Branching Networks II

Last updated: 2018/03/23, 12:08:15

Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2018

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

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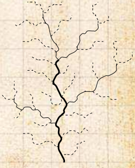

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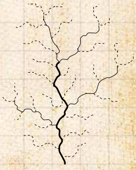
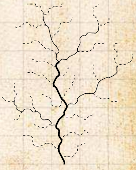


Outline




THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
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THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
THE CONTAGION
THE CONFUSION OF CONTAGIONS
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THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
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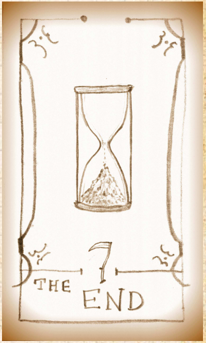
THE END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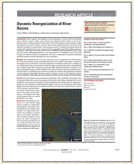
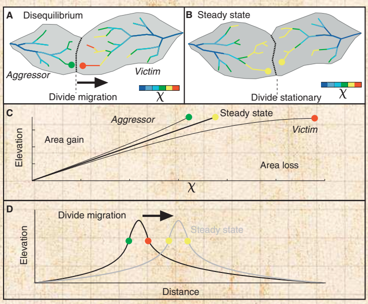
Piracy on the high χ's:



"Dynamic Reorganization of River Basins"
Willett et al.,
Science Magazine, **343**, 1248765, 2014. [21]



$$\frac{\partial z(x,t)}{\partial t} = U - K A^m \left| \frac{\partial z(x,t)}{\partial x} \right|^n$$

$$z(x) = z_{\mathrm{b}} + \left(\frac{U}{K A_0^m} \right)^{1/n} \chi$$

$$\chi = \int_{x_{\mathrm{b}}}^{x} \left(\frac{A_0}{A(x')} \right)^{m/n} \mathrm{d}x'$$

Piracy on the high χ's:

http://www.youtube.com/watch?v=FnroL1_-l2c?rel=0

More: How river networks move across a landscape (Science Daily)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The Manifesto
The Well of Scales
The Laws of Decaying Power
The Unexpected Transfiguration
Ministry of Random Walks
The Rich Get Richer
The Dream of Efficient Language
The Law of Digits First
The Surprise of Being Robust yet Fragile
The Misreading of the Book of Sand
The Many Tales of Many Tails
The Measuring
The Data Poor
The Pouring Data
The Charms of the Many Wrong Paths
The Knowledge of Dead Monks
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The Emergence of Structure
The Emergence of Destruction
The Emergence of Thinking
The Emergence of Stories
The Mechanics Statistical
The Thresholds of Percolation
The Networks Complex
The Elements of Webs
The Birds of a Feather
The Birds of Not a Feather
The Ark of the Random Network
The Strangeness of Friends
The Smallness of Worlds
The Theory of Six Degrees
The Networks Free of Scale
Classified
Phoenix
The Contagion
The Confusion of Contagions
The Disease of Random Mixing
The Unending Pandemics
The Thresholds of the Mind
The Social Wild
The Social Contagion
The Ratio of Gain
The Mechanics of Fame
The Talk of Fame
The Stories Contagious
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The Paths of Universality
The Dependent Paths
The Trousers of Bifurcating Realities
The Unknown Mechanism
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The Networks of Blood
The Trees of Reality
The Orders of Streams
The Laws of Branching
The Braid or the Branch
The Law of Optimal Forks
The Church of Quarterology
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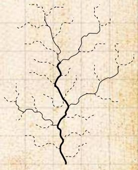
The Truthicide
The Family of Pancake Cows
The Family of Spherical Cows
The Improbable Pachyderm
The Ill-Scaling of Allometric Bodies
The Supply Most Fair
The Supply to the Most
The Dark Art of Generating Functions
The Random Sum of Randomness
The Two-sided Worlds
The Stories and the Tropes
The Truth of Diffusion
The Unveiling of Hidden Powers
The Scrying of Shapes of Things
The Limits of Organizations
The Overcoming of Catastrophe
The Troubles Ill-Formed
The Networks of Many Ilks
The Chaos of Fettered Contagion
The End


The Scaling of Broken Hierarchies

Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:

- In terms of network achitecture, Horton's laws appear to contain less detailed information than Tokunaga's law.
- Oddly, Horton's laws have four parameters and Tokunaga has two parameters.
- R_n, R_a, R_ℓ, and R_s versus T_1 and R_T. One simple redundancy: $R_\ell = R_s$.
  Insert question from assignment 1
- To make a connection, clearest approach is to start with Tokunaga's law ...
- Known result: Tokunaga $\rightarrow$ Horton [18, 19, 20, 9, 2]



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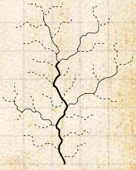

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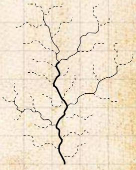

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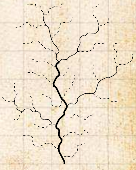

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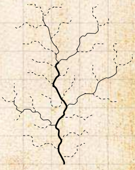

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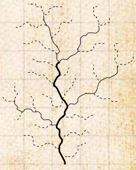


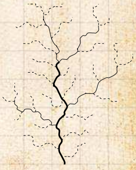
Let us make them happy

We need one more ingredient:

Space-fillingness

- A network is space-filling if the average distance between adjacent streams is roughly constant.
- Reasonable for river and cardiovascular networks
- For river networks:
  Drainage density ρ_{dd} = inverse of typical distance between channels in a landscape.
- In terms of basin characteristics:

$$\rho_{dd} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \frac{\sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega}{a_\Omega}$$



The University of Vermont

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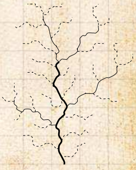

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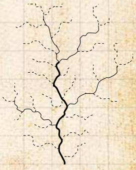

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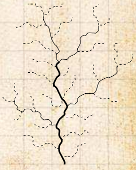

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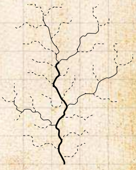

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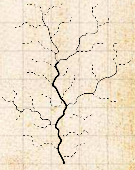

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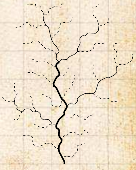
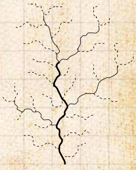


More with the happy-making thing



Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law: $n_\omega / n_{\omega+1} = R_n$.
- Estimate n_ω, the number of streams of order ω in terms of other $n_{\omega'}$, $\omega' > \omega$.
- Observe that each stream of order ω terminates by either:

1. Running into another stream of order ω and generating a stream of order $\omega + 1$...
   - $2n_{\omega+1}$ streams of order ω do this
2. Running into and being absorbed by a stream of higher order $\omega' > \omega$...
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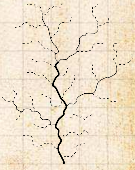

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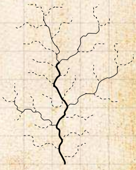

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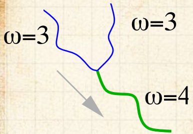

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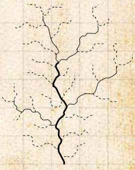

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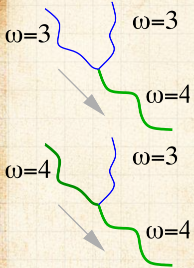
- Start looking for Horton's stream number law: $n_\omega / n_{\omega+1} = R_n$.
- Estimate n_ω, the number of streams of order ω in terms of other $n_{\omega'}$, $\omega' > \omega$.
- Observe that each stream of order ω terminates by either:



1. Running into another stream of order ω and generating a stream of order $\omega + 1$ ...
   - $2n_{\omega+1}$ streams of order ω do this
2. Running into and being absorbed by a stream of higher order $\omega' > \omega$ ...
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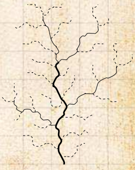

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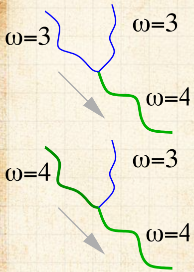

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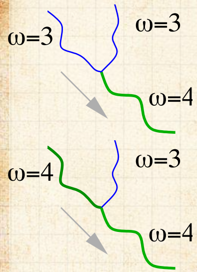

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More with the happy-making thing

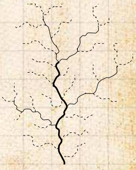
Putting things together:

$$n_\omega = \underbrace{2 n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega} n_{\omega'}}_{\text{absorption}}$$

- Use Tokunaga's law and manipulate expression to find Horton's law for stream numbers follows and hence obtain R_n.
- Insert question from assignment 1
- Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

(The larger value is the one we want.)



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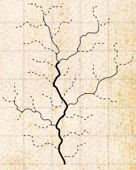

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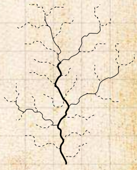

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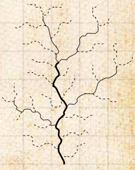

Finding other Horton ratios

Connect Tokunaga to R_s

- Now use uniform drainage density ρ_{dd}.
- Assume side streams are roughly separated by distance $1/\rho_{\text{dd}}$.
- For an order ω stream segment, expected length is

$$\bar{s}_\omega \simeq \rho_{\text{dd}}^{-1}\left(1+\sum_{k=1}^{\omega-1} T_k\right)$$

- Substitute in Tokunaga's law $T_k = T_1 R_T^{k-1}$:

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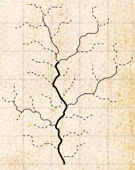

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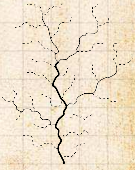

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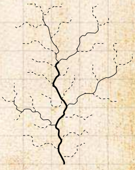

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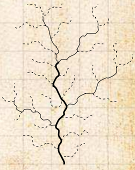

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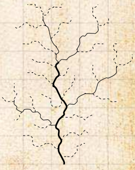

Horton and Tokunaga are happy

Altogether then:

- $$\Rightarrow \bar{s}_{\omega}/\bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$

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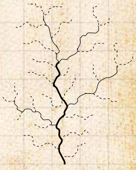

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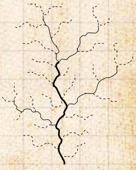

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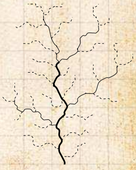

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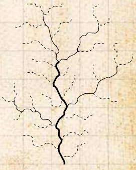

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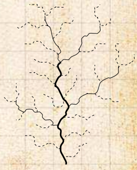
Some observations:

- R_n and R_ℓ depend on T_1 and R_T.
- Seems that R_a must as well ...
- Suggests Horton's laws must contain some redundancy
- We'll in fact see that $R_a = R_n$.
- Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between non-trivial statistical distributions. [3, 4]



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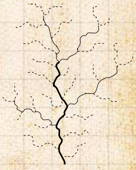

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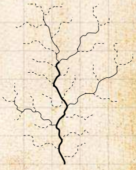

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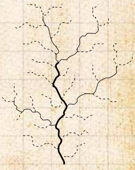

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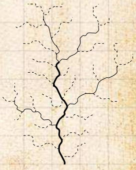

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The other way round

- Note: We can invert the expresssions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.
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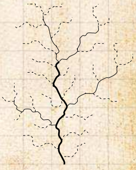

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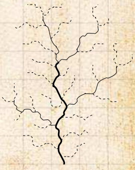

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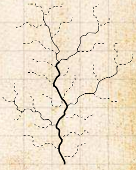

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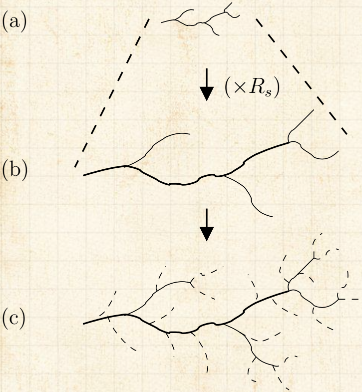

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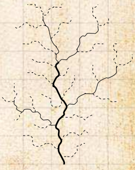

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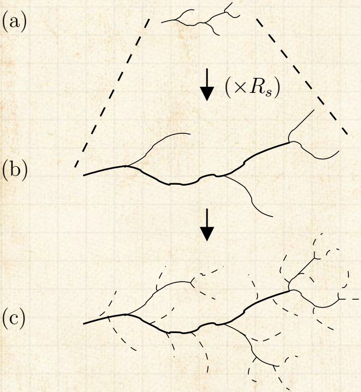

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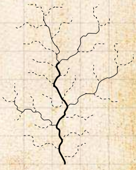

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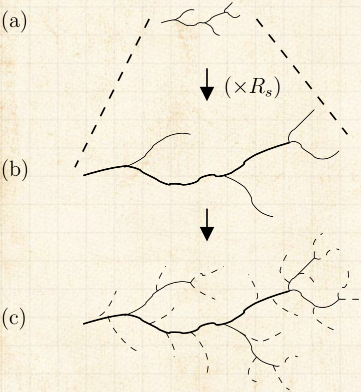

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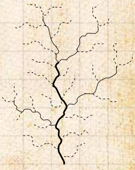

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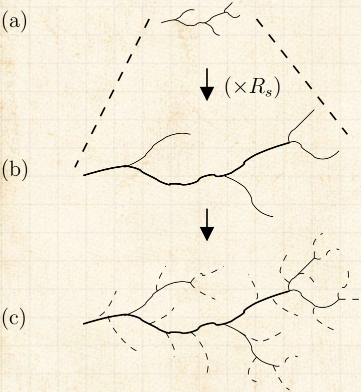

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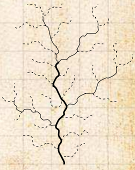

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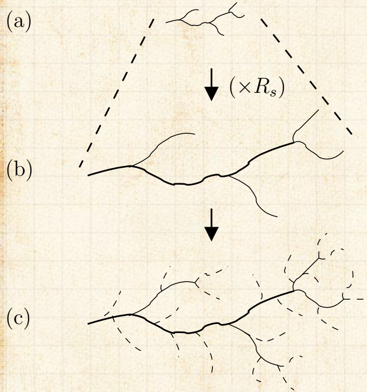

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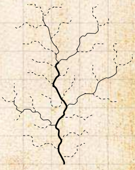

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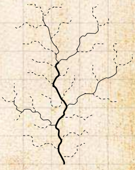

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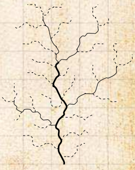

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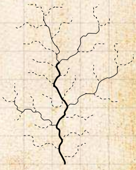

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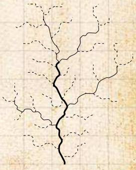

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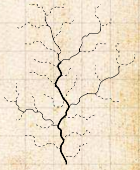

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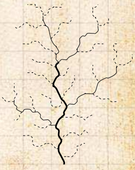

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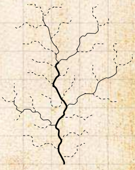

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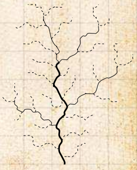
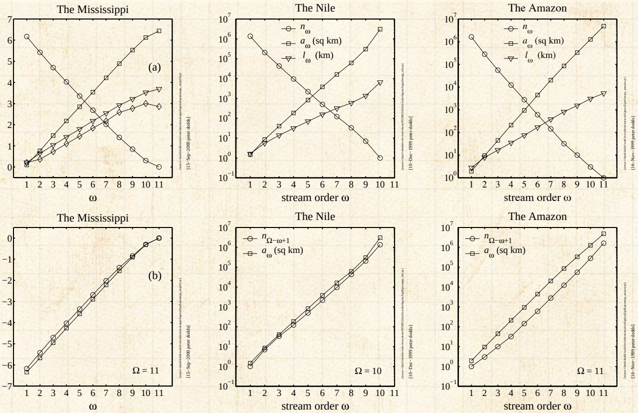
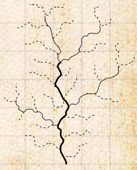


Horton's laws of area and number:

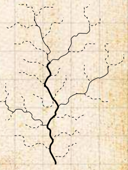




- In bottom plots, stream number graph has been flipped vertically.
- Highly suggestive that $R_n \equiv R_a$ ...

Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?
- Rule of thumb: discard data for two smallest and two largest orders.



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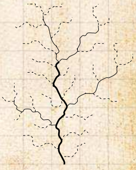
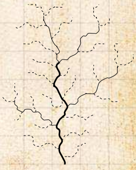


Mississippi:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
$[2, 3]$	5.27	5.26	2.48	2.30	1.00
$[2, 5]$	4.86	4.96	2.42	2.31	1.02
$[2, 7]$	4.77	4.88	2.40	2.31	1.02
$[3, 4]$	4.72	4.91	2.41	2.34	1.04
$[3, 6]$	4.70	4.83	2.40	2.35	1.03
$[3, 8]$	4.60	4.79	2.38	2.34	1.04
$[4, 6]$	4.69	4.81	2.40	2.36	1.02
$[4, 8]$	4.57	4.77	2.38	2.34	1.05
$[5, 7]$	4.68	4.83	2.36	2.29	1.03
$[6, 7]$	4.63	4.76	2.30	2.16	1.03
$[7, 8]$	4.16	4.67	2.41	2.56	1.12
mean μ	4.69	4.85	2.40	2.33	1.04
std dev σ	0.21	0.13	0.04	0.07	0.03
σ/μ	0.045	0.027	0.015	0.031	0.024



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Amazon:

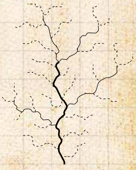
ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
$[2,3]$	4.78	4.71	2.47	2.08	0.99
$[2,5]$	4.55	4.58	2.32	2.12	1.01
$[2,7]$	4.42	4.53	2.24	2.10	1.02
$[3,5]$	4.45	4.52	2.26	2.14	1.01
$[3,7]$	4.35	4.49	2.20	2.10	1.03
$[4,6]$	4.38	4.54	2.22	2.18	1.03
$[5,6]$	4.38	4.62	2.22	2.21	1.06
$[6,7]$	4.08	4.27	2.05	1.83	1.05
mean μ	4.42	4.53	2.25	2.10	1.02
std dev σ	0.17	0.10	0.10	0.09	0.02
σ/μ	0.038	0.023	0.045	0.042	0.019

Reducing Horton's laws:

Rough first effort to show $R_n \equiv R_a$:

- $a_\Omega \propto$ sum of all stream segment lengths in a order Ω basin (assuming uniform drainage density)
- So:

$$a_\Omega \simeq \sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega / \rho_{\mathrm{dd}}$$



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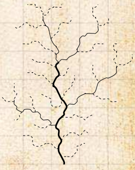

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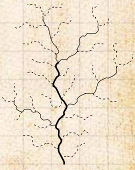

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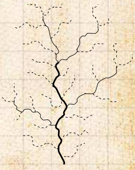

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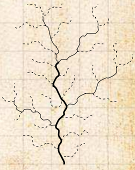

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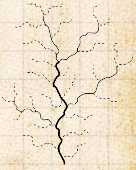

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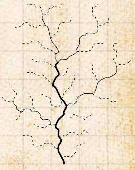

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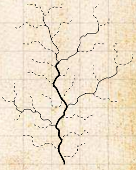

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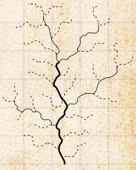

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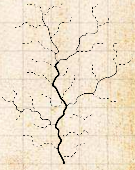
- ...But this only a rough argument as Horton's laws do not imply a strict hierarchy
- Need to account for sidebranching.
- Insert question from assignment 2



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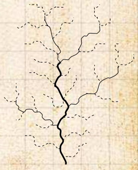

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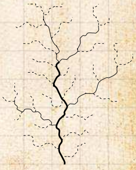

Equipartitioning:

Intriguing division of area:

- Observe: Combined area of basins of order ω independent of ω.
- Not obvious: basins of low orders not necessarily contained in basis on higher orders.
- Story:

$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$

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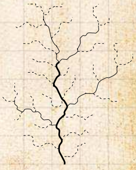

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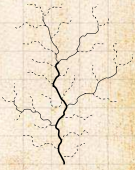

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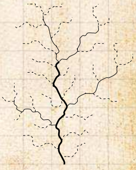

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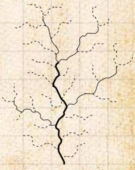

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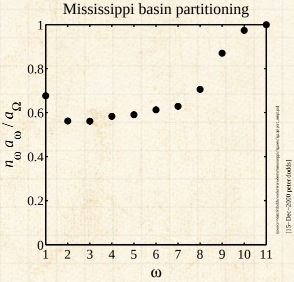
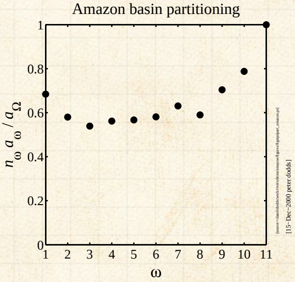
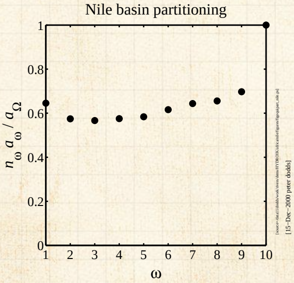
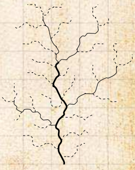
Equipartitioning:

Some examples:



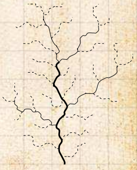






Neural Reboot: Fwoompf

http://www.youtube.com/watch?v=5mUs70SqD4o?rel=0



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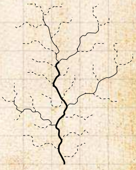
Scaling laws

The story so far:

- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- Tokunaga's law describes detailed architecture: $T_k = T_1 R_T^{k-1}$.
- We have connected Tokunaga's and Horton's laws
- Only two Horton laws are independent ($R_n = R_a$)
- Only two parameters are independent: $(T_1, R_T) \Leftrightarrow (R_n, R_s)$



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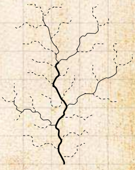

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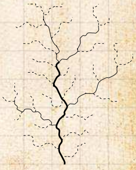

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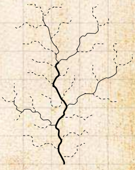

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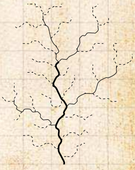

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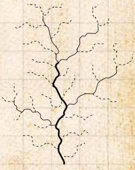

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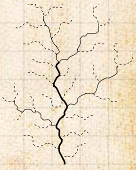

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- Ignore stream ordering for the moment
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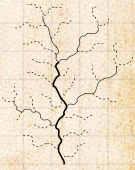

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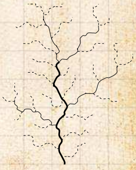

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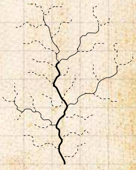

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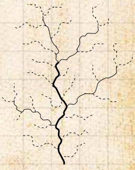

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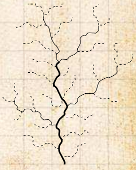

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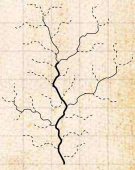

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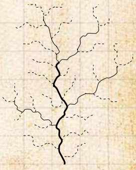

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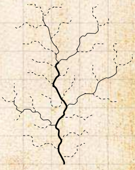

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Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)
  - Word frequency (Zipf's law)[22]
  - Wealth (maybe not—at least heavy tailed)
  - Statistical mechanics (phase transitions)[5]
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- Arise from mechanisms: growth, randomness, optimization, ...
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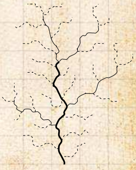

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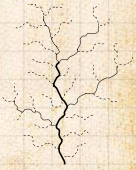

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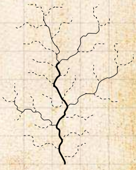

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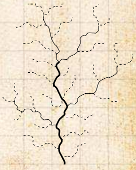

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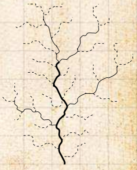

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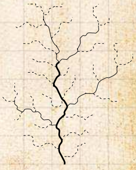

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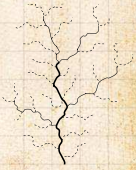

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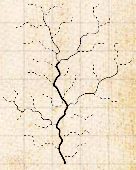

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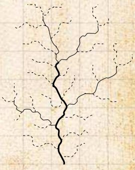

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- Let's work on $P(\ell)$ ...
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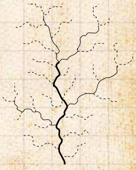
- Next: place stick between teeth. Bite stick. Proceed.



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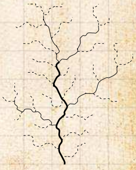

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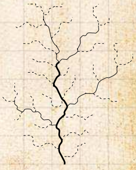

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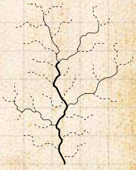

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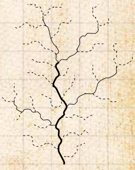

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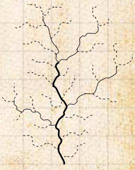

Scaling laws

Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- Plan: Derive $P(a) \propto a^{-\tau}$ and $P(\ell) \propto \ell^{-\gamma}$ starting with Tokunaga/Horton story [17, 1, 2]
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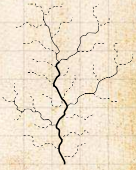

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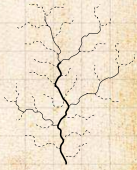

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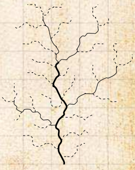

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Finding γ:

- Often useful to work with cumulative distributions, especially when dealing with power-law distributions.
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$$P_{>}(\ell_*) = P(\ell > \ell_*) = \int_{\ell=\ell_*}^{\ell_{\max}} P(\ell)\mathrm{d}\ell$$

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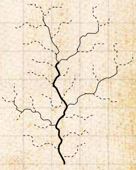

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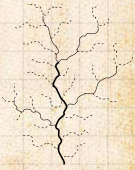

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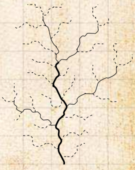

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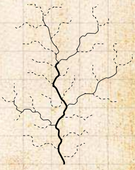

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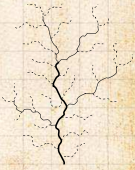

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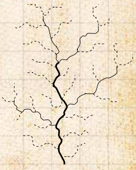

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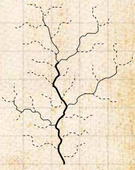

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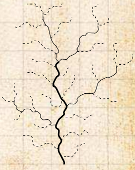

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Finding γ:

- Aim: determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
- Assume some spatial sampling resolution Δ
- Landscape is broken up into grid of $\Delta \times \Delta$ sites
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$$P_>(\ell_*) = \frac{N_>(\ell_*; \Delta)}{N_>(0; \Delta)}.$$

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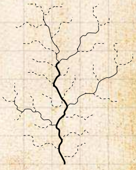

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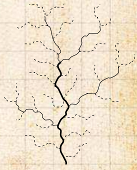

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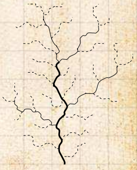

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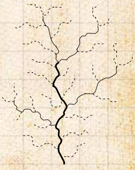

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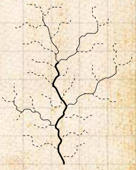

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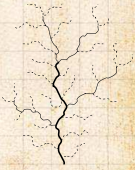

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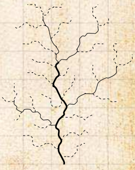

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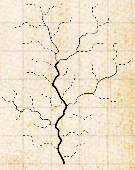

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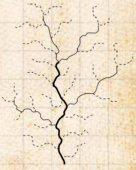

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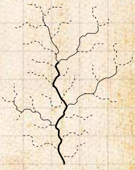

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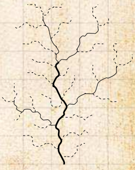

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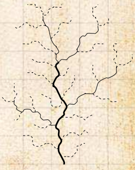

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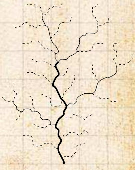

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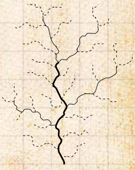

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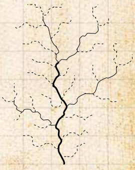

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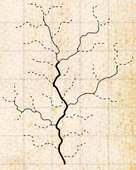

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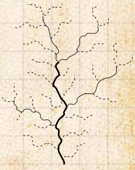

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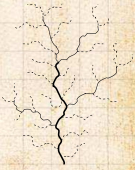

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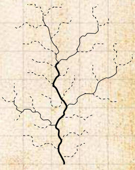

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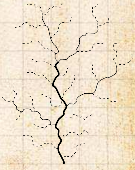

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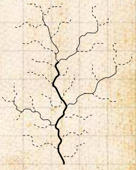

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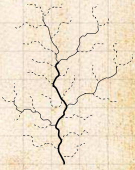

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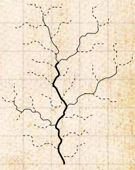

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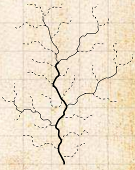

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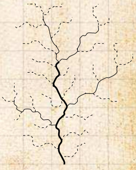

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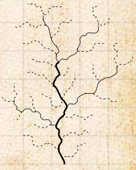

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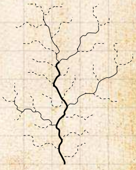

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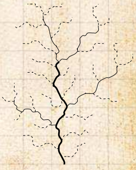

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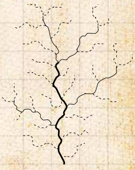

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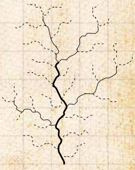

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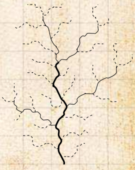

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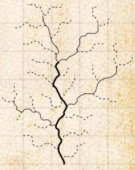

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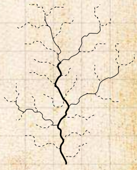

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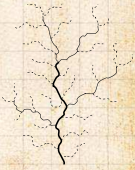

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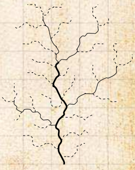

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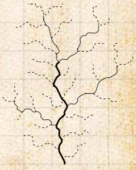
- And so we have:

$$\boxed{\gamma = \ln R_n / \ln R_s}$$

- Proceeding in a similar fashion, we can show

$$\boxed{\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma}$$

  Insert question from assignment 2
- Such connections between exponents are called scaling relations
- Let's connect to one last relationship: Hack's law



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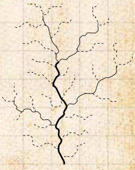

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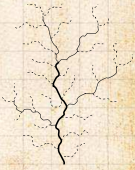

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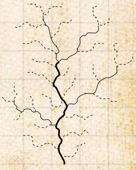

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- Typically observed that $0.5 \lesssim h \lesssim 0.7$.
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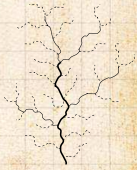

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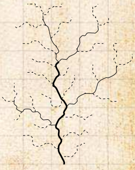

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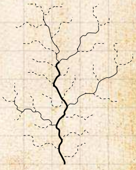

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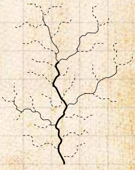

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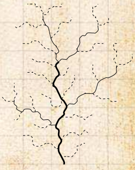

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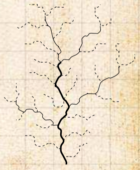

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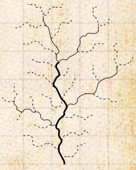

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Hack's law: [6]

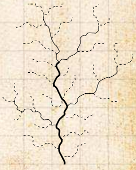
- $$\ell \propto a^h$$
- Typically observed that $0.5 \lesssim h \lesssim 0.7$.
- Use Horton laws to connect h to Horton ratios:

$$\bar{\ell}_\omega \propto R_s^\omega \text{ and } \bar{a}_\omega \propto R_n^\omega$$

- Observe:

$$\bar{\ell}_\omega \propto e^{\,\omega \ln R_s} \propto \left(e^{\,\omega \ln R_n}\right)^{\ln R_s / \ln R_n}$$

$$\propto (R_n^\omega)^{\ln R_s / \ln R_n} \propto \bar{a}_\omega^{\ln R_s / \ln R_n} \Rightarrow h = \ln R_s / \ln R_n$$



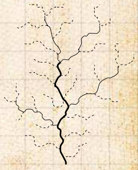
We mentioned there were a good number of ‘laws’: [2]

Relation:	Name or description:
$T_k = T_1 (R_T)^{k-1}$	Tokunaga’s law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega / n_{\omega+1} = R_n$	Horton’s law of stream numbers
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	Horton’s law of main stream lengths
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	Horton’s law of basin areas
$\bar{s}_{\omega+1} / \bar{s}_\omega = R_s$	Horton’s law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack’s law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein’s law
$\lambda \sim L^\varphi$	variation of Langbein’s law

Connecting exponents

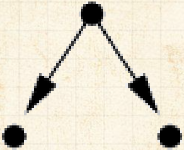
Only 3 parameters are independent:
e.g., take d, R_n, and R_s

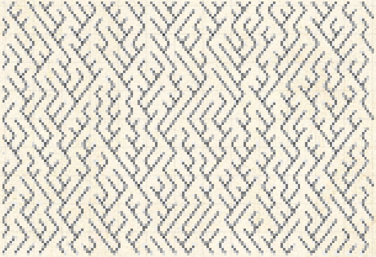
relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d
$T_k = T_1 (R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_\omega / n_{\omega+1} = R_n$	R_n
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	$R_a = R_n$
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \ln R_s / \ln R_n$
$a \sim L^D$	$D = d/h$
$L_\perp \sim L^H$	$H = d/h - 1$
$P(a) \sim a^{-\tau}$	$\tau = 2 - h$
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim a^\beta$	$\beta = 1 + h$
$\lambda \sim L^\varphi$	$\varphi = d$



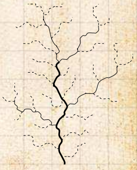
Scheidegger's model

Directed random networks [11, 12]





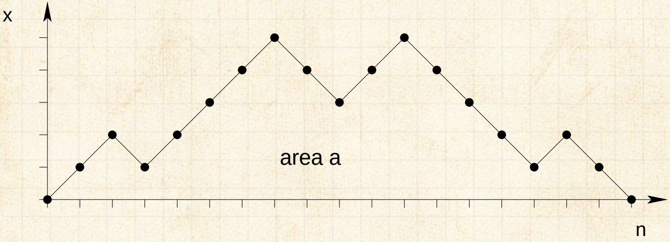
- $$P(\searrow) = P(\swarrow) = 1/2$$
- Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]
- Useful and interesting test case

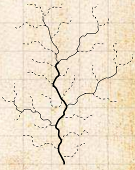


A toy model—Scheidegger's model

Random walk basins:

- Boundaries of basins are random walks





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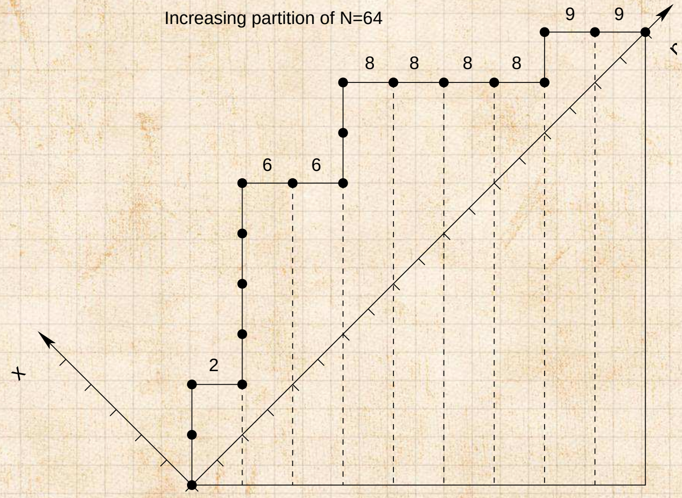


Scheidegger's model

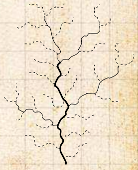
Prob for first return of a random walk in (1+1) dimensions (from CSYS/MATH 300):

- $$P(n) \sim \frac{1}{2\sqrt{\pi}}\, n^{-3/2}.$$

  and so $P(\ell) \propto \ell^{-3/2}$.
- Typical area for a walk of length n is $\propto n^{3/2}$:

  $$\ell \propto a^{2/3}.$$

- Find $\tau = 4/3$, $h = 2/3$, $\gamma = 3/2$, $d = 1$.
- Note $\tau = 2 - h$ and $\gamma = 1/h$.
- R_n and R_ℓ have not been derived analytically.



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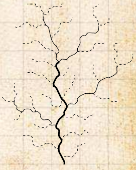

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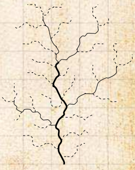

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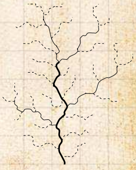

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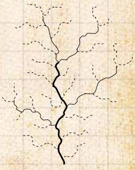

The University of Vermont

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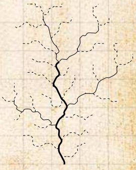
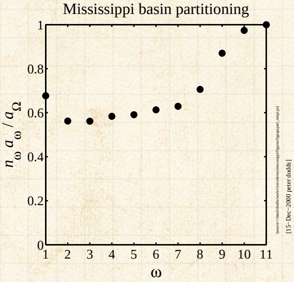
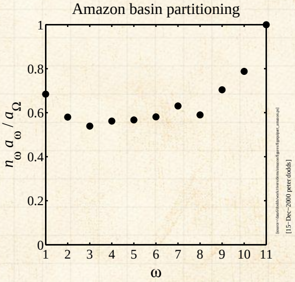
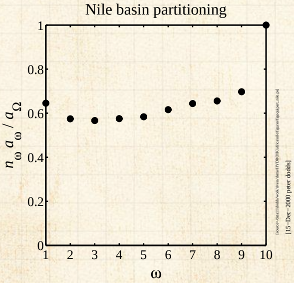


Equipartitioning reexamined:

Recall this story:







Equipartitioning

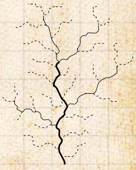
- What about

$$P(a) \sim a^{-\tau} \qquad ?$$

- Since $\tau > 1$, suggests no equipartitioning:

$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$

- $P(a)$ overcounts basins within basins ...
- while stream ordering separates basins ...



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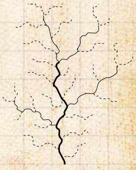

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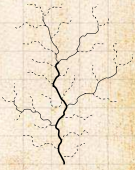

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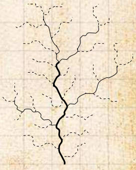


Hard neural reboot (sound matters):

https://twitter.com/round_boys/status/951873765964681216

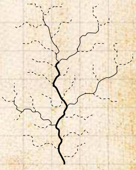
Fluctuations

Moving beyond the mean:

- Both Horton's laws and Tokunaga's law relate average properties, e.g.,

$$\bar{s}_\omega / \bar{s}_{\omega-1} = R_s$$

- Natural generalization to consider relationships between probability distributions
- Yields rich and full description of branching network structure
- See into the heart of randomness ...



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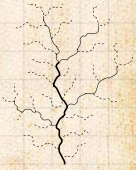

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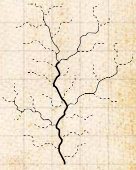

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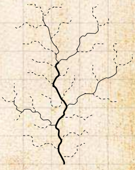

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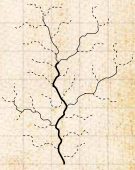
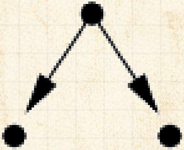
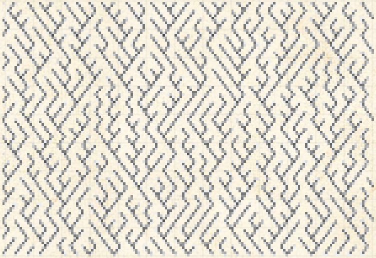


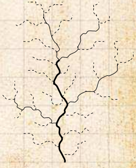
A toy model—Scheidegger's model

Directed random networks [11, 12]





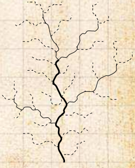
- $$P(\searrow) = P(\swarrow) = 1/2$$
- Flow is directed downwards



Generalizing Horton's laws

- $\bar{\ell}_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell|\omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell/R_\ell^\omega)$
- $\bar{a}_\omega \propto (R_a)^\omega \Rightarrow N(a|\omega) = (R_n^2)^{-\omega} F_a(a/R_n^\omega)$

- Scaling collapse works well for intermediate orders
- All moments grow exponentially with order



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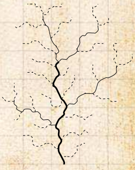

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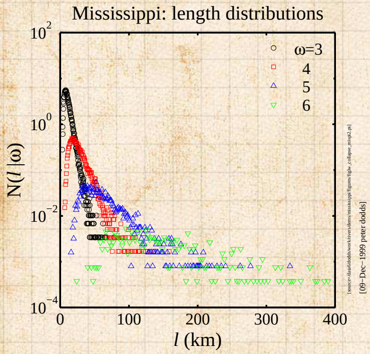

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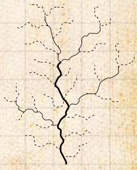

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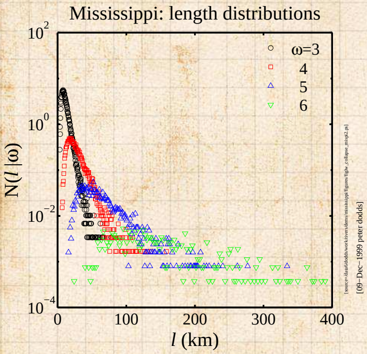

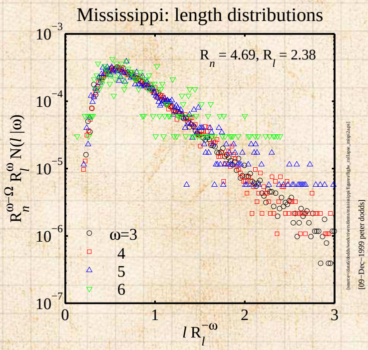

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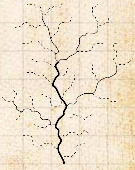

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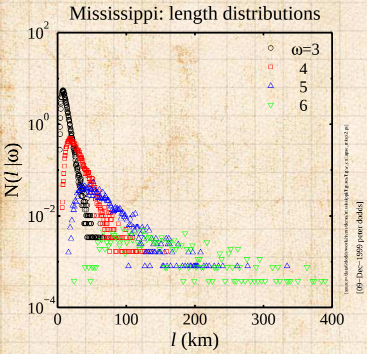

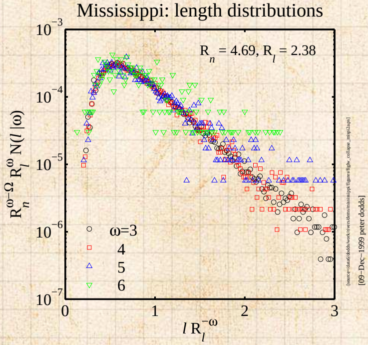

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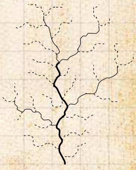
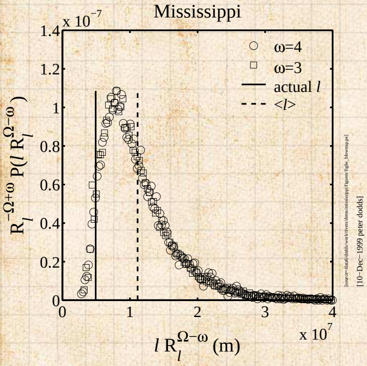
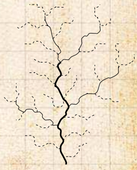


Generalizing Horton’s laws

- How well does overall basin fit internal pattern?



- Actual length = 4920 km (at 1 km res)
- Predicted Mean length = 11100 km
- Predicted Std dev = 5600 km
- Actual length/Mean length = 44 %
- Okay.



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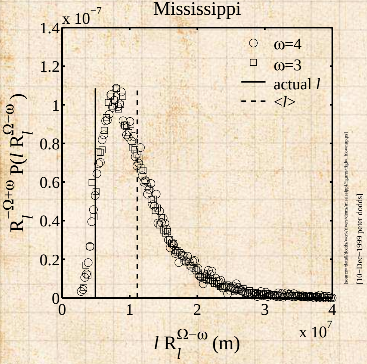

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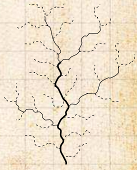
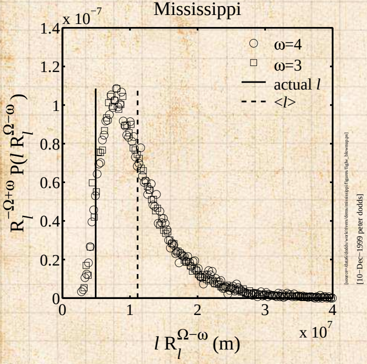


Generalizing Horton's laws

- How well does overall basin fit internal pattern?



- Actual length = 4920 km (at 1 km res)
- Predicted Mean length = 11100 km
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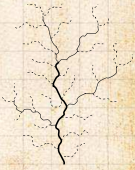

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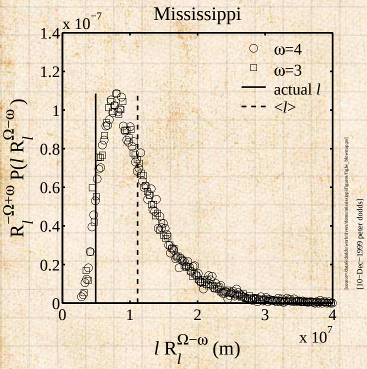

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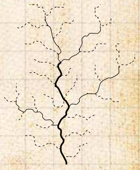

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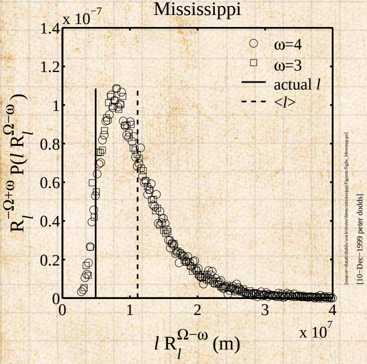

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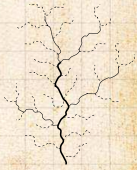

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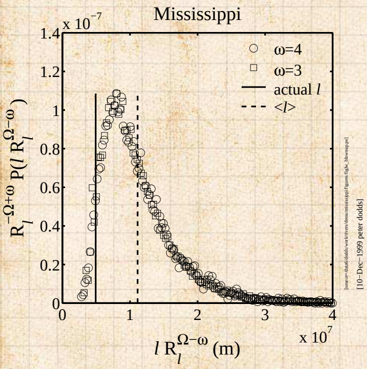

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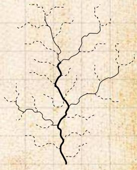


The UNIVERSITY of VERMONT

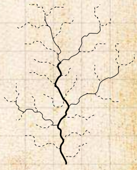
Generalizing Horton's laws

Comparison of predicted versus measured main stream lengths for large scale river networks (in 10^3 km):

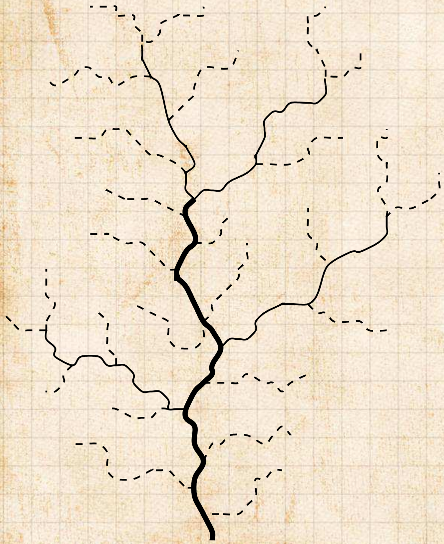
basin:	ℓ_Ω	$\bar{\ell}_\Omega$	σ_ℓ	$\ell_\Omega/\bar{\ell}_\Omega$	$\sigma_\ell/\bar{\ell}_\Omega$
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73

	a_Ω	$\bar{a}_\Omega$	σ_a	$a_\Omega/\bar{a}_\Omega$	$\sigma_a/\bar{a}_\Omega$
Mississippi	2.74	7.55	5.58	0.36	0.74
Amazon	5.40	9.07	8.04	0.60	0.89
Nile	3.08	0.96	0.79	3.19	0.82
Congo	3.70	10.09	8.28	0.37	0.82
Kansas	0.14	0.49	0.42	0.28	0.86

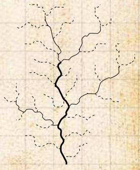
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations
Models
Nutshell
References



Combining stream segments distributions:



- Stream segments sum to give main stream lengths
- $$\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$$
- $P(\ell_\omega)$ is a convolution of distributions for the s_μ



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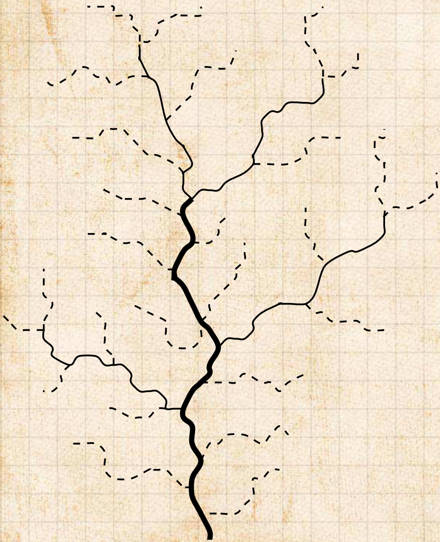

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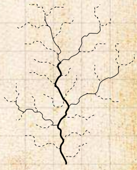
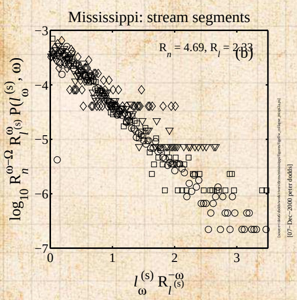


Generalizing Horton's laws

- Sum of variables $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$ leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \cdots * N(s|\omega)$$



$$N(s|\omega) = \frac{1}{R_n^\omega R_\ell^\omega} F(s/R_\ell^\omega)$$

$$F(x) = e^{-x/\xi}$$

Mississippi: $\xi \simeq 900$ m.

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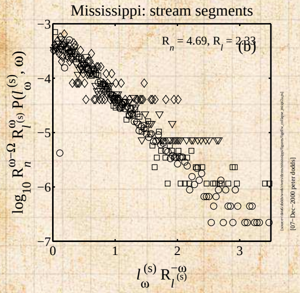

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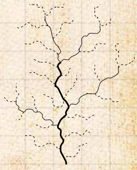
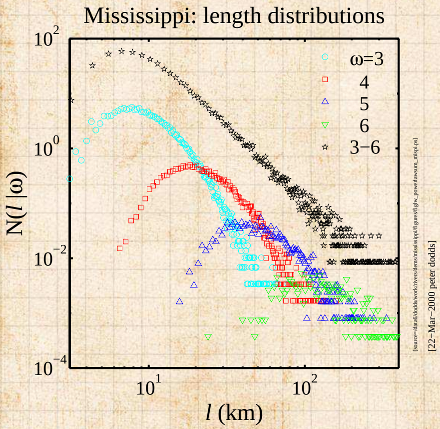
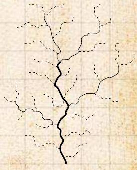


Generalizing Horton's laws

- Next level up: Main stream length distributions must combine to give overall distribution for stream length



- $P(\ell) \sim \ell^{-\gamma}$
- Another round of convolutions [3]
- Interesting ...



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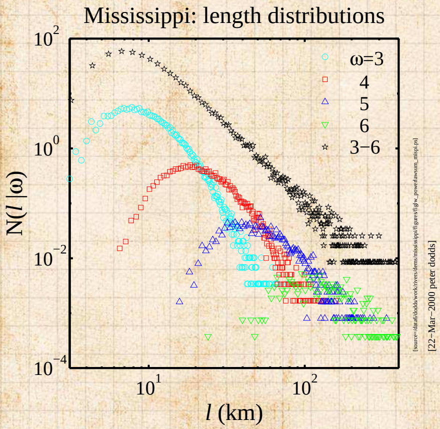

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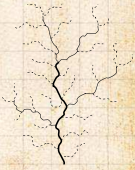
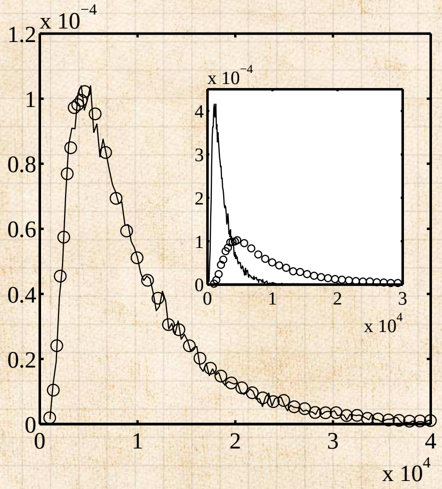
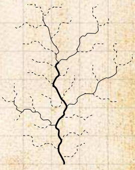


Generalizing Horton's laws

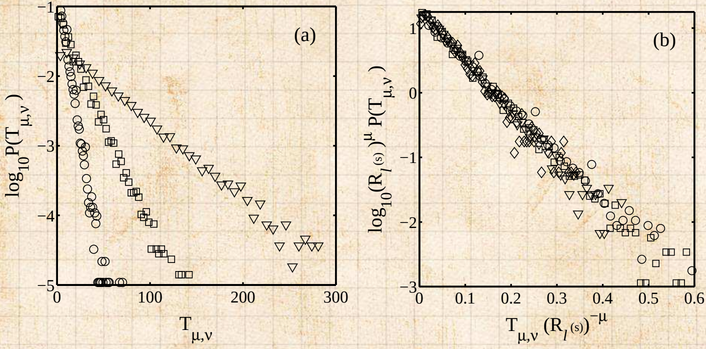
- Number and area distributions for the Scheidegger model [3]
- $P(n_{1,6})$ versus $P(a_6)$ for a randomly selected $\omega = 6$ basin.





Generalizing Tokunaga's law

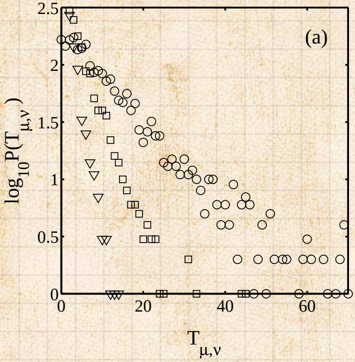
Scheidegger:

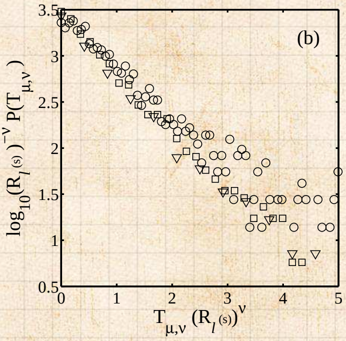


- Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using R_s

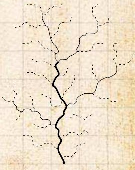
Generalizing Tokunaga's law

Mississippi:





- Same data collapse for Mississippi ...



Generalizing Tokunaga's law

So

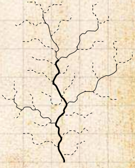
$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t \left[T_{\mu,\nu}/(R_s)^{\mu-\nu-1}\right]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

$$\boxed{P(s_\mu) \Leftrightarrow P(T_{\mu,\nu})}$$

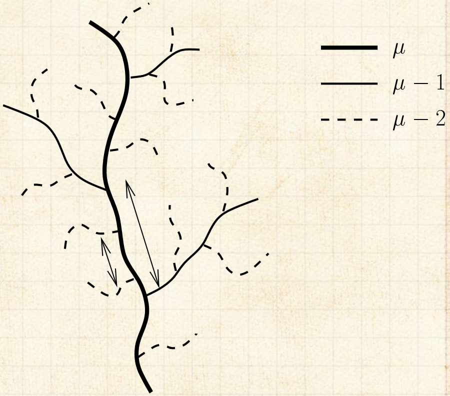
- Exponentials arise from randomness.
- Look at joint probability $P(s_\mu, T_{\mu,\nu})$.

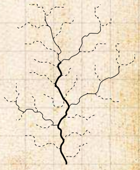


Generalizing Tokunaga's law

Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments



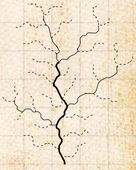


Generalizing Tokunaga's law

- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is constant:

$$\tilde{p}_{\mu} \simeq 1/(R_s)^{\mu-1}\xi_s$$

- Probability decays exponentially with stream order
- Inter-tributary lengths exponentially distributed
- ⇒ random spatial distribution of stream segments



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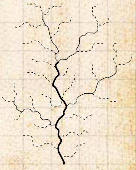

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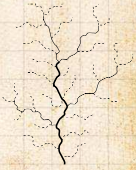

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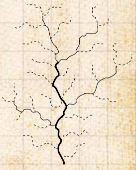
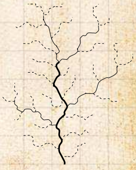


Generalizing Tokunaga's law

- Joint distribution for generalized version of Tokunaga's law:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

  where
  - p_ν = probability of absorbing an order ν side stream
  - $\tilde{p}_\mu$ = probability of an order μ stream terminating
- Approximation: depends on distance units of s_μ
- In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.



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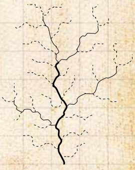

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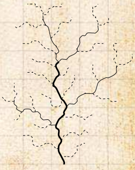

Generalizing Tokunaga's law

- Now deal with this thing:

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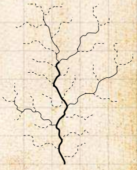
- Set $(x, y) = (s_\mu, T_{\mu,\nu})$ and $q = 1 - p_\nu - \tilde{p}_\mu$, approximate liberally.
- Obtain

$$P(x, y) = N x^{-1/2} \left[F(y/x)\right]^x$$

where

$$F(v) = \left(\frac{1-v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$



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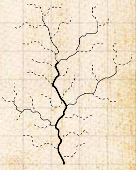
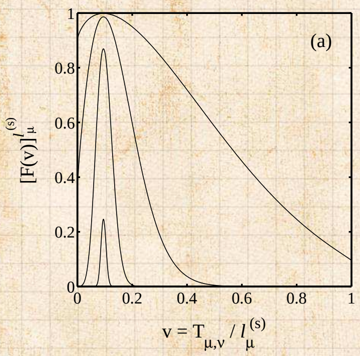
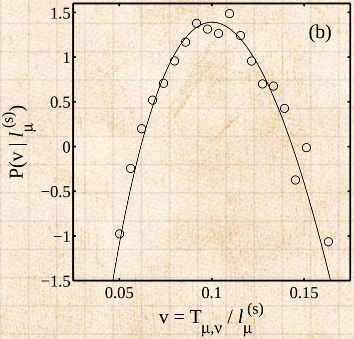


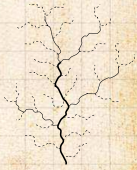
Generalizing Tokunaga's law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Scheidegger:







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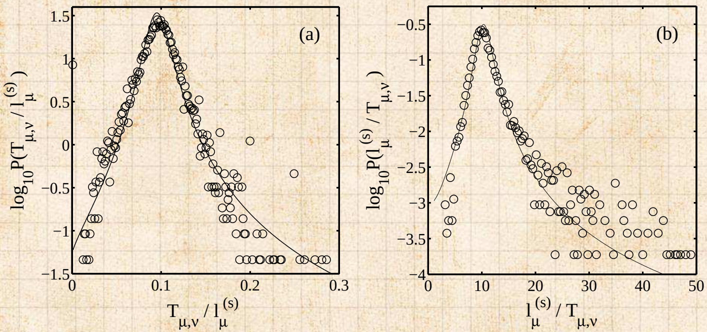

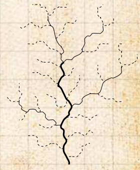

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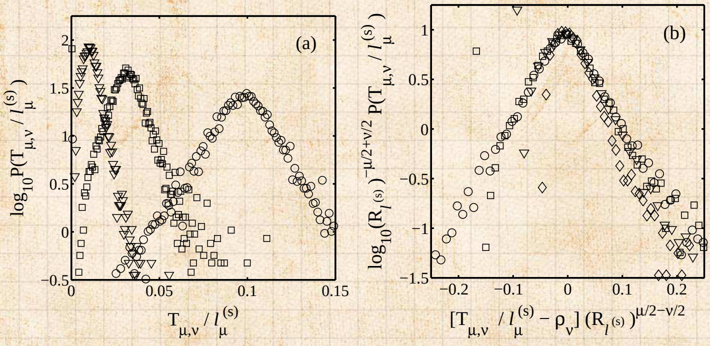

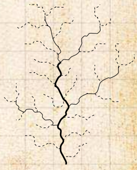

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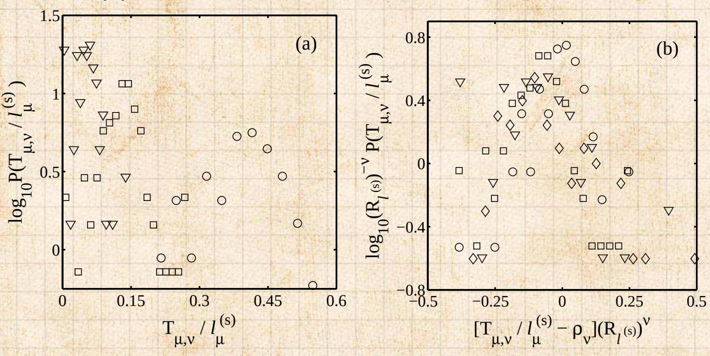
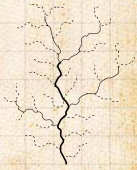
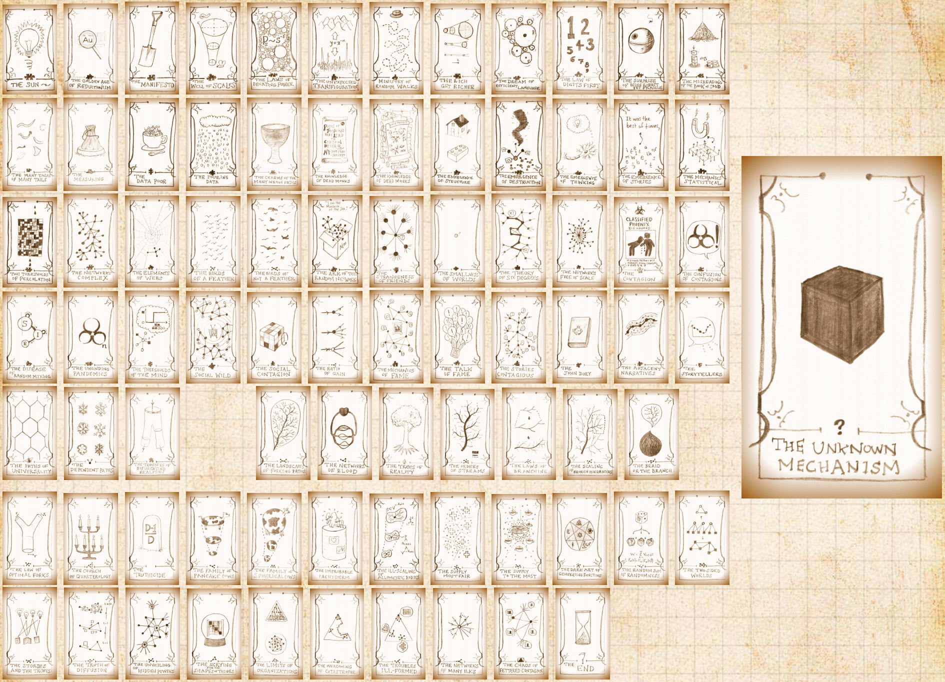
Mississippi:






THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
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?
THE UNKNOWN MECHANISM
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
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THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
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THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
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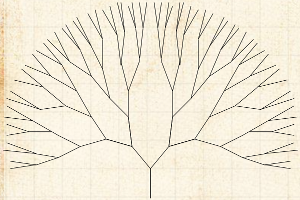
THE END

Random subnetworks on a Bethe lattice [13]



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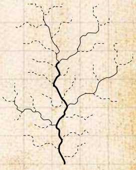

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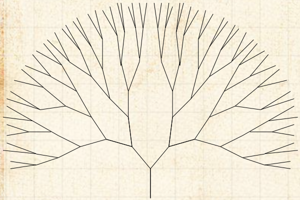

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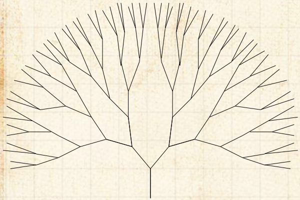

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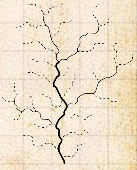

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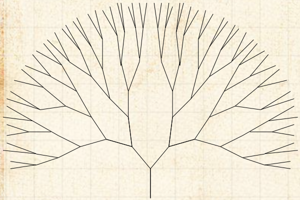

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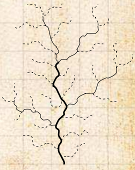

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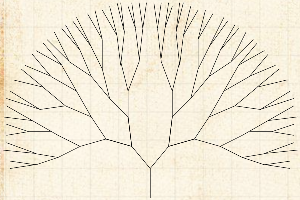

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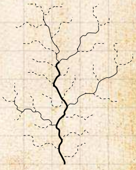

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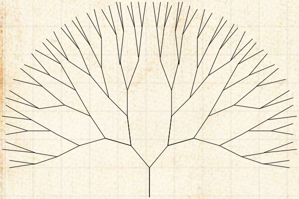

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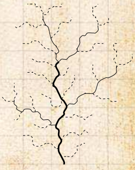

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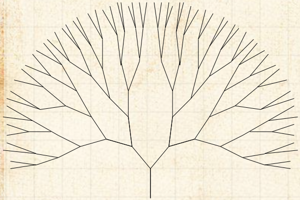

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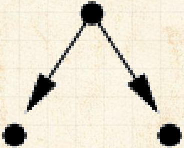
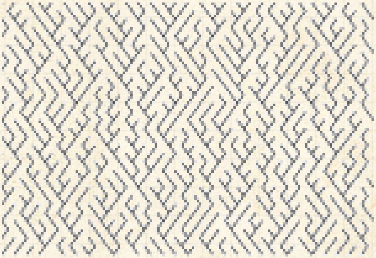
Scheidegger's model

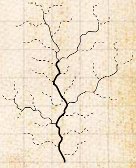
Directed random networks [11, 12]





$$P(\searrow) = P(\swarrow) = 1/2$$

- Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]



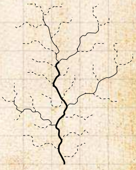
Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. [10]

- Landscapes $h(\vec{x})$ evolve such that energy dissipation $\dot{\varepsilon}$ is minimized, where

$$\dot{\varepsilon} \propto \int \mathrm{d}\vec{r}\,(\text{flux}) \times (\text{force}) \sim \sum_i I_i \nabla h_i \sim \sum_i I_i^{0.5}$$

- Landscapes obtained numerically give exponents near that of real networks.
- But: numerical method used matters.
- And: Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network [9]



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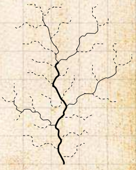

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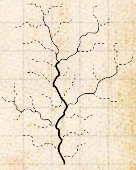

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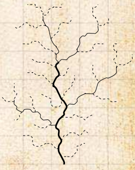

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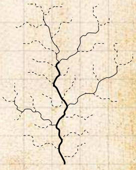

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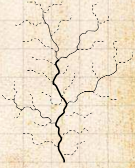

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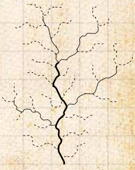

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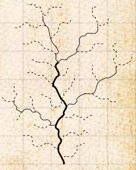


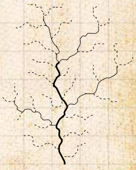
Theoretical networks

Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

$h \Rightarrow \ell \propto a^h$ (Hack's law).

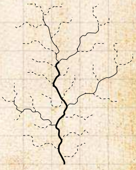
$d \Rightarrow \ell \propto L_{\parallel}^d$ (stream self-affinity).



Nutshell

Branching networks II Key Points:

- Horton's laws and Tokunaga law all fit together.
- For 2-d networks, these laws are 'planform' laws and ignore slope.
- Abundant scaling relations can be derived.
- Can take R_n, R_ℓ, and d as three independent parameters necessary to describe all 2-d branching networks.
- For scaling laws, only $h = \ln R_\ell / \ln R_n$ and d are needed.
- Laws can be extended nicely to laws of distributions.
- Numerous models of branching network evolution exist: nothing rock solid yet.



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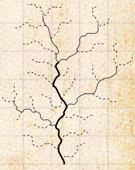

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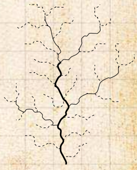

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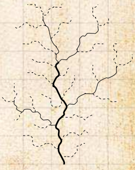

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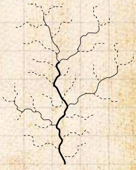

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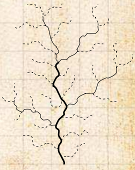

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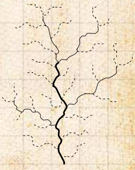

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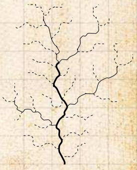

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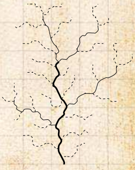

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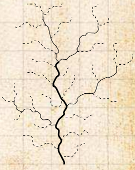

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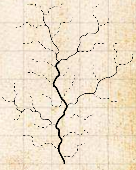

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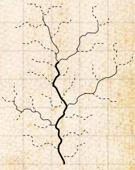

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