Mixed, correlated random networks

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Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2018

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Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

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Outline


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
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THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
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THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
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THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
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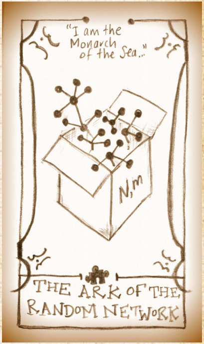
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


"I am the Monarch of the Sea."
N, m
THE ARK OF THE RANDOM NETWORK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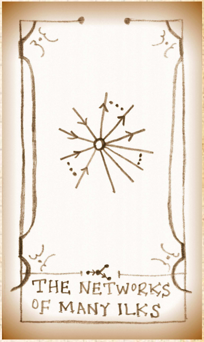
THE NETWORKS OF MANY ILKS

Random directed networks:

- So far, we've largely studied networks with undirected, unweighted edges.
- Now consider directed, unweighted edges.
- Nodes have k_i and k_o incoming and outgoing edges, otherwise random.

- Network defined by joint in- and out-degree distribution: P_{k_i,k_o}
- Normalization: $\sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} P_{k_i,k_o} = 1$
- Marginal in-degree and out-degree distributions:

$$P_{k_i} = \sum_{k_o=0}^{\infty} P_{k_i,k_o} \text{ and } P_{k_o} = \sum_{k_i=0}^{\infty} P_{k_i,k_o}$$

- Required balance:

$$\langle k_i \rangle = \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} k_i P_{k_i,k_o} = \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} k_o P_{k_i,k_o} = \langle k_o \rangle$$

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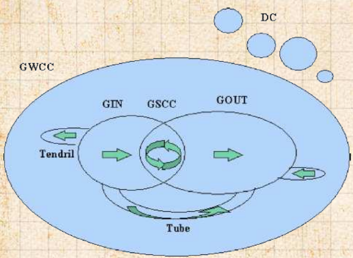
Directed network structure:



From Boguñá and Serano. [1]

- GWCC = Giant Weakly Connected Component (directions removed);
- GIN = Giant In-Component;
- GOUT = Giant Out-Component;
- GSCC = Giant Strongly Connected Component;
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- When moving through a family of increasingly connected directed random networks, GWCC usually appears before GIN, GOUT, and GSCC which tend to appear together. [4, 1]

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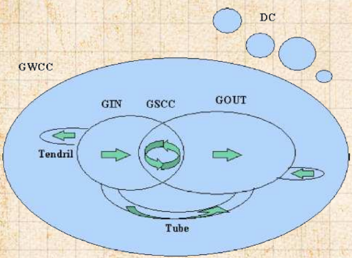

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Important observation:

- Directed and undirected random networks are separate families ...
- ...and analyses are also disjoint.
- Need to examine a larger family of random networks with mixed directed and undirected edges.

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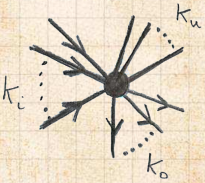

- Consider nodes with three types of edges:
  1. k_u undirected edges,
  2. k_i incoming directed edges,
  3. k_o outgoing directed edges.
- Define a node by generalized degree:

$$\vec{k} = [\, k_\mathrm{u} \;\; k_\mathrm{i} \;\; k_\mathrm{o} \,]^\mathrm{T}.$$

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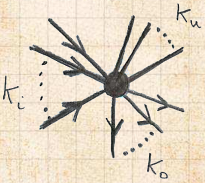

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- Joint degree distribution:

$$P_{\vec{k}} \text{ where } \vec{k} = [\, k_{\mathrm{u}} \;\; k_{\mathrm{i}} \;\; k_{\mathrm{o}} \,]^{\mathsf{T}}.$$

- As for directed networks, require in- and out-degree averages to match up:

$$\langle k_{\mathrm{i}} \rangle = \sum_{k_{\mathrm{u}}=0}^{\infty} \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{i}} P_{\vec{k}} = \sum_{k_{\mathrm{u}}=0}^{\infty} \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{o}} P_{\vec{k}} = \langle k_{\mathrm{o}} \rangle$$

- Otherwise, no other restrictions and connections are random.
- Directed and undirected random networks are disjoint subfamilies:

$$\text{Undirected: } P_{\vec{k}} = P_{k_{\mathrm{u}}} \delta_{k_{\mathrm{i}},0} \delta_{k_{\mathrm{o}},0}$$

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Correlations:

- Now add correlations (two point or Markovian):
  1. $P^{(u)}(\vec{k}\,|\,\vec{k}')$ = probability that an undirected edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node.
  2. $P^{(i)}(\vec{k}\,|\,\vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an in-directed edge relative to the destination node.
  3. $P^{(o)}(\vec{k}\,|\,\vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an out-directed edge relative to the destination node.
- Now require more refined (detailed) balance.
- Conditional probabilities cannot be arbitrary.
  1. $P^{(u)}(\vec{k}\,|\,\vec{k}')$ must be related to $P^{(u)}(\vec{k}'\,|\,\vec{k})$.
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- Randomly choose an edge, and randomly choose one end.
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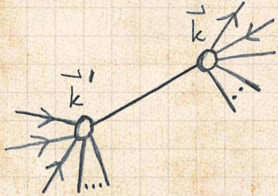

- Conditional probability connection:

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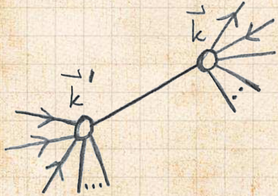

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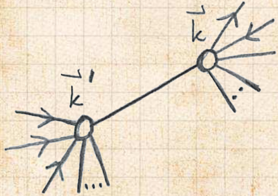

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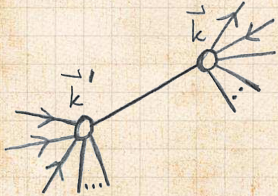

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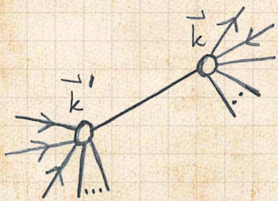

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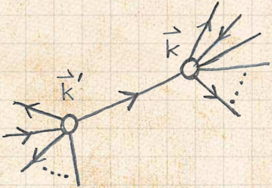
- The quantities

$$\frac{k_{\mathrm{o}} P(\vec{k})}{\langle k_{\mathrm{o}} \rangle} \text{ and } \frac{k_{\mathrm{i}} P(\vec{k})}{\langle k_{\mathrm{i}} \rangle}$$

give the probabilities that in starting at a random end of a randomly selected edge, we begin at a degree $\vec{k}$ node and then find ourselves travelling:

1. along an outgoing edge, or
2. against the direction of an incoming edge.



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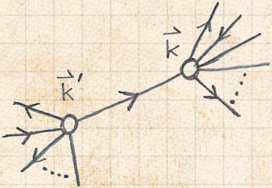

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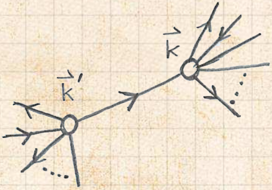

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Outline

Global spreading condition: [2]

When are cascades possible?:

- Consider uncorrelated mixed networks first.
- Recall our first result for undirected random networks, that edge gain ratio must exceed 1:

$$\mathbf{R} = \sum_{k_{\mathrm{u}}=0}^{\infty} \frac{k_{\mathrm{u}} P_{k_{\mathrm{u}}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}},1} > 1.$$

- Similar form for purely directed networks:

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Local growth equation:

- Define number of infected edges leading to nodes a distance d away from the original seed as $f(d)$.
- Infected edge growth equation:

$$f(d+1) = \mathbf{R} f(d).$$

- Applies for discrete time and continuous time contagion processes.
- Now see $B_{k_{u},1}$ is the probability that an infected edge eventually infects a node.
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Mixed, uncorrelated random netwoks:

- Now have two types of edges spreading infection: directed and undirected.
- Gain ratio now more complicated:
  1. Infected directed edges can lead to infected directed or undirected edges.
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$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} \frac{k_u P_{\vec{k}}}{\langle k_u \rangle} \bullet (k_u - 1) & \frac{k_i P_{\vec{k}}}{\langle k_i \rangle} \bullet k_u \\ \frac{k_u P_{\vec{k}}}{\langle k_u \rangle} \bullet k_o & \frac{k_i P_{\vec{k}}}{\langle k_i \rangle} \bullet k_o \end{bmatrix} \bullet B_{k_u + k_i, 1}$$

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$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) & \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{u}} \\ \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet k_{\mathrm{o}} & \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1}$$

- Spreading condition: max eigenvalue of $\mathbf{R} > 1$.

- Gain ratio now has a matrix form:

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- Two separate gain equations:

$$f^{(\mathrm{u})}(d{+}1) = \sum_{\vec{k}} \left[\frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{u})}(d) + \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{u}} \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1} f^{(\mathrm{o})}(\right.$$

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Global spreading condition:

- Useful change of notation for making results more general: write $P^{(\mathrm{u})}(\vec{k}\,|\,*) = \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle}$ and $P^{(\mathrm{i})}(\vec{k}\,|\,*) = \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle}$ where $*$ indicates the starting node's degree is irrelevant (no correlations).
- Also write $B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$ to indicate a more general infection probability, but one that does not depend on the edge's origin.
- Now have, for the example of mixed, uncorrelated random networks:

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet (k_{\mathrm{u}} - 1) & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{u}} \\ P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$$

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Summary of contagion conditions for uncorrelated networks:

- I. Undirected, Uncorrelated—$f(d+1) = \mathbf{f}(d)$:

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$$\begin{bmatrix} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{bmatrix}$$

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Correlated version:

- Now have to think of transfer of infection from edges emanating from degree $\vec{k}'$ nodes to edges emanating from degree $\vec{k}$ nodes.
- Replace $P^{(\mathrm{i})}(\vec{k} \mid *)$ with $P^{(\mathrm{i})}(\vec{k} \mid \vec{k}')$ and so on.
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Summary of contagion conditions for correlated networks:

- IV. Undirected, Correlated—$f_{k_u}(d+1) = \sum_{k_u'} R_{k_u k_u'} f_{k_u'}(d)$

$$R_{k_u k_u'} = P^{(u)}(k_u \mid k_u') \bullet (k_u - 1) \bullet B_{k_u k_u'}$$

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Outline

Directed random networks

Mixed random networks

- Definition
- Correlations

Mixed Random Network Contagion

- Spreading condition
- **Full generalization**
- Triggering probabilities

Nutshell

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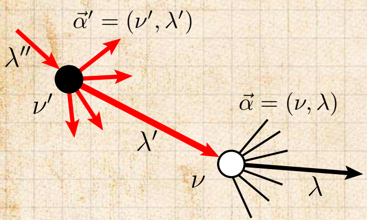
Full generalization:



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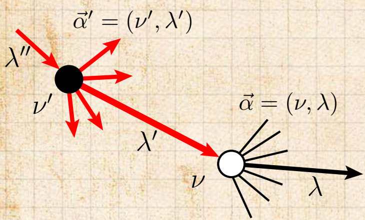

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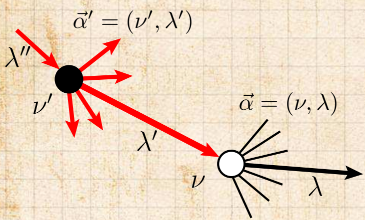

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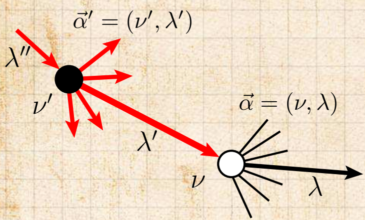

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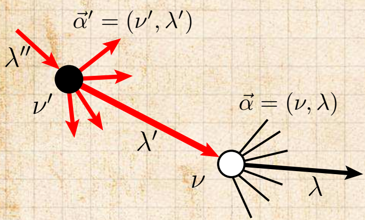

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- Two good things:

$$Q_{\text{trig}} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet B_{k1} \bullet \left[1 - \left(1 - Q_{\text{trig}}\right)^{k-1}\right],$$

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- Generating functions arguably make some kinds of calculations easier (but perhaps we don't care about component sizes that much).
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- As we saw earlier, the triggering probability for simple contagion on random networks can be determined with a straightforward physical argument.
- Two good things:

$$Q_{\mathrm{trig}} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet B_{k1} \bullet \left[1 - \left(1 - Q_{\mathrm{trig}} \right)^{k-1} \right],$$

$$P_{\mathrm{trig}} = S_{\mathrm{trig}} = \sum_k P_k \bullet \left[1 - (1 - Q_{\mathrm{trig}})^k \right].$$

- Equivalent to result found via the eldritch route of generating functions.
- Generating functions arguably make some kinds of calculations easier (but perhaps we don't care about component sizes that much).
- On the other hand, a plainspoken physical argument helps us generalize to correlated networks more easily.

Summary of triggering probabilities for uncorrelated networks: [3]

- I. Undirected, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_{\text{u}}} P^{(\text{u})}(k'_{\text{u}} \,|\, \cdot) B_{k'_{\text{u}}1} \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{u}}-1}\right]$$

$$P_{\text{trig}} = S_{\text{trig}} = \sum_{k'_{\text{u}}} P(k'_{\text{u}}) \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{u}}}\right]$$

- II. Directed, Uncorrelated—

$$Q_{\text{trig}} = \sum_{k'_{\text{i}}, k'_{\text{o}}} P^{(\text{u})}(k'_{\text{i}}, k'_{\text{o}} \,|\, \cdot) B_{k'_{\text{i}}1} \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{o}}}\right]$$

$$S_{\text{trig}} = \sum_{k'_{\text{i}}, k'_{\text{o}}} P(k'_{\text{i}}, k'_{\text{o}}) \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{o}}}\right]$$

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Summary of triggering probabilities for uncorrelated networks:

- III. Mixed Directed and Undirected, Uncorrelated—

$$Q^{(\mathrm{u})}_{\mathrm{trig}} = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}' \,|\, \cdot) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}})^{k'_{\mathrm{u}}-1}(1 - Q^{(\mathrm{o})}_{\mathrm{trig}})^{k'_{\mathrm{o}}}\right]$$

$$Q^{(\mathrm{o})}_{\mathrm{trig}} = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}' \,|\, \cdot) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}})^{k'_{\mathrm{u}}}(1 - Q^{(\mathrm{o})}_{\mathrm{trig}})^{k'_{\mathrm{o}}}\right]$$

$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}})^{k'_{\mathrm{u}}}(1 - Q^{(\mathrm{o})}_{\mathrm{trig}})^{k'_{\mathrm{o}}}\right]$$

Summary of triggering probabilities for correlated networks:

- IV. Undirected, Correlated— $Q_{\text{trig}}(k_{\text{u}}) = \sum_{k'_{\text{u}}} P^{(\text{u})}(k'_{\text{u}} \,|\, k_{\text{u}}) B_{k'_{\text{u}}1} \left[1 - (1 - Q_{\text{trig}}(k'_{\text{u}}))^{k'_{\text{u}}-1}\right]$

$$S_{\text{trig}} = \sum_{k'_{\text{u}}} P(k'_{\text{u}}) \left[1 - (1 - Q_{\text{trig}}(k'_{\text{u}}))^{k'_{\text{u}}}\right]$$

- V. Directed, Correlated— $Q_{\text{trig}}(k_{\text{i}}, k_{\text{o}}) = \sum_{k'_{\text{i}}, k'_{\text{o}}} P^{(\text{u})}(k'_{\text{i}}, k'_{\text{o}} \,|\, k_{\text{i}}, k_{\text{o}}) B_{k'_{\text{i}}1} \left[1 - (1 - Q_{\text{trig}}(k'_{\text{i}}, k'_{\text{o}}))^{k'_{\text{o}}}\right]$

$$S_{\text{trig}} = \sum_{k'_{\text{i}}, k'_{\text{o}}} P(k'_{\text{i}}, k'_{\text{o}}) \left[1 - (1 - Q_{\text{trig}}(k'_{\text{i}}, k'_{\text{o}}))^{k'_{\text{o}}}\right]$$

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Summary of triggering probabilities for correlated networks:

- VI. Mixed Directed and Undirected, Correlated—

$$Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}'\,|\,\vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}-1}(1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

$$Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}'\,|\,\vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}}(1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

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Nutshell:

- Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.
- Spreading conditions and triggering probabilities of contagion processes can be determined using a direct, physical approach.
- These conditions can be generalized to arbitrary random networks with arbitrary node and edge types.
- More generalizations: bipartite affiliation graphs and multilayer networks.

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