

Generating Functions and Networks

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Complex Networks | @networksvox
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Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples

Average Component Size

References

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Generating Functions

Definitions

Basic Properties

Giant Component
Condition

Component sizes

Useful results

Size of the Giant
Component

A few examples

Average Component Size

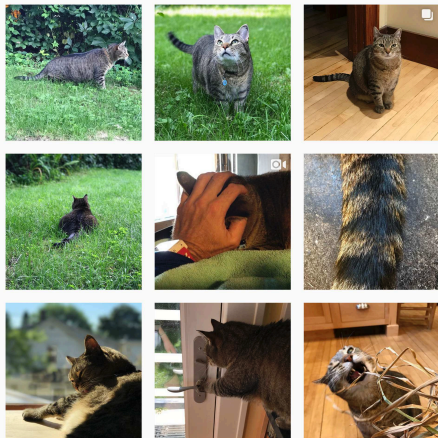
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Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results



Size of the Giant Component

A few examples

Average Component Size

References



 On Instagram at [pratchett_the_cat](https://www.instagram.com/pratchett_the_cat) 

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples

Average Component Size

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples


Average Component Size


References

References







 **Idea:** Given a sequence a_0, a_1, a_2, \dots , associate each element with a distinct function or other mathematical object.


 Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

Definition:

 The **generating function** (g.f.) for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

 Roughly: transforms a vector in R^∞ into a function defined on R^1 .

 Related to Fourier, Laplace, Mellin, ...

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples


Average Component Size

References




Simple examples:


Rolling dice and flipping coins:


 $p_k^{(\text{die})} = \mathbf{Pr}(\text{throwing a } k) = 1/6 \text{ where } k = 1, 2, \dots, 6.$

$$F^{(\text{die})}(x) = \sum_{k=1}^6 p_k^{(\text{die})} x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

 $p_0^{(\text{coin})} = \mathbf{Pr}(\text{head}) = 1/2, p_1^{(\text{coin})} = \mathbf{Pr}(\text{tail}) = 1/2.$

$$F^{(\text{coin})}(x) = p_0^{(\text{coin})}x^0 + p_1^{(\text{coin})}x^1 = \frac{1}{2}(1 + x).$$

 A generating function for a probability distribution is called a **Probability Generating Function (p.g.f.)**.

 We'll come back to these simple examples as we derive various delicious properties of generating functions.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Example

Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where geometrically, we have $c = 1 - e^{-\lambda}$

The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}.$$

Notice that $F(1) = c/(1 - e^{-\lambda}) = 1$.

For probability distributions, we must always have $F(1) = 1$ since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k = \sum_{k=0}^{\infty} P_k = 1.$$

Check die and coin p.g.f.'s.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component


A few examples

Average Component Size


References




Properties:

 Average degree:

$$\begin{aligned}\langle k \rangle &= \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \Big|_{x=1} \\ &= \frac{d}{dx} F(x) \Big|_{x=1} = F'(1)\end{aligned}$$

 In general, many calculations become simple, if a little abstract.

 For our exponential example:

$$F'(x) = \frac{(1 - e^{-\lambda})e^{-\lambda}}{(1 - xe^{-\lambda})^2}.$$



$$\text{So: } \langle k \rangle = F'(1) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})}.$$

 Check for die and coin p.g.f.'s.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples

Average Component Size


References




Useful pieces for probability distributions:

 Normalization:


$$F(1) = 1$$

 First moment:

$$\langle k \rangle = F'(1)$$

 Higher moments:

$$\langle k^n \rangle = \left(x \frac{d}{dx} \right)^n F(x) \Big|_{x=1}$$

 k th element of sequence (general):

$$P_k = \frac{1}{k!} \frac{d^k}{dx^k} F(x) \Big|_{x=0}$$

Generating Functions

Definitions

Basic Properties

•••••
Giant Component Condition

Component sizes

Useful results

Size of the Giant Component


A few examples

Average Component Size

References





A beautiful, fundamental thing:


-  The generating function for the sum of two random variables

$$W = U + V$$

is

$$F_W(x) = F_U(x)F_V(x).$$

-  Conolve yourself with Convolutions:
Insert question from assignment 5  .

-  Try with die and coin p.g.f.'s.
 1. Add two coins (tail=0, head=1).
 2. Add two dice.
 3. Add a coin flip to one die roll.

Generating Functions

Definitions

Basic Properties

.....
 Giant Component Condition

Component sizes

Useful results

Size of the Giant Component


A few examples

Average Component Size


References




Edge-degree distribution

 Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$


 Let's re-express our condition in terms of generating functions.

 We first need the g.f. for R_k .

 We'll now use this notation:

$F_P(x)$ is the g.f. for P_k .

$F_R(x)$ is the g.f. for R_k .

 Giant component condition in terms of g.f. is:

$$\langle k \rangle_R = F'_R(1) > 1.$$

 Now find how F_R is related to F_P ...

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

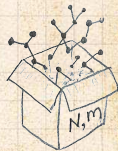
Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Edge-degree distribution



We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to $j = k + 1$ and pull out $\frac{1}{\langle k \rangle}$:

$$\begin{aligned} F_R(x) &= \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{d}{dx} x^j \\ &= \frac{1}{\langle k \rangle} \frac{d}{dx} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{d}{dx} (F_P(x) - P_0) = \frac{1}{\langle k \rangle} F'_P(x). \end{aligned}$$

Finally, since $\langle k \rangle = F'_P(1)$,

$$F_R(x) = \frac{F'_P(x)}{F'_P(1)}$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

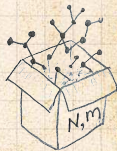
Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Recall giant component condition is $\langle k \rangle_R = F'_R(1) > 1$.

Since we have $F_R(x) = F'_P(x)/F'_P(1)$,

$$F'_R(x) = \frac{F''_P(x)}{F'_P(1)}.$$

Setting $x = 1$, our condition becomes

$$\frac{F''_P(1)}{F'_P(1)} > 1$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

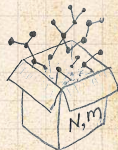
Useful results

Size of the Giant Component

A few examples

Average Component Size


References




Size distributions

To figure out the **size of the largest component** (S_1), we need more resolution on component sizes.

Definitions:

 π_n = probability that a random node belongs to a finite component of size $n < \infty$.

 ρ_n = probability that a random end of a random link leads to a finite subcomponent of size $n < \infty$.

Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

neighbors \Leftrightarrow components

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

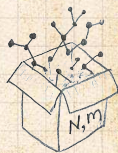
Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Connecting probabilities:

n nodes

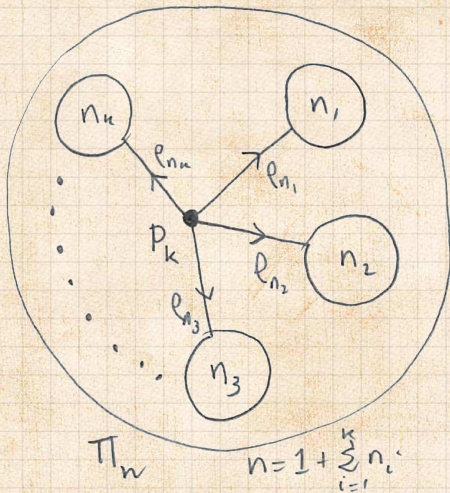


π_n



k edges

P_k



Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

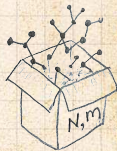
Useful results

Size of the Giant Component

A few examples

Average Component Size

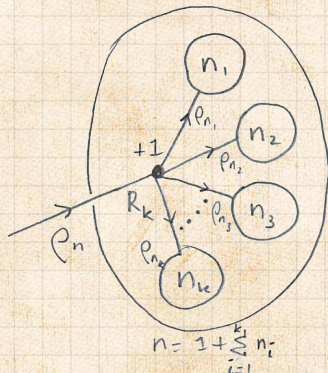
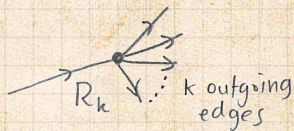
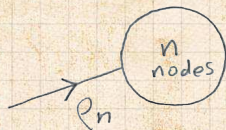
References



Markov property of random networks connects

π_n , ρ_n , and P_k .

Connecting probabilities:



Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes


Useful results

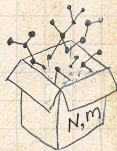
Size of the Giant Component

A few examples

Average Component Size

References

 Markov property of random networks connects ρ_n and R_k .



G.f.'s for component size distributions:



$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

The largest component:

Subtle key: $F_{\pi}(1)$ is the probability that a node belongs to a **finite** component.

Therefore: $S_1 = 1 - F_{\pi}(1)$.

Our mission, which we accept:

Determine and connect the four generating functions

$$F_P, F_R, F_{\pi}, \text{ and } F_{\rho}.$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

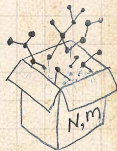
Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Useful results we'll need for g.f.'s

Sneaky Result 1:

- Consider two random variables U and V whose values may be 0, 1, 2, ...
- Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V .
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^U V^{(i)} \text{ with each } V^{(i)} \stackrel{d}{=} V$$

then

$$F_W(x) = F_U(F_V(x))$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Proof of SR1:

Write probability that variable W has value k as W_k .

$$W_k = \sum_{j=0}^{\infty} U_j \times \Pr(\text{sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} V_{i_2} \dots V_{i_j}$$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} V_{i_2} \dots V_{i_j} x^k$$

$$= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j}$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Proof of SR1:

With some concentration, observe:

$$\begin{aligned}
 F_W(x) &= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j} \\
 &= \underbrace{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j}_{\left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j = (F_V(x))^j} \\
 &= \sum_{j=0}^{\infty} U_j (F_V(x))^j \\
 &= F_U(F_V(x))
 \end{aligned}$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Alternate, groovier proof in the accompanying assignment.

Useful results we'll need for g.f.'s

Sneaky Result 2:

Start with a random variable U with distribution U_k ($k = 0, 1, 2, \dots$)

SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

$$= x \sum_{j=0}^{\infty} U_j x^j = xF_U(x).$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples


Average Component Size

References




Useful results we'll need for g.f.'s

Generalization of SR2:

 (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

 (2) If $V = U - i$ then

$$F_V(x) = x^{-i} F_U(x)$$

$$= x^{-i} \sum_{k=0}^{\infty} U_k x^k$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component


A few examples

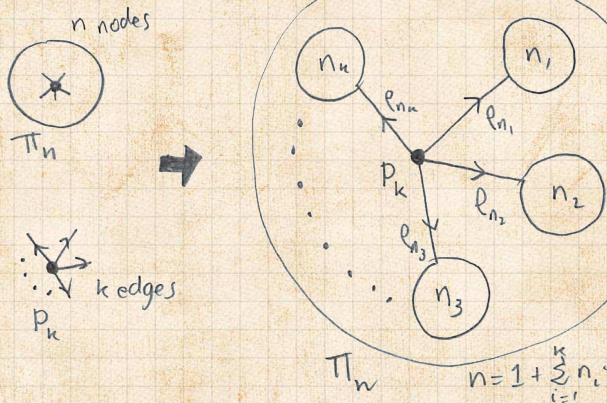
Average Component Size

References



Connecting generating functions:

 **Goal:** figure out forms of the component generating functions, F_π and F_ρ .



Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

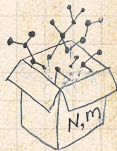
Useful results


Size of the Giant Component

A few examples


Average Component Size

References



 Relate π_n to P_k and ρ_n through one step of recursion.

Connecting generating functions:


 π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$



Therefore:

$$F_{\pi}(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_P(F_{\rho}(x))}_{\text{SR1}}$$

 Extra factor of x accounts for random node itself.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

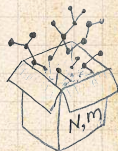
Useful results

Size of the Giant Component

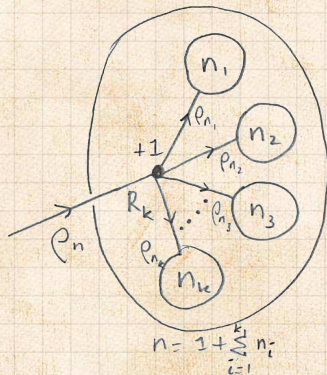
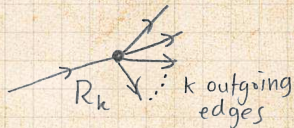
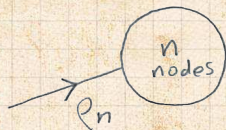
A few examples

Average Component Size

References



Connecting generating functions:



Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

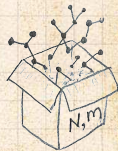
A few examples

Average Component Size


References




Relate ρ_n to R_k and ρ_n through one step of recursion.



Connecting generating functions:

 ρ_n = probability that a random link leads to a finite subcomponent of size n .

 Invoke one step of recursion:


ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n - 1$,

$$= \sum_{k=0}^{\infty} R_k \times \Pr \left(\text{sum of sizes of subcomponents at end of } k \text{ random links} = n - 1 \right)$$



Therefore:

$$F_{\rho}(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_R(F_{\rho}(x))}_{\text{SR1}}$$

 Again, extra factor of x accounts for random node itself.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

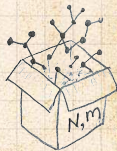
Useful results

Size of the Giant Component


A few examples

Average Component Size


References






Connecting generating functions:

-  We now have two functional equations connecting our generating functions:

$$F_{\pi}(x) = xF_{\mathcal{P}}(F_{\rho}(x)) \quad \text{and} \quad F_{\rho}(x) = xF_{\mathcal{R}}(F_{\rho}(x))$$

-  Taking stock: We know $F_{\mathcal{P}}(x)$ and $F_{\mathcal{R}}(x) = F'_{\mathcal{P}}(x)/F'_{\mathcal{P}}(1)$.

-  We first untangle the **second equation** to find F_{ρ}
-  We can do this because it **only involves** F_{ρ} and $F_{\mathcal{R}}$.
-  The first equation then immediately gives us F_{π} in terms of F_{ρ} and $F_{\mathcal{R}}$.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

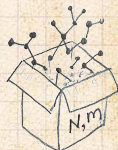
Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Remembering vaguely what we are doing:

Finding F_π to obtain the **fractional size of the largest component** $S_1 = 1 - F_\pi(1)$.



Set $x = 1$ in our two equations:

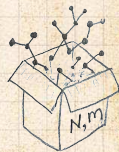
$$F_\pi(1) = F_P(F_\rho(1)) \quad \text{and} \quad F_\rho(1) = F_R(F_\rho(1))$$



Solve second equation numerically for $F_\rho(1)$.



Plug $F_\rho(1)$ into first equation to obtain $F_\pi(1)$.



Component sizes


Example: Standard random graphs.


 We can show $F_P(x) = e^{-\langle k \rangle(1-x)}$

$$\Rightarrow F_R(x) = F'_P(x)/F'_P(1)$$


$$= \langle k \rangle e^{-\langle k \rangle(1-x)} / \langle k \rangle e^{-\langle k \rangle(1-x')} \Big|_{x'=1}$$

$$= e^{-\langle k \rangle(1-x)} = F_P(x) \quad \dots\text{aha!}$$

 RHS's of our two equations are the same.

 So $F_\pi(x) = F_\rho(x) = xF_R(F_\rho(x)) = xF_R(F_\pi(x))$

 Consistent with how our dirty (but wrong) trick worked earlier ...

 $\pi_n = \rho_n$ just as $P_k = R_k$.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

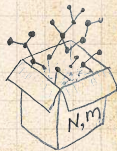
Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Component sizes



We are down to

$$F_{\pi}(x) = xF_R(F_{\pi}(x)) \text{ and } F_R(x) = e^{-\langle k \rangle(1-x)}.$$



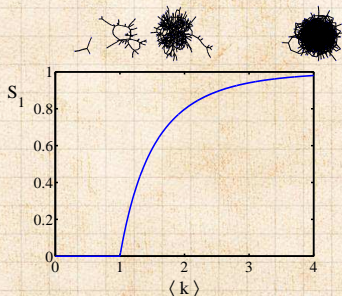
$$\therefore F_{\pi}(x) = xe^{-\langle k \rangle(1-F_{\pi}(x))}$$



We're first after $S_1 = 1 - F_{\pi}(1)$ so set $x = 1$ and replace $F_{\pi}(1)$ by $1 - S_1$:

$$1 - S_1 = e^{-\langle k \rangle S_1}$$

$$\text{Or: } \langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$$



Just as we found with our dirty trick ...



Again, we (usually) have to resort to numerics ...

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

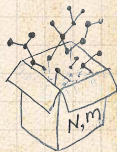
Useful results

Size of the Giant Component



A few examples


Average Component Size


References





A few simple random networks to contemplate and play around with:


 **Notation:** The Kronecker delta function  $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.


 $P_k = \delta_{k1}$.


 $P_k = \delta_{k2}$.


 $P_k = \delta_{k3}$.

 $P_k = \delta_{kk'}$ for some fixed $k' \geq 0$.

 $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.

 $P_k = a\delta_{k1} + (1-a)\delta_{k3}$, with $0 \leq a \leq 1$.

 $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{kk'}$ for some fixed $k' \geq 2$.

 $P_k = a\delta_{k1} + (1-a)\delta_{kk'}$ for some fixed $k' \geq 2$ with $0 \leq a \leq 1$.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

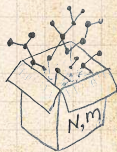
Useful results

Size of the Giant Component

A few examples


Average Component Size


References




A joyful example \square :

$$P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}.$$


 We find (two ways): $R_k = \frac{1}{4}\delta_{k0} + \frac{3}{4}\delta_{k2}$.


 A giant component exists because:
 $\langle k \rangle_R = 0 \times 1/4 + 2 \times 3/4 = 3/2 > 1$.


 Generating functions for P_k and R_k :

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2$$

 Check for goodness:

 $F_R(x) = F'_P(x)/F'_P(1)$ and $F_P(1) = F_R(1) = 1$.

 $F'_P(1) = \langle k \rangle_P = 2$ and $F'_R(1) = \langle k \rangle_R = \frac{3}{2}$.

 Things to figure out: Component size generating functions for π_n and ρ_n , and the size of the giant component.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

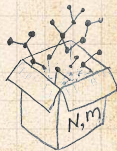
Useful results

Size of the Giant Component


A few examples

Average Component Size

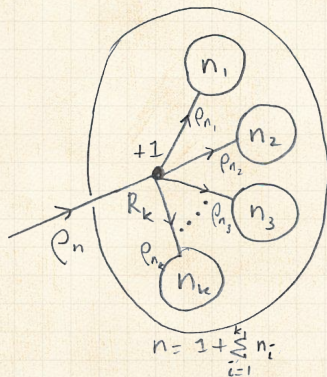
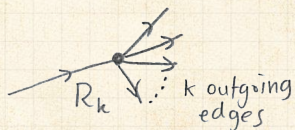
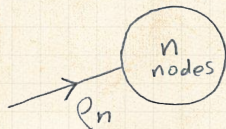
References



Find $F_\rho(x)$ first:

 We know:

$$F_\rho(x) = xF_R(F_\rho(x)).$$



Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

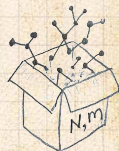
Useful results


Size of the Giant Component

A few examples


Average Component Size

References




 Sticking things in things, we have:


$$F_\rho(x) = x \left(\frac{1}{4} + \frac{3}{4} [F_\rho(x)]^2 \right).$$


 Rearranging:


$$3x [F_\rho(x)]^2 - 4F_\rho(x) + x = 0.$$

 Please and thank you:

$$F_\rho(x) = \frac{2}{3x} \left(1 \pm \sqrt{1 - \frac{3}{4}x^2} \right)$$

 Time for a Taylor series expansion.

 The promise: non-negative powers of x with non-negative coefficients.

 First: which sign do we take?

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

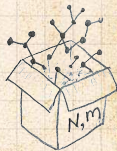
Useful results


Size of the Giant Component


A few examples

Average Component Size

References




 Because ρ_n is a probability distribution, we know $F_\rho(1) \leq 1$ and $F_\rho(x) \leq 1$ for $0 \leq x \leq 1$.


 Thinking about the limit $x \rightarrow 0$ in

$$F_\rho(x) = \frac{2}{3x} \left(1 \pm \sqrt{1 - \frac{3}{4}x^2} \right),$$

we see that the positive sign solution blows to smithereens, and the negative one is okay.

 So we must have:

$$F_\rho(x) = \frac{2}{3x} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right),$$

 We can now deploy the Taylor expansion:

$$(1+z)^\theta = \binom{\theta}{0} z^0 + \binom{\theta}{1} z^1 + \binom{\theta}{2} z^2 + \binom{\theta}{3} z^3 + \dots$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

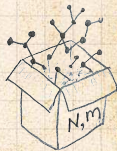
Useful results


Size of the Giant Component

A few examples


Average Component Size

References




 Let's define a binomial for arbitrary θ and $k = 0, 1, 2, \dots$:

$$\binom{\theta}{k} = \frac{\Gamma(\theta + 1)}{\Gamma(k + 1)\Gamma(\theta - k + 1)}$$

 For $\theta = \frac{1}{2}$, we have:

$$\begin{aligned} (1 + z)^{\frac{1}{2}} &= \binom{\frac{1}{2}}{0} z^0 + \binom{\frac{1}{2}}{1} z^1 + \binom{\frac{1}{2}}{2} z^2 + \dots \\ &= \frac{\Gamma(\frac{3}{2})}{\Gamma(1)\Gamma(\frac{3}{2})} z^0 + \frac{\Gamma(\frac{3}{2})}{\Gamma(2)\Gamma(\frac{1}{2})} z^1 + \frac{\Gamma(\frac{3}{2})}{\Gamma(3)\Gamma(-\frac{1}{2})} z^2 + \dots \\ &= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \dots \end{aligned}$$

where we've used $\Gamma(x + 1) = x\Gamma(x)$ and noted that $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.

 Note: $(1 + z)^\theta \sim 1 + \theta z$ always.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

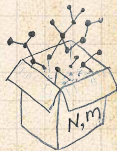
Useful results


Size of the Giant Component

A few examples


Average Component Size

References




 Totally psyched, we go back to here:

$$F_\rho(x) = \frac{2}{3x} \left(1 - \sqrt{1 - \frac{3}{4}x^2} \right).$$

 Setting $z = -\frac{3}{4}x^2$ and expanding, we have:


$$F_\rho(x) =$$


$$\frac{2}{3x} \left(1 - \left[1 + \frac{1}{2} \left(-\frac{3}{4}x^2 \right)^1 - \frac{1}{8} \left(-\frac{3}{4}x^2 \right)^2 + \frac{1}{16} \left(-\frac{3}{4}x^2 \right)^3 \right] + \dots \right)$$

 Giving:


$$F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n =$$


$$\frac{1}{4}x + \frac{3}{64}x^3 + \frac{9}{512}x^5 + \dots + \frac{2}{3} \left(\frac{3}{4} \right)^k \frac{(-1)^{k+1} \Gamma\left(\frac{3}{2}\right)}{\Gamma(k+1) \Gamma\left(\frac{3}{2} - k\right)} x^{2k-1} + \dots$$


 Do odd powers make sense?

 We can now find $F_\pi(x)$ with:

$$\begin{aligned}
 F_\pi(x) &= xF_P(F_\rho(x)) \\
 &= x\frac{1}{2}\left((F_\rho(x))^1 + (F_\rho(x))^3\right) \\
 &= x\frac{1}{2}\left[\frac{2}{3x}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right) + \frac{2^3}{(3x)^3}\left(1 - \sqrt{1 - \frac{3}{4}x^2}\right)^3\right].
 \end{aligned}$$

 Delicious.

 In principle, we can now extract all the π_n .

 But let's just find the size of the giant component.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

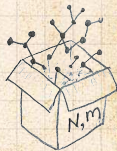
Useful results


Size of the Giant Component

A few examples


Average Component Size


References




 First, we need $F_\rho(1)$:


$$F_\rho(x)|_{x=1} = \frac{2}{3 \cdot 1} \left(1 - \sqrt{1 - \frac{3}{4}1^2} \right) = \frac{1}{3}.$$

 This is the probability that a random edge leads to a sub-component of finite size.

 Next:

$$F_\pi(1) = 1 \cdot F_P(F_\rho(1)) = F_P\left(\frac{1}{3}\right) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \left(\frac{1}{3}\right)^3 = \frac{5}{27}.$$

 This is the probability that a random chosen node belongs to a finite component.

 Finally, we have

$$S_1 = 1 - F_\pi(1) = 1 - \frac{5}{27} = \frac{22}{27}.$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

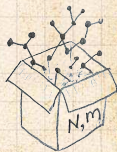
Useful results

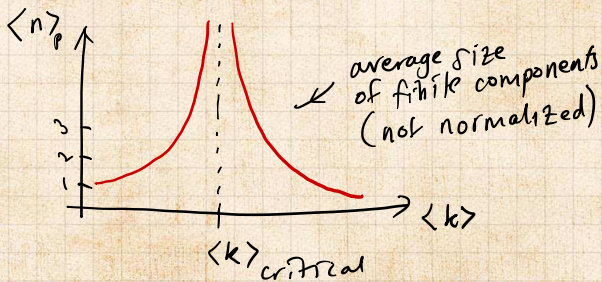
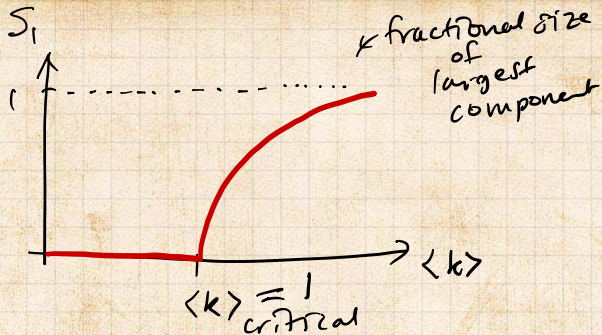
Size of the Giant Component

A few examples

Average Component Size

References





Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

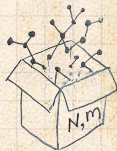
Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Average component size

Next: find **average size** of **finite** components $\langle n \rangle$.

Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.

Try to avoid finding $F_\pi(x)$...

Starting from $F_\pi(x) = xF_P(F_\rho(x))$, we differentiate:

$$F'_\pi(x) = F_P(F_\rho(x)) + xF'_\rho(x)F'_P(F_\rho(x))$$

While $F_\rho(x) = xF_R(F_\rho(x))$ gives

$$F'_\rho(x) = F_R(F_\rho(x)) + xF'_\rho(x)F'_R(F_\rho(x))$$

Now set $x = 1$ in both equations.

We solve the second equation for $F'_\rho(1)$ (we must already have $F_\rho(1)$).

Plug $F'_\rho(1)$ and $F_\rho(1)$ into first equation to find $F'_\pi(1)$.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

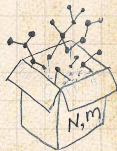
Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Average component size

Example: Standard random graphs.



Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.



Two differentiated equations reduce to only one:

$$F'_\pi(x) = F_P(F_\pi(x)) + xF'_\pi(x)F'_P(F_\pi(x))$$

$$\text{Rearrange: } F'_\pi(x) = \frac{F_P(F_\pi(x))}{1 - xF'_P(F_\pi(x))}$$



Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$



Replace $F_P(F_\pi(x))$ using $F_\pi(x) = xF_P(F_\pi(x))$.



Set $x = 1$ and replace $F_\pi(1)$ with $1 - S_1$.

$$\text{End result: } \langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

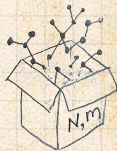
Useful results

Size of the Giant Component


A few examples

Average Component Size


References





Average component size

 Our result for standard random networks:


$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$


 Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.

 Look at what happens when we increase $\langle k \rangle$ to 1 from below.

 We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

 This blows up as $\langle k \rangle \rightarrow 1$.

 **Reason:** we have a power law distribution of component sizes at $\langle k \rangle = 1$.

 Typical critical point behavior ...

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Average component size

Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$

As $\langle k \rangle \rightarrow 0$, $S_1 = 0$, and $\langle n \rangle \rightarrow 1$.

All nodes are isolated.

As $\langle k \rangle \rightarrow \infty$, $S_1 \rightarrow 1$ and $\langle n \rangle \rightarrow 0$.

No nodes are outside of the giant component.

Extra on largest component size:

For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}/N$.

For $\langle k \rangle < 1$, $S_1 \sim (\log N)/N$.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

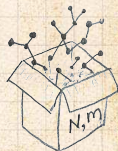
Useful results

Size of the Giant Component

A few examples

Average Component Size

References



Let's return to our example: $P_k = \frac{1}{2}\delta_{k1} + \frac{1}{2}\delta_{k3}$.

We're after:

$$\langle n \rangle = F'_\pi(1) = F_P(F_\rho(1)) + F'_\rho(1)F'_P(F_\rho(1))$$

where we first need to compute

$$F'_\rho(1) = F_R(F_\rho(1)) + F'_\rho(1)F'_R(F_\rho(1)).$$

Place stick between teeth, and recall that we have:

$$F_P(x) = \frac{1}{2}x + \frac{1}{2}x^3 \text{ and } F_R(x) = \frac{1}{4}x^0 + \frac{3}{4}x^2.$$

Differentiation gives us:

$$F'_P(x) = \frac{1}{2} + \frac{3}{2}x^2 \text{ and } F'_R(x) = \frac{3}{2}x.$$

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

Useful results


Size of the Giant Component

A few examples


Average Component Size

References




 We bite harder and use $F_\rho(1) = \frac{1}{3}$ to find:

$$\begin{aligned} F'_\rho(1) &= F_R(F_\rho(1)) + F'_\rho(1)F'_R(F_\rho(1)) \\ &= F_R\left(\frac{1}{3}\right) + F'_\rho(1)F'_R\left(\frac{1}{3}\right) \\ &= \frac{1}{4} + \cancel{\frac{3}{4}} \frac{1}{3\cancel{2}} + F'_\rho(1) \cancel{\frac{3}{2}} \frac{1}{\cancel{3}}. \end{aligned}$$

 After some reallocation of objects, we have $F'_\rho(1) = \frac{13}{2}$.



$$\begin{aligned} \text{Finally: } \langle n \rangle &= F'_\pi(1) = F_P\left(\frac{1}{3}\right) + \frac{13}{2}F'_P\left(\frac{1}{3}\right) \\ &= \frac{1}{2} \frac{1}{3} + \frac{1}{2} \frac{1}{3^3} + \frac{13}{2} \left(\frac{1}{2} + \cancel{\frac{3}{2}} \frac{1}{3\cancel{2}} \right) = \frac{5}{27} + \frac{13}{3} = \frac{122}{27}. \end{aligned}$$

 So, kinda small.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

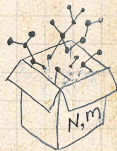
Useful results

Size of the Giant Component

A few examples

Average Component Size

References



- Generating functions allow us to strangely calculate features of random networks.
- They're a bit scary and magical.
- We'll find generating functions useful for contagion.
- But we'll also see that more direct, physics-bearing calculations are possible.

Generating Functions

Definitions

Basic Properties

Giant Component Condition

Component sizes

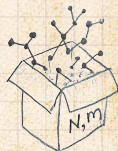
Useful results

Size of the Giant Component

A few examples

Average Component Size

References




Generating Functions

- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- A few examples
- Average Component Size

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