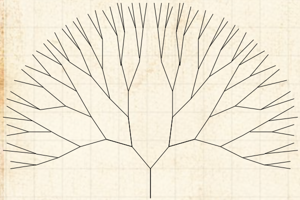

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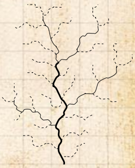

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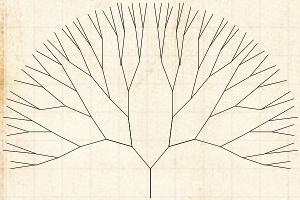

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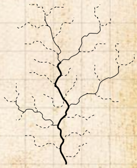

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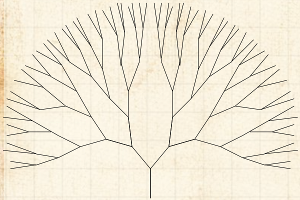

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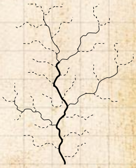

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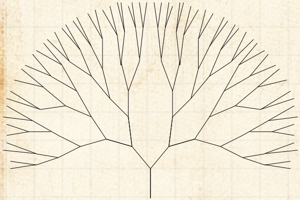

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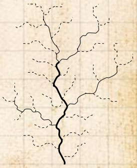

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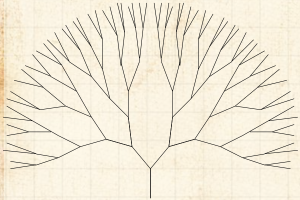

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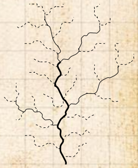
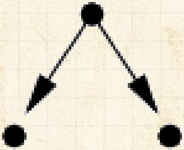
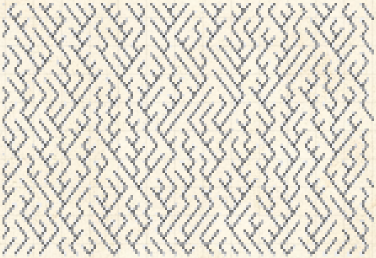


Scheidegger's model

Directed random networks [11, 12]





- $$P(\searrow) = P(\swarrow) = 1/2$$
- Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]

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Rodríguez-Iturbe, Rinaldo, et al. [10]

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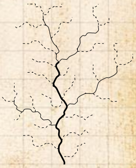

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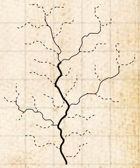

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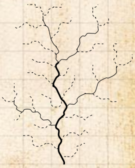

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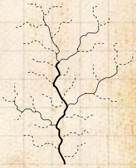

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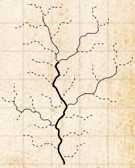
- And: Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network [8]



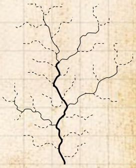
Theoretical networks

Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

$h \Rightarrow \ell \propto a^h$ (Hack's law).

$d \Rightarrow \ell \propto L_{\parallel}^d$ (stream self-affinity).



Nutshell

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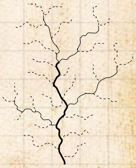

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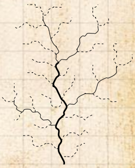

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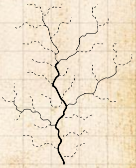
- For scaling laws, only $h = \ln R_\ell / \ln R_n$ and d are needed.
- Laws can be extended nicely to laws of distributions.



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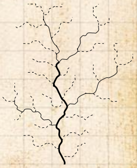
- Laws can be extended nicely to laws of distributions.
- Numerous models of branching network evolution exist: nothing rock solid yet.



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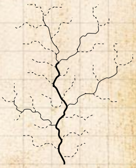

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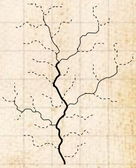

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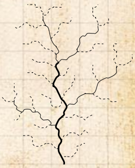

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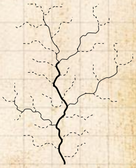

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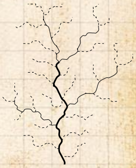

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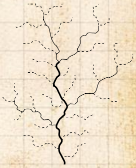

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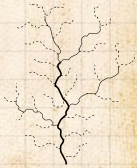

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