Random walks and diffusion on networks

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Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2018

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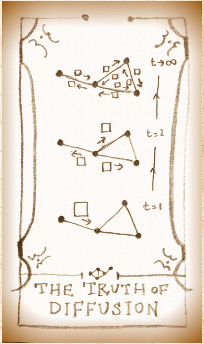
Outline


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
D-1
D
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


t→∞
t=2
t=1
THE TRUTH OF DIFFUSION

Random walks on networks—basics:

- Imagine a single random walker moving around on a network.
- At $t = 0$, start walker at node j and take time to be discrete.
- Q: What's the long term probability distribution for where the walker will be?
- Define $p_i(t)$ as the probability that at time step t, our walker is at node i.
- We want to characterize the evolution of $\vec{p}(t)$.
- First task: connect $\vec{p}(t+1)$ to $\vec{p}(t)$.

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- Let's call our walker Barry.
- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Worse still: Barry is texting.

Where is Barry?

- Consider simple undirected, ergodic (strongly connected) networks.
- As usual, represent network by adjacency matrix A where

$$a_{ij} = 1 \text{ if } i \text{ has an edge leading to } j,$$
$$a_{ij} = 0 \text{ otherwise.}$$

- Barry is at node j at time t with probability $p_j(t)$.
- In the next time step, he randomly lurches toward one of j's neighbors.
- Barry arrives at node i from node j with probability $\frac{1}{k_j}$ if an edge connects j to i.
- Equation-wise:

$$p_i(t+1) = \sum_{j=1}^{n} \frac{1}{k_j} a_{ji} p_j(t).$$

where k_j is j's degree.

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where k_j is j's degree. Note: $k_i = \sum_{j=1}^{n} a_{ij}$.

Inebriation and diffusion:

- Excellent observation: The same equation applies for stuff moving around a network, such that at each time step all material at node i is sent to its neighbors.
- $x_i(t)$ = amount of stuff at node i at time t.
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$$x_i(t+1) = \sum_{j=1}^{n} \frac{1}{k_j} a_{ji} x_j(t).$$
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- Linear algebra-based excitement: $p_i(t+1) = \sum_{j=1}^{n} a_{ji} \frac{1}{k_j} p_j(t)$ is more usefully viewed as

$$\vec{p}(t+1) = A^{\mathsf{T}} K^{-1} \vec{p}(t)$$

  where $[K_{ij}] = [\delta_{ij} k_i]$ has node degrees on the main diagonal and zeros everywhere else.
- So... we need to find the dominant eigenvalue of $A^{\mathsf{T}} K^{-1}$.
- Expect this eigenvalue will be 1 (doesn't make sense for total probability to change).
- The corresponding eigenvector will be the limiting probability distribution (or invariant measure).
- Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.

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Where is Barry?

- By inspection, we see that

$$\vec{p}(\infty) = \frac{1}{\sum_{i=1}^{n} k_i} \vec{k}$$

satisfies $\vec{p}(\infty) = A^{\mathsf{T}} K^{-1} \vec{p}(\infty)$ with eigenvalue 1.

- We will find Barry at node i with probability proportional to its degree k_i.
- Beautiful implication: probability of finding Barry travelling along any edge is uniform.
- Diffusion in real space smooths things out.
- On networks, uniformity occurs on edges.
- So in fact, diffusion in real space is about the edges too but we just don't see that.

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Other pieces:

- Goodness: $A^{\mathsf{T}}K^{-1}$ is similar to a real symmetric matrix if $A = A^{\mathsf{T}}$.
- Consider the transformation $M = K^{-1/2}$:

$$K^{-1/2} A^{\mathsf{T}} K^{-1} K^{1/2} = K^{-1/2} A^{\mathsf{T}} K^{-1/2}.$$

- Since $A^{\mathsf{T}} = A$, we have

$$(K^{-1/2} A K^{-1/2})^{\mathsf{T}} = K^{-1/2} A K^{-1/2}.$$

- Upshot: $A^{\mathsf{T}}K^{-1} = AK^{-1}$ has real eigenvalues and a complete set of orthogonal eigenvectors.
- Can also show that maximum eigenvalue magnitude is indeed 1.

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