Generating Functions and Networks

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Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2018

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Vermont Advanced Computing Core | University of Vermont

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Outline


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TAILS OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
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THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED PHOENIX BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
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THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
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THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
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THE TREES OF REALITY
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THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
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THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
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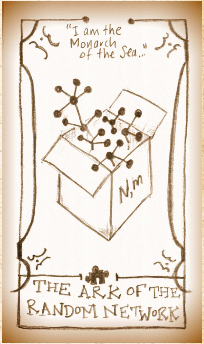
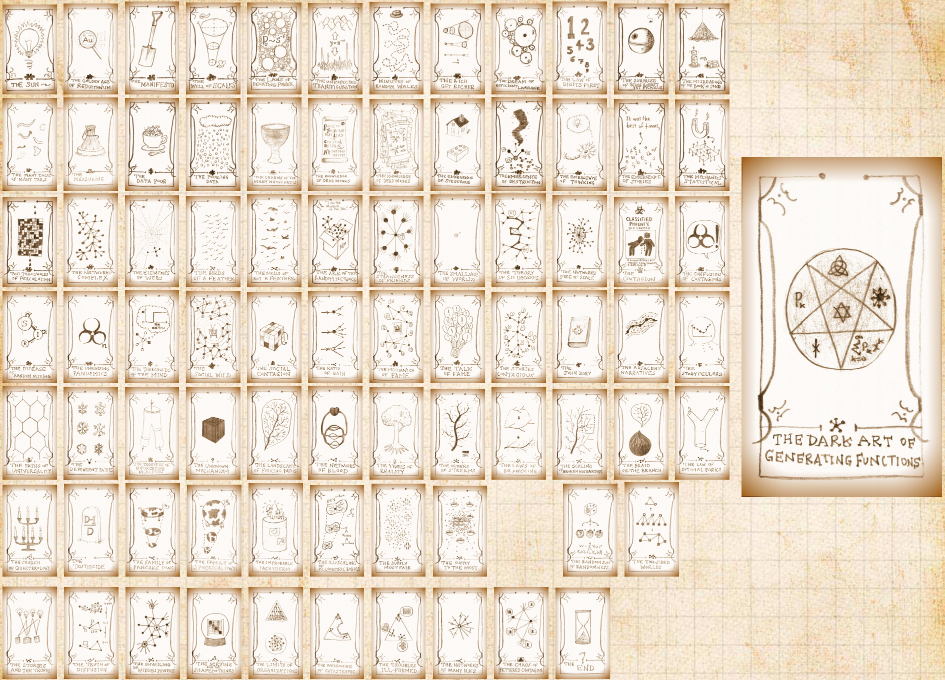
THE CHAOS OF FETTERED CONTAGION
THE END


"I am the Monarch of the Sea."
N,m
THE ARK OF THE RANDOM NETWORK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THE DARK ART OF GENERATING FUNCTIONS

Outline

Generatingfunctionology [1]

- Idea: Given a sequence $a_0, a_1, a_2, \dots,$ associate each element with a distinct function or other mathematical object.
- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

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Definition:

- The generating function (g.f.) for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- Roughly: transforms a vector in R^∞ into a function defined on R^1.
- Related to Fourier, Laplace, Mellin, ...

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Simple examples:

Rolling dice and flipping coins:

- $p_k^{(⚂)} = \mathbf{Pr}(\text{throwing a } k) = 1/6$ where $k = 1, 2, \ldots, 6$.

$$F^{(⚂)}(x) = \sum_{k=1}^{6} p_k^{(⚂)} x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

- $p_0^{(\text{coin})} = \mathbf{Pr}(\text{head}) = 1/2$, $p_1^{(\text{coin})} = \mathbf{Pr}(\text{tail}) = 1/2$.

$$F^{(\text{coin})}(x) = p_0^{(\text{coin})} x^0 + p_1^{(\text{coin})} x^1 = \frac{1}{2}(1 + x).$$

- A generating function for a probability distribution is called a Probability Generating Function (p.g.f.).
- We'll come back to these simple examples as we derive various delicious properties of generating functions.

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Example

- Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where geometricsumfully, we have $c = 1 - e^{-\lambda}$

- The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}.$$

- Notice that $F(1) = c/(1 - e^{-\lambda}) = 1$.
- For probability distributions, we must always have $F(1) = 1$ since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k = \sum_{k=0}^{\infty} P_k = 1.$$

- Check die and coin p.g.f.'s.

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Outline

Generating Functions

- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- A few examples
- Average Component Size

References

Properties:

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} F(x) \bigg|_{x=1} = F'(1)$$

- In general, many calculations become simple, if a little abstract.
- For our exponential example:

$$F'(x) = \frac{(1-e^{-\lambda})e^{-\lambda}}{(1-xe^{-\lambda})^2}.$$

- So: $\langle k \rangle = F'(1) = \frac{e^{-\lambda}}{(1-e^{-\lambda})}$.
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Useful pieces for probability distributions:

- Normalization:

$$F(1) = 1$$

- First moment:

$$\langle k \rangle = F'(1)$$

- Higher moments:

$$\langle k^n \rangle = \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^n F(x) \bigg|_{x=1}$$

- kth element of sequence (general):

$$P_k = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) \bigg|_{x=0}$$

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A beautiful, fundamental thing:

- The generating function for the sum of two random variables

$$W = U + V$$

is

$$F_W(x) = F_U(x) F_V(x).$$

- Convolve yourself with Convolutions: Insert question from assignment 5 ↗ .
- Try with die and coin p.g.f.'s.
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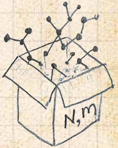

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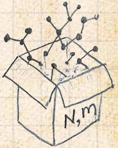

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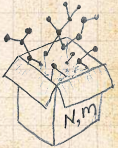

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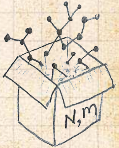

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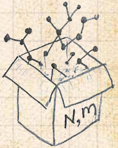

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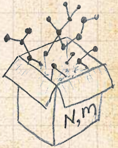

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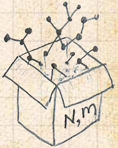

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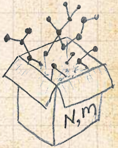

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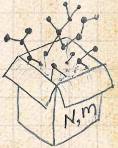

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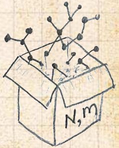

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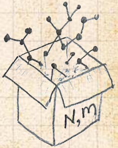

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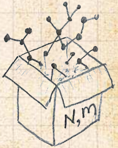

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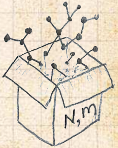

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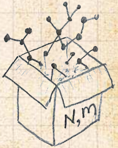

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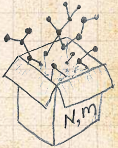

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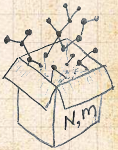

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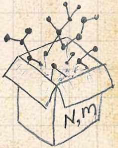

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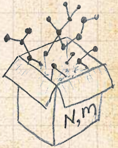

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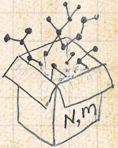

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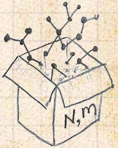

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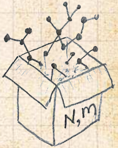

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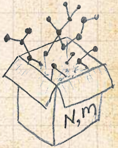

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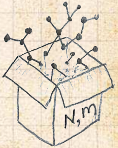

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Definitions:

- π_n = probability that a random node belongs to a finite component of size $n < \infty$.
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$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

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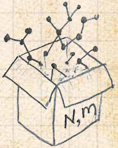

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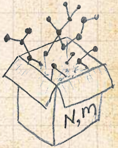

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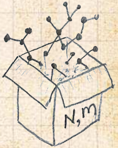
$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

neighbors $\Leftrightarrow$ components



Size distributions

To figure out the size of the largest component (S_1), we need more resolution on component sizes.

Definitions:

- π_n = probability that a random node belongs to a finite component of size $n < \infty$.
- ρ_n = probability that a random end of a random link leads to a finite subcomponent of size $n < \infty$.

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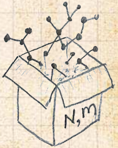
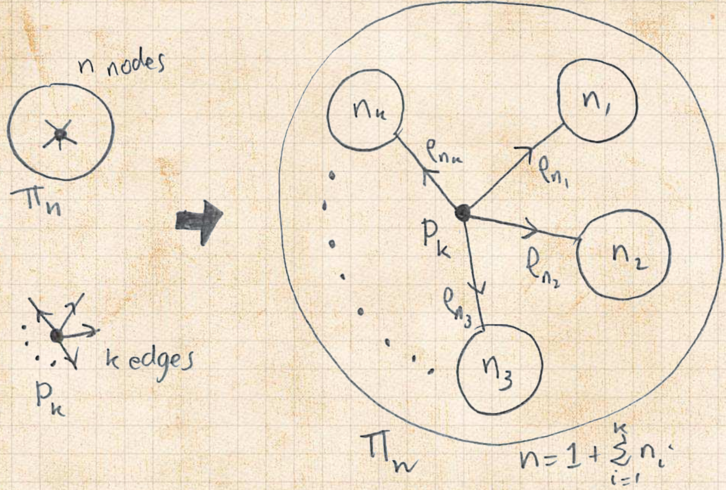
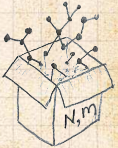


Connecting probabilities:





- Markov property of random networks connects π_n, ρ_n, and P_k.

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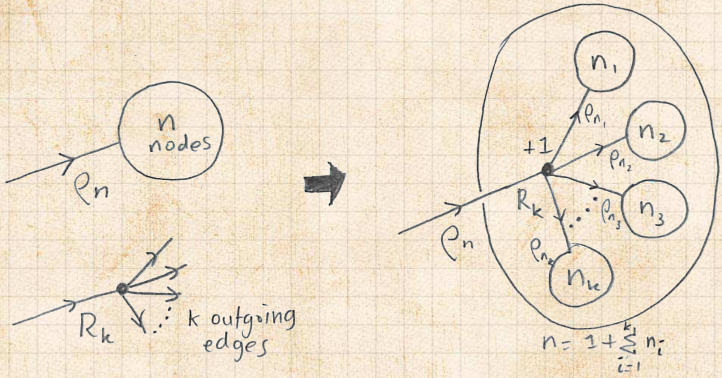

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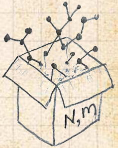


G.f.'s for component size distributions:

$$F_\pi(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

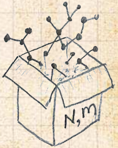
The largest component:

- Subtle key: $F_\pi(1)$ is the probability that a node belongs to a finite component.
- Therefore: $S_1 = 1 - F_\pi(1)$.

Our mission, which we accept:

- Determine and connect the four generating functions

$$F_P, F_R, F_\pi, \text{ and } F_\rho.$$



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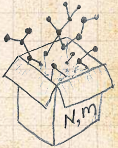

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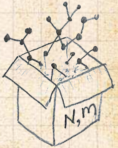

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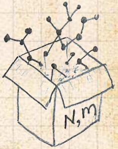

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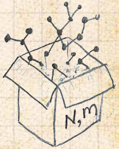

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Outline

Useful results we'll need for g.f.'s

Sneaky Result 1:

- Consider two random variables U and V whose values may be $0, 1, 2, \ldots$
- Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V.
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^{U} V^{(i)} \text{ with each } V^{(i)} \stackrel{d}{=} V$$

$$F_W(x) = F_U(F_V(x))$$

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THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
1 2 5 4 3 6 7 8 9
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
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THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARKS OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
D-1
D
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
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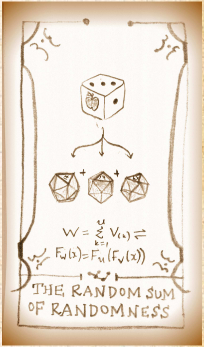
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END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THE RANDOM SUM OF RANDOMNESS

Proof of SR1:

Write probability that variable W has value k as W_k.

$$W_k = \sum_{j=0}^{\infty} U_j \times \Pr(\text{sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\} \mid \\ i_1 + i_2 + \ldots + i_j = k}} V_{i_1} V_{i_2} \cdots V_{i_j}$$

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- Alternate, groovier proof in the accompanying assignment.

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Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \ldots)$
- SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

- **Reason:** $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.
-

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

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- SR2: If a second random variable is defined as

$$V = U + 1 \quad \text{then} \quad \boxed{F_V(x) = x F_U(x)}$$

- Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.

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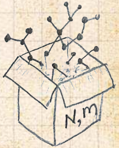
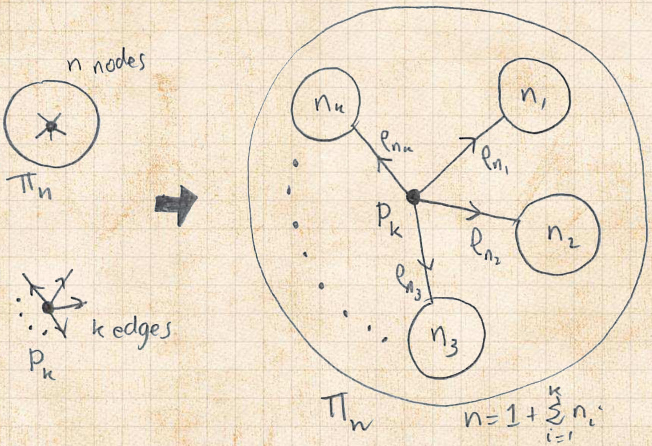
Outline

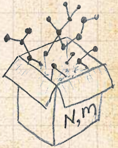


Connecting generating functions:

- Goal: figure out forms of the component generating functions, F_π and F_ρ.



- Relate π_n to P_k and ρ_n through one step of recursion.



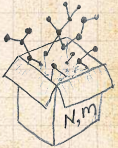
Connecting generating functions:

- π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr\left(\begin{array}{c}\text{sum of sizes of subcomponents}\\ \text{at end of } k \text{ random links} = n-1\end{array}\right)$$

- Therefore: $\boxed{F_{\pi}(x) = x\, F_P\left(F_{\rho}(x)\right)}$

- Extra factor of x accounts for random node itself.



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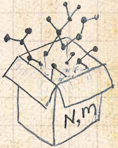

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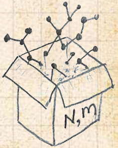

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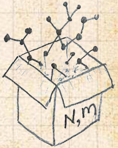

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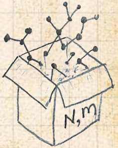

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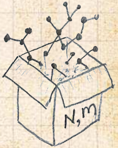

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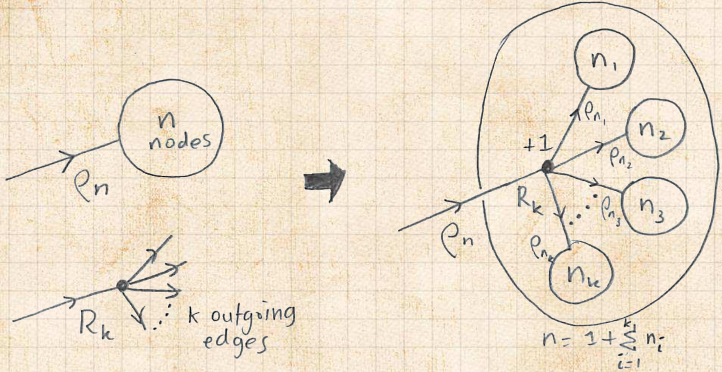

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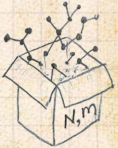

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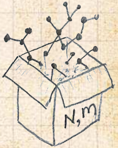

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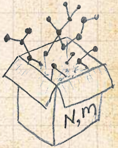

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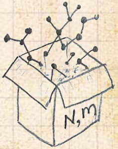

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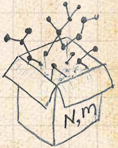

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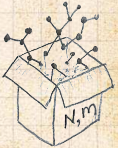

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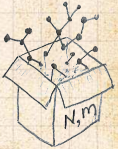

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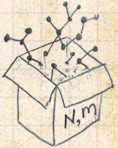

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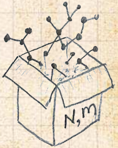

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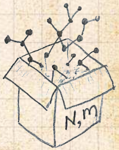

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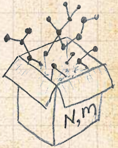

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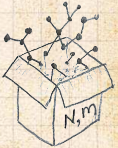

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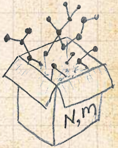

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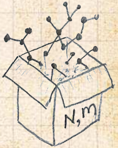

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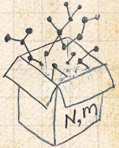

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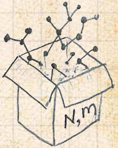

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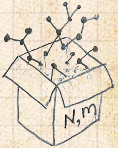

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  Finding F_π to obtain the fractional size of the largest component $S_1 = 1 - F_\pi(1)$.

- Set $x = 1$ in our two equations:

$$F_\pi(1) = F_P\left(F_\rho(1)\right) \quad \text{and} \quad F_\rho(1) = F_R\left(F_\rho(1)\right)$$

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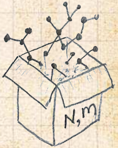

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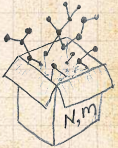

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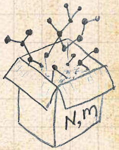

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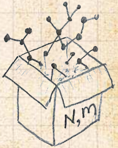

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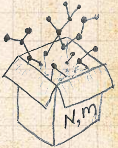

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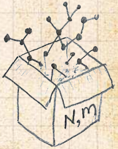

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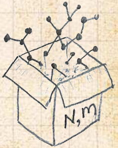

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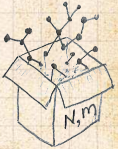

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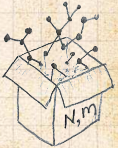

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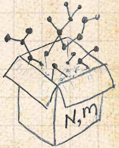

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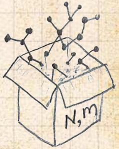

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  Or: $\langle k\rangle = \frac{1}{S_1}\ln\frac{1}{1-S_1}$
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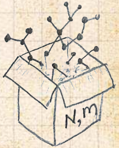

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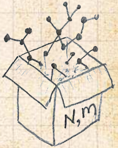

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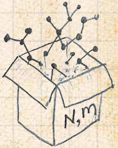

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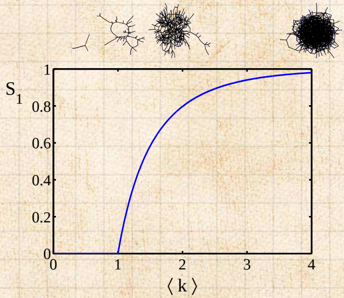

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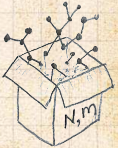

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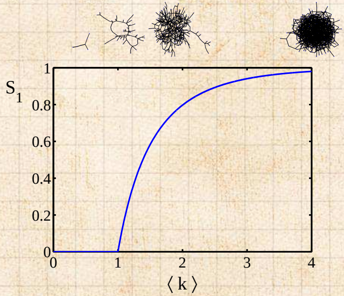

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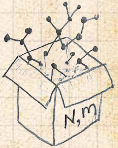

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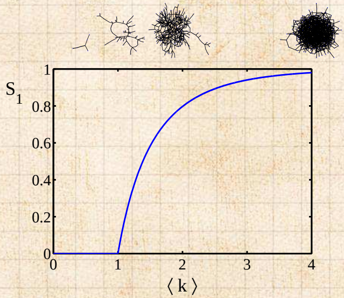

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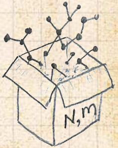
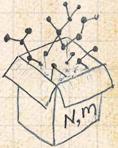


Outline



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- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
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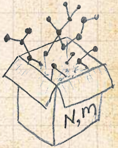

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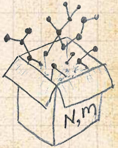

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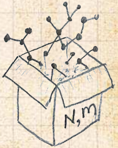

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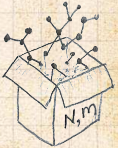

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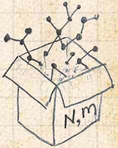

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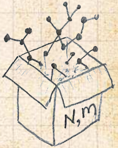

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- Notation: The Kronecker delta function $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
- $P_k = \delta_{k1}$.
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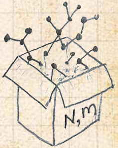

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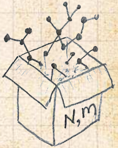

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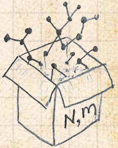

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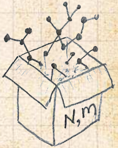

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$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$

- We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}$.
- A giant component exists because: $\langle k \rangle_R = 0 \times 1/4 + 2 \times 3/4 = 3/2 > 1$.
- Generating functions for P_k and R_k:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2$$

- Check for goodness:
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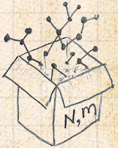

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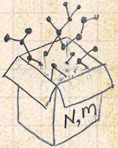

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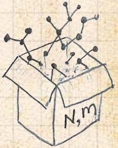

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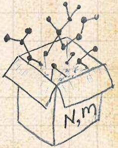

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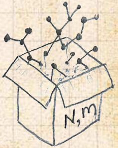

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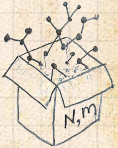

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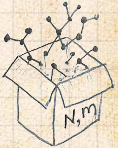
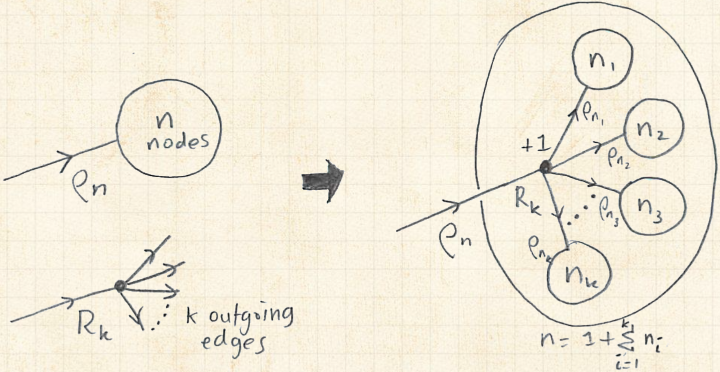


Find $F_{\rho}(x)$ first:

- We know:

$$F_{\rho}(x) = xF_R\left(F_{\rho}(x)\right).$$



- Sticking things in things, we have:

$$F_\rho(x) = x\left(\frac{1}{4} + \frac{3}{4}\left[F_\rho(x)\right]^2\right).$$

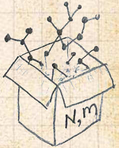
- Rearranging:

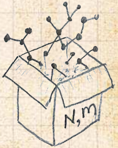
$$3x\left[F_\rho(x)\right]^2 - 4F_\rho(x) + x = 0.$$

- Please and thank you:

$$F_\rho(x) = \frac{2}{3x}\left(1 \pm \sqrt{1 - \frac{3}{4}x^2}\right)$$

- Time for a Taylor series expansion.
- The promise: non-negative powers of x with non-negative coefficients.
- First: which sign do we take?





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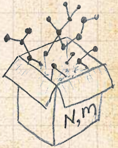

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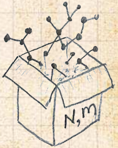

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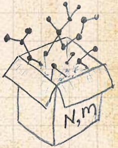

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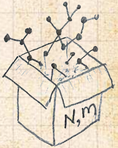

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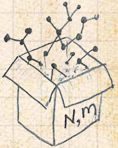
we see that the positive sign solution blows to smithereens, and the negative one is okay.
- So we must have:

$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right),$$

- We can now deploy the Taylor expansion:

$$(1+z)^\theta = \binom{\theta}{0}z^0 + \binom{\theta}{1}z^1 + \binom{\theta}{2}z^2 + \binom{\theta}{3}z^3 + \ldots$$



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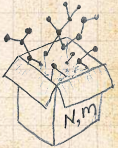

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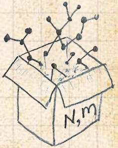

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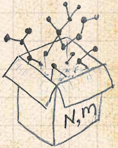


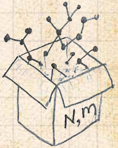
Let's define a binomial for arbitrary θ and $k = 0, 1, 2, \ldots$:

$$\binom{\theta}{k} = \frac{\Gamma(\theta+1)}{\Gamma(k+1)\Gamma(\theta-k+1)}$$

For $\theta = \frac{1}{2}$, we have:

$$(1+z)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} z^0 + \binom{\frac{1}{2}}{1} z^1 + \binom{\frac{1}{2}}{2} z^2 + \ldots$$

Note: $(1+z)^\theta \sim 1 + \theta z$ always.



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$$(1+z)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} z^0 + \binom{\frac{1}{2}}{1} z^1 + \binom{\frac{1}{2}}{2} z^2 + \ldots$$

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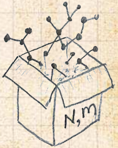
- Note: $(1+z)^\theta \sim 1 + \theta z$ always.



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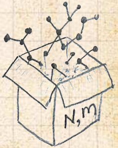

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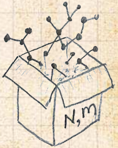

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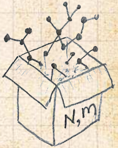

- Totally psyched, we go back to here:

$$F_\rho(x) = \frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right).$$

- Setting $z = -\frac{3}{4}x^2$ and expanding, we have:

$$F_\rho(x) =$$

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$$F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n =$$

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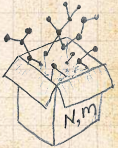
- Delicious.
- In principle, we can now extract all the π_n.
- But let's just find the size of the giant component.



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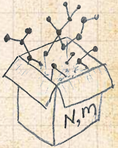

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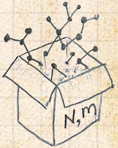

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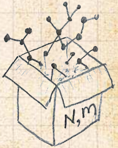

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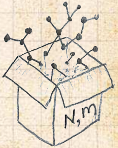

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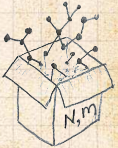

First, we need $F_\rho(1)$:

$$F_\rho(x)\big|_{x=1} = \frac{2}{3\cdot 1}\left(1 - \sqrt{1 - \frac{3}{4}1^2}\right) = \frac{1}{3}.$$

This is the probability that a random edge leads to a sub-component of finite size.

Next:

$$F_\pi(1) = 1 \cdot F_P\left(F_\rho(1)\right) = F_P\left(\frac{1}{3}\right) = \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\left(\frac{1}{3}\right)^3 = \frac{5}{27}.$$

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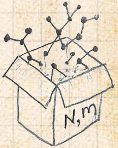
Finally, we have

$$S_1 = 1 - F_\pi(1) = 1 - \frac{5}{27} = \frac{22}{27}.$$



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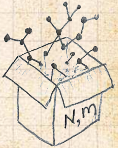

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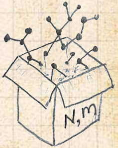

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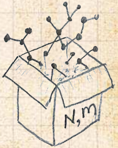

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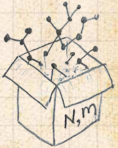

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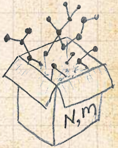

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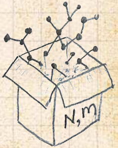

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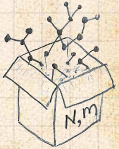
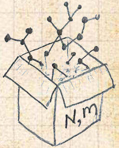
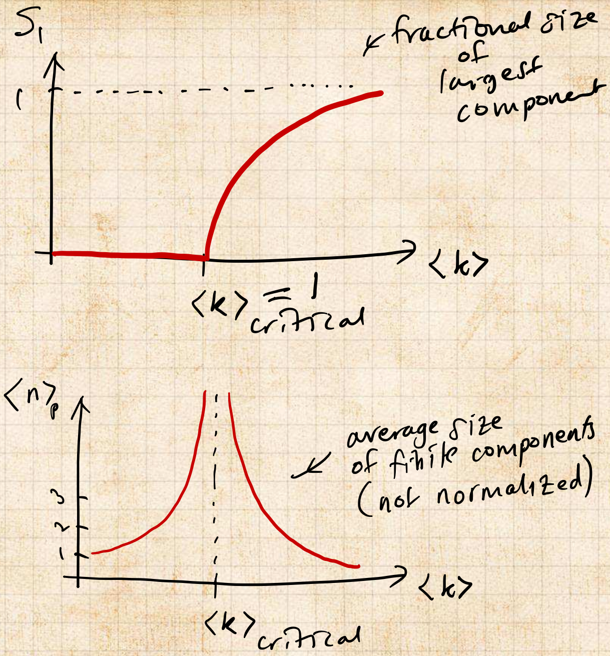


Outline

Generating Functions




S_1
1
fractional size of largest component
$\langle k \rangle$
$\langle k \rangle_{critical} = 1$
$\langle n \rangle_p$
3
2
1
average size of finite components (not normalized)
$\langle k \rangle$
$\langle k \rangle_{critical}$

Average component size

- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.
- Try to avoid finding $F_{\pi}(x)$ ...
- Starting from $F_{\pi}(x) = x F_P(F_{\rho}(x))$, we differentiate:

$$F'_{\pi}(x) = F_P(F_{\rho}(x)) + x F'_{\rho}(x) F'_P(F_{\rho}(x))$$

- While $F_{\rho}(x) = x F_R(F_{\rho}(x))$ gives

$$F'_{\rho}(x) = F_R(F_{\rho}(x)) + x F'_{\rho}(x) F'_R(F_{\rho}(x))$$

- Now set $x = 1$ in both equations.
- We solve the second equation for $F'_{\rho}(1)$ (we must already have $F_{\rho}(1)$).
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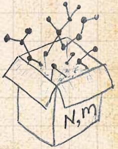

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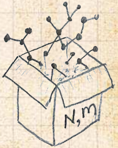

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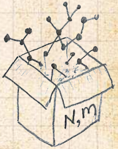

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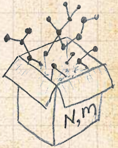

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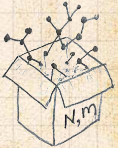

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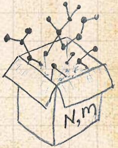

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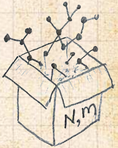

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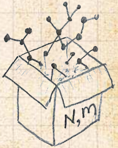

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- Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$
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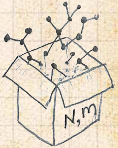

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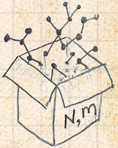

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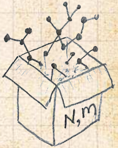

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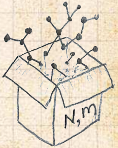

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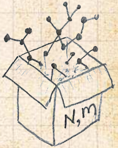

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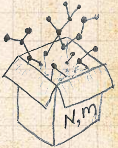

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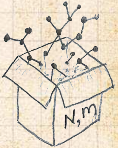

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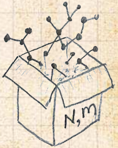

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Average component size

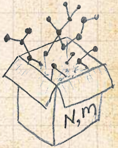
- Our result for standard random networks:

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- This blows up as $\langle k \rangle \to 1$.
- Reason: we have a power law distribution of component sizes at $\langle k \rangle = 1$.
- Typical critical point behavior ...



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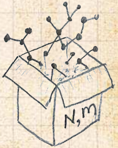

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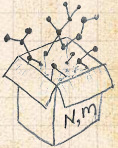

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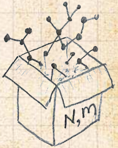

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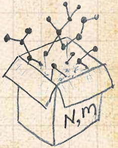

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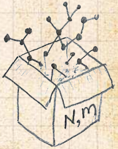

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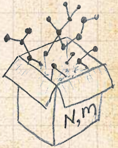

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Average component size

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Extra on largest component size:

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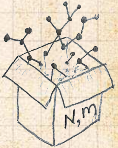

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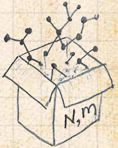

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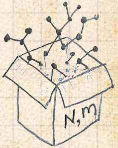

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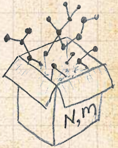

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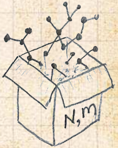

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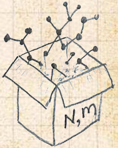

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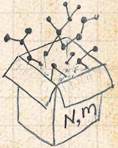

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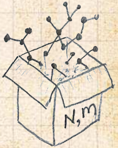

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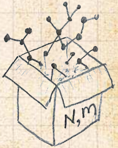

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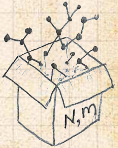

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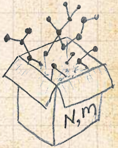

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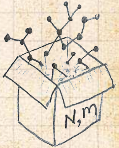

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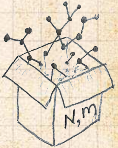

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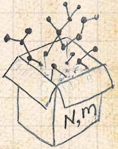

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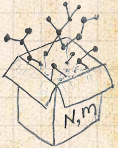

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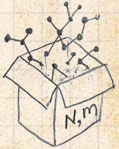

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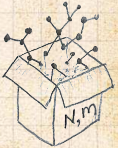

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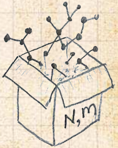

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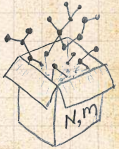

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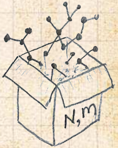

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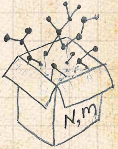

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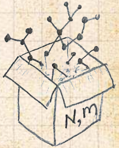

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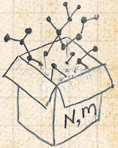

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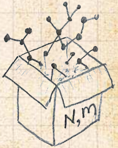

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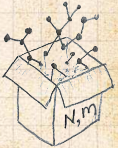
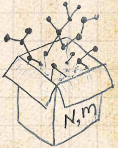


Neural reboot (NR):

Elevation:

https://www.youtube.com/watch?v=bGBoZbT7cR8?rel=0



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