Branching Networks II

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Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2018

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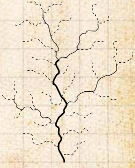
Vermont Advanced Computing Core | University of Vermont



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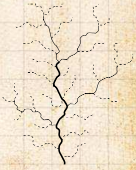

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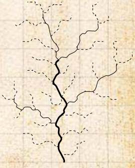


Outline

Horton ⇔ Tokunaga

Reducing Horton

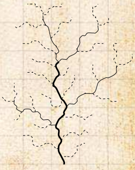
Scaling relations

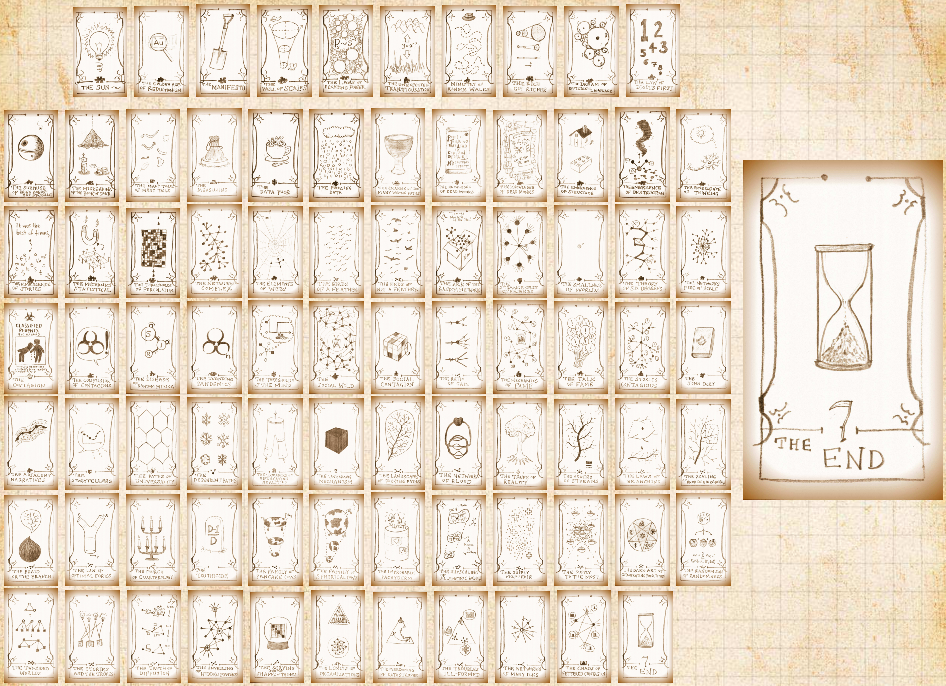
Fluctuations

Models

Nutshell

References

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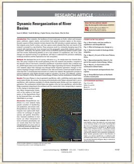
THE END
THE END

Piracy on the high χ's:



"Dynamic Reorganization of River Basins"
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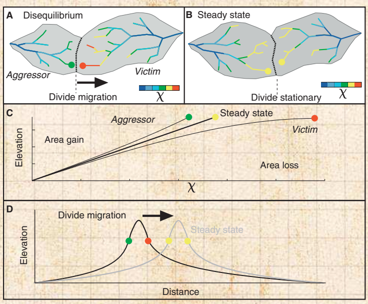

$$\frac{\partial z(x,t)}{\partial t} = U - K A^m \left| \frac{\partial z(x,t)}{\partial x} \right|^n$$

$$z(x) = z_\mathrm{b} + \left(\frac{U}{K A_0^m} \right)^{1/n} \chi$$

$$\chi = \int_{x_\mathrm{b}}^{x} \left(\frac{A_0}{A(x')} \right)^{m/n} \mathrm{d}x'$$

Piracy on the high χ's:

http://www.youtube.com/watch?v=FnroL1_-l2c?rel=0

More: How river networks move across a landscape (Science Daily)


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THE LAWS OF BRANCHING


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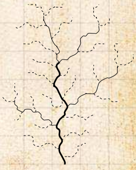
THE SCALING OF BROKEN HIERARCHIES

Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:

- In terms of network achitecture, Horton's laws appear to contain less detailed information than Tokunaga's law.
- Oddly, Horton's laws have four parameters and Tokunaga has two parameters.
- R_n, R_a, R_ℓ, and R_s versus T_1 and R_T. One simple redundancy: $R_\ell = R_s$.
  Insert question from assignment 1
- To make a connection, clearest approach is to start with Tokunaga's law ...
- Known result: Tokunaga $\rightarrow$ Horton [18, 19, 20, 9, 2]



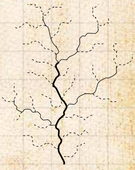
Let us make them happy

We need one more ingredient:

Space-fillingness

- A network is space-filling if the average distance between adjacent streams is roughly constant.
- Reasonable for river and cardiovascular networks
- For river networks:
  Drainage density ρ_{dd} = inverse of typical distance between channels in a landscape.
- In terms of basin characteristics:

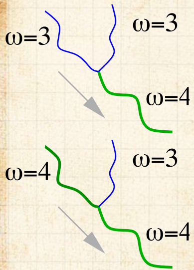
$$\rho_{\mathrm{dd}} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \frac{\sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega}{a_\Omega}$$



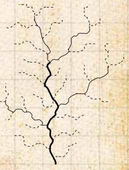
More with the happy-making thing

Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law: $n_\omega / n_{\omega+1} = R_n$.
- Estimate n_ω, the number of streams of order ω in terms of other $n_{\omega'}$, $\omega' > \omega$.
- Observe that each stream of order ω terminates by either:



1. Running into another stream of order ω and generating a stream of order $\omega + 1$ ...
   - $2n_{\omega+1}$ streams of order ω do this
2. Running into and being absorbed by a stream of higher order $\omega' > \omega$ ...
   - $n_{\omega'} T_{\omega'-\omega}$ streams of order ω do this



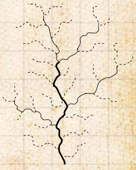
More with the happy-making thing

Putting things together:

- $$n_\omega = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega} n_{\omega'}}_{\text{absorption}}$$
- Use Tokunaga's law and manipulate expression to find Horton's law for stream numbers follows and hence obtain R_n.
- Insert question from assignment 1
- Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

(The larger value is the one we want.)



Finding other Horton ratios

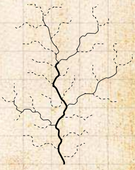
Connect Tokunaga to R_s

- Now use uniform drainage density ρ_{dd}.
- Assume side streams are roughly separated by distance $1/\rho_{\text{dd}}$.
- For an order ω stream segment, expected length is

$$\bar{s}_\omega \simeq \rho_{\text{dd}}^{-1} \left(1 + \sum_{k=1}^{\omega-1} T_k \right)$$

- Substitute in Tokunaga's law $T_k = T_1 R_T^{k-1}$:

$$\bar{s}_\omega \simeq \rho_{\text{dd}}^{-1} \left(1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1} \right) \propto R_T^{\omega}$$



Horton and Tokunaga are happy

Altogether then:

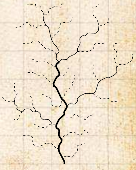
- $$\Rightarrow \bar{s}_{\omega}/\bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$

- Recall $R_\ell = R_s$ so

$$\boxed{R_\ell = R_s = R_T}$$

- And from before:

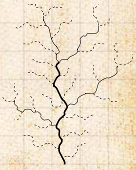
$$\boxed{R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}}$$



Horton and Tokunaga are happy

Some observations:

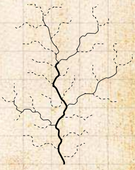
- R_n and R_ℓ depend on T_1 and R_T.
- Seems that R_a must as well ...
- Suggests Horton's laws must contain some redundancy
- We'll in fact see that $R_a = R_n$.
- Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between non-trivial statistical distributions. [3, 4]



Horton and Tokunaga are happy

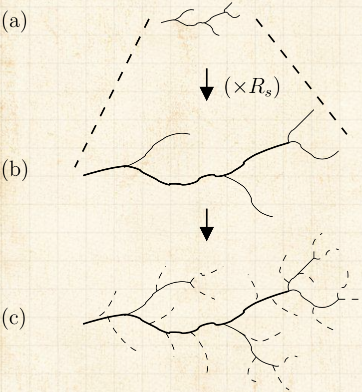
The other way round

- Note: We can invert the expresssions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.
- $$R_T = R_\ell,$$
- $$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n.$$
- Suggests we should be able to argue that Horton's laws imply Tokunaga's laws (if drainage density is uniform) ...

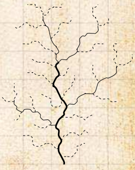


Horton and Tokunaga are friends

From Horton to Tokunaga [2]



- Assume Horton's laws hold for number and length
- Start with picture showing an order ω stream and order $\omega - 1$ generating and side streams.
- Scale up by a factor of R_ℓ, orders increment to $\omega + 1$ and ω.
- Maintain drainage density by adding new order $\omega - 1$ streams



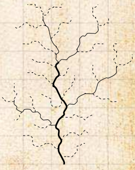
Horton and Tokunaga are friends

...and in detail:

- Must retain same drainage density.
- Add an extra $(R_\ell - 1)$ first order streams for each original tributary.
- Since by definition, an order $\omega + 1$ stream segment has T_ω order 1 side streams, we have:

$$T_k = (R_\ell - 1)\left(1 + \sum_{i=1}^{k-1} T_i\right).$$

- For large ω, Tokunaga's law is the solution—let's check ...



Horton and Tokunaga are friends

Just checking:

- Substitute Tokunaga's law $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$ into

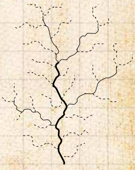
$$T_k = (R_\ell - 1)\left(1 + \sum_{i=1}^{k-1} T_i\right)$$

-

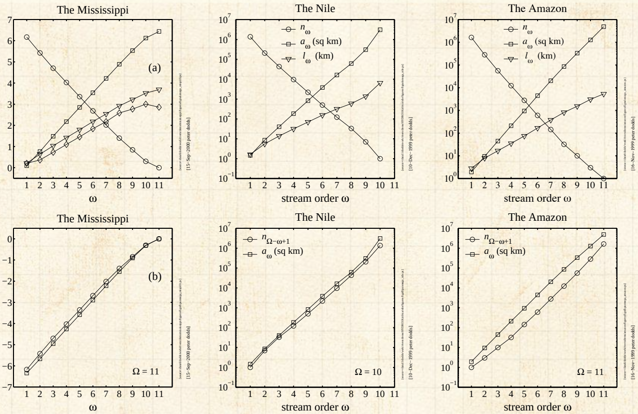
$$T_k = (R_\ell - 1)\left(1 + \sum_{i=1}^{k-1} T_1 R_\ell^{i-1}\right)$$

$$= (R_\ell - 1)\left(1 + T_1 \frac{R_\ell^{k-1} - 1}{R_\ell - 1}\right)$$

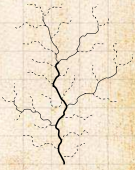
$$\simeq (R_\ell - 1) T_1 \frac{R_\ell^{k-1}}{R_\ell - 1} = T_1 R_\ell^{k-1}$$ ...yep.



Horton's laws of area and number:



- In bottom plots, stream number graph has been flipped vertically.
- Highly suggestive that $R_n \equiv R_a$ ...

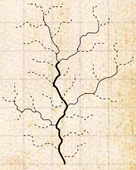


Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?
- Rule of thumb: discard data for two smallest and two largest orders.

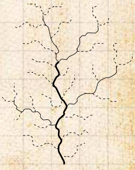
Mississippi:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
$[2,3]$	5.27	5.26	2.48	2.30	1.00
$[2,5]$	4.86	4.96	2.42	2.31	1.02
$[2,7]$	4.77	4.88	2.40	2.31	1.02
$[3,4]$	4.72	4.91	2.41	2.34	1.04
$[3,6]$	4.70	4.83	2.40	2.35	1.03
$[3,8]$	4.60	4.79	2.38	2.34	1.04
$[4,6]$	4.69	4.81	2.40	2.36	1.02
$[4,8]$	4.57	4.77	2.38	2.34	1.05
$[5,7]$	4.68	4.83	2.36	2.29	1.03
$[6,7]$	4.63	4.76	2.30	2.16	1.03
$[7,8]$	4.16	4.67	2.41	2.56	1.12
mean μ	4.69	4.85	2.40	2.33	1.04
std dev σ	0.21	0.13	0.04	0.07	0.03
σ/μ	0.045	0.027	0.015	0.031	0.024



Amazon:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
$[2,3]$	4.78	4.71	2.47	2.08	0.99
$[2,5]$	4.55	4.58	2.32	2.12	1.01
$[2,7]$	4.42	4.53	2.24	2.10	1.02
$[3,5]$	4.45	4.52	2.26	2.14	1.01
$[3,7]$	4.35	4.49	2.20	2.10	1.03
$[4,6]$	4.38	4.54	2.22	2.18	1.03
$[5,6]$	4.38	4.62	2.22	2.21	1.06
$[6,7]$	4.08	4.27	2.05	1.83	1.05
mean μ	4.42	4.53	2.25	2.10	1.02
std dev σ	0.17	0.10	0.10	0.09	0.02
σ/μ	0.038	0.023	0.045	0.042	0.019



Reducing Horton's laws:

Rough first effort to show $R_n \equiv R_a$:

- $a_\Omega \propto$ sum of all stream segment lengths in a order Ω basin (assuming uniform drainage density)
- So:

$$a_\Omega \simeq \sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega / \rho_{\text{dd}}$$

$$\propto \sum_{\omega=1}^{\Omega} \underbrace{R_n^{\Omega-\omega} \cdot \overbrace{1}^{n_\Omega}}_{n_\omega} \underbrace{\bar{s}_1 \cdot R_s^{\omega-1}}_{\bar{s}_\omega}$$

$$= \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n} \right)^{\omega}$$

Reducing Horton's laws:

Continued ...

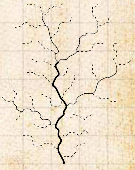
$$a_\Omega \propto \frac{R_n^\Omega}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n} \right)^\omega$$

$$= \frac{R_n^\Omega}{R_s} \bar{s}_1 \frac{R_s}{R_n} \frac{1-(R_s/R_n)^\Omega}{1-(R_s/R_n)}$$

$$\sim R_n^{\Omega-1} \bar{s}_1 \frac{1}{1-(R_s/R_n)} \text{ as } \Omega \nearrow$$

- So, a_Ω is growing like R_n^Ω and therefore:

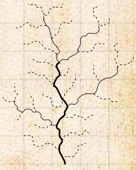
$$\boxed{R_n \equiv R_a}$$



Reducing Horton's laws:

Not quite:

- ...But this only a rough argument as Horton's laws do not imply a strict hierarchy
- Need to account for sidebranching.
- Insert question from assignment 2



Equipartitioning:

Intriguing division of area:

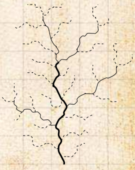
- Observe: Combined area of basins of order ω independent of ω.
- Not obvious: basins of low orders not necessarily contained in basis on higher orders.
- Story:

$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$

- Reason:

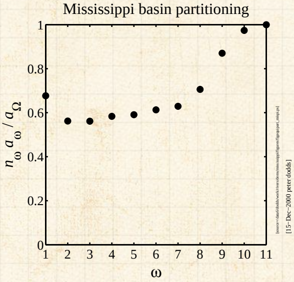
$$n_\omega \propto (R_n)^{-\omega}$$

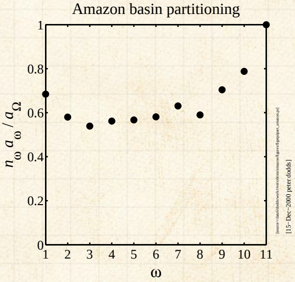
$$\bar{a}_\omega \propto (R_a)^{\omega} \propto n_\omega^{-1}$$

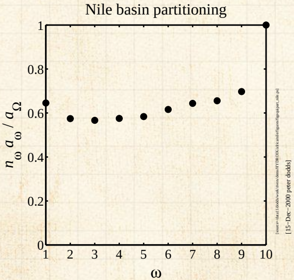


Equipartitioning:

Some examples:







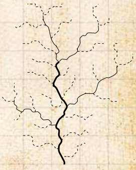
Neural Reboot: Fwoompf

http://www.youtube.com/watch?v=5mUs70SqD4o?rel=0

Scaling laws

The story so far:

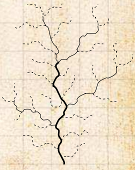
- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- Tokunaga's law describes detailed architecture: $T_k = T_1 R_T^{k-1}$.
- We have connected Tokunaga's and Horton's laws
- Only two Horton laws are independent ($R_n = R_a$)
- Only two parameters are independent: $(T_1, R_T) \Leftrightarrow (R_n, R_s)$



Scaling laws

A little further ...

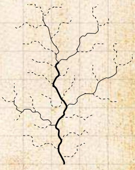
- Ignore stream ordering for the moment
- Pick a random location on a branching network p.
- Each point p is associated with a basin and a longest stream length
- Q: What is probability that the p's drainage basin has area a? $P(a) \propto a^{-\tau}$ for large a
- Q: What is probability that the longest stream from p has length ℓ? $P(\ell) \propto \ell^{-\gamma}$ for large ℓ
- Roughly observed: $1.3 \lesssim \tau \lesssim 1.5$ and $1.7 \lesssim \gamma \lesssim 2.0$



Scaling laws

Probability distributions with power-law decays

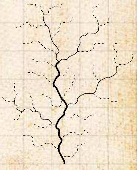
- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)
  - Word frequency (Zipf's law) [22]
  - Wealth (maybe not—at least heavy tailed)
  - Statistical mechanics (phase transitions) [5]
- A big part of the story of complex systems
- Arise from mechanisms: growth, randomness, optimization, ...
- Our task is always to illuminate the mechanism ...



Scaling laws

Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- Plan: Derive $P(a) \propto a^{-\tau}$ and $P(\ell) \propto \ell^{-\gamma}$ starting with Tokunaga/Horton story [17, 1, 2]
- Let's work on $P(\ell)$ ...
- Our first fudge: assume Horton's laws hold throughout a basin of order Ω.
- (We know they deviate from strict laws for low ω and high ω but not too much.)
- Next: place stick between teeth. Bite stick. Proceed.



Scaling laws

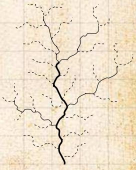
Finding γ:

- Often useful to work with cumulative distributions, especially when dealing with power-law distributions.
- The complementary cumulative distribution turns out to be most useful:

$$P_{>}(\ell_*) = P(\ell > \ell_*) = \int_{\ell=\ell_*}^{\ell_{\max}} P(\ell)\mathrm{d}\ell$$

- $$P_{>}(\ell_*) = 1 - P(\ell < \ell_*)$$

- Also known as the exceedance probability.



Scaling laws

Finding γ:

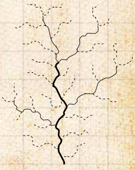
- The connection between $P(x)$ and $P_>(x)$ when $P(x)$ has a power law tail is simple:
- Given $P(\ell) \sim \ell^{-\gamma}$ large ℓ then for large enough ℓ_*

$$P_>(\ell_*) = \int_{\ell=\ell_*}^{\ell_{\text{max}}} P(\ell)\,\mathrm{d}\ell$$

$$\sim \int_{\ell=\ell_*}^{\ell_{\text{max}}} \ell^{-\gamma} \mathrm{d}\ell$$

$$= \left.\frac{\ell^{-(\gamma-1)}}{-(\gamma-1)}\right|_{\ell=\ell_*}^{\ell_{\text{max}}}$$

$$\propto \ell_*^{-(\gamma-1)} \quad \text{for } \ell_{\text{max}} \gg \ell_*$$



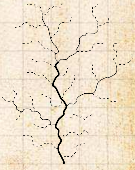
Scaling laws

Finding γ:

- Aim: determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
- Assume some spatial sampling resolution Δ
- Landscape is broken up into grid of $\Delta \times \Delta$ sites
- Approximate $P_>(\ell_*)$ as

$$P_>(\ell_*) = \frac{N_>(\ell_*;\Delta)}{N_>(0;\Delta)}.$$

  where $N_>(\ell_*;\Delta)$ is the number of sites with main stream length $> \ell_*$.
- Use Horton's law of stream segments: $\bar{s}_\omega / \bar{s}_{\omega-1} = R_s$ ...

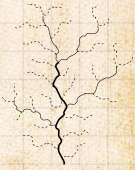


Scaling laws

Finding γ:

- Set $\ell_* = \bar{\ell}_\omega$ for some $1 \ll \omega \ll \Omega$.
- $$P_>(\bar{\ell}_\omega) = \frac{N_>(\bar{\ell}_\omega; \Delta)}{N_>(0; \Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \cancel{\Delta}}{\sum_{\omega'=1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} / \cancel{\Delta}}$$
- Δ's cancel
- Denominator is $a_\Omega \rho_{\mathrm{dd}}$, a constant.
- So ...using Horton's laws ...

$$P_>(\bar{\ell}_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} \bar{s}_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'})(\bar{s}_1 \cdot R_s^{\omega'-1})$$



Scaling laws

Finding γ:

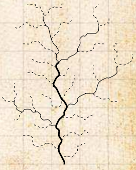
- We are here:

$$P_{>}(\bar{\ell}_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'})(\bar{s}_1 \cdot R_s^{\omega'-1})$$

- Cleaning up irrelevant constants:

$$P_{>}(\bar{\ell}_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} \left(\frac{R_s}{R_n}\right)^{\omega'}$$

- Change summation order by substituting $\omega'' = \Omega - \omega'$.
- Sum is now from $\omega'' = 0$ to $\omega'' = \Omega - \omega - 1$ (equivalent to $\omega' = \Omega$ down to $\omega' = \omega + 1$)



Scaling laws

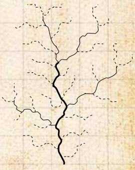
Finding γ:

$$P_>(\bar{\ell}_\omega) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_s}{R_n}\right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_n}{R_s}\right)^{\omega''}$$

- Since $R_n > R_s$ and $1 \ll \omega \ll \Omega$,

$$P_>(\bar{\ell}_\omega) \propto \left(\frac{R_n}{R_s}\right)^{\Omega-\omega} \propto \left(\frac{R_n}{R_s}\right)^{-\omega}$$

again using $\sum_{i=0}^{n-1} a^i = (a^n - 1)/(a - 1)$



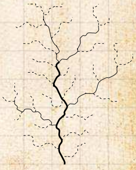
Scaling laws

Finding γ:

- Nearly there:

$$P_{>}(\bar{\ell}_\omega) \propto \left(\frac{R_n}{R_s}\right)^{-\omega} = e^{-\omega \ln(R_n/R_s)}$$

- Need to express right hand side in terms of $\bar{\ell}_\omega$.
- Recall that $\bar{\ell}_\omega \simeq \bar{\ell}_1 R_\ell^{\omega-1}$.
- $$\bar{\ell}_\omega \propto R_\ell^\omega = R_s^\omega = e^{\omega \ln R_s}$$



Scaling laws

Finding γ:

- Therefore:

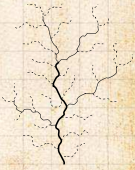
$$P_>(\bar{\ell}_\omega) \propto e^{-\omega \ln(R_n/R_s)} = \left(e^{\omega \ln R_s}\right)^{-\ln(R_n/R_s)/\ln(R_s)}$$

- $$\propto \bar{\ell}_\omega^{\,-\ln(R_n/R_s)/\ln R_s}$$

- $$= \bar{\ell}_\omega^{\,-(\ln R_n - \ln R_s)/\ln R_s}$$

- $$= \bar{\ell}_\omega^{\,-\ln R_n/\ln R_s + 1}$$

- $$= \bar{\ell}_\omega^{\,-\gamma+1}$$



Scaling laws

Finding γ:

- And so we have:

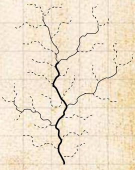
$$\gamma = \ln R_n / \ln R_s$$

- Proceeding in a similar fashion, we can show

$$\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma$$

Insert question from assignment 2

- Such connections between exponents are called scaling relations
- Let's connect to one last relationship: Hack's law



Scaling laws

Hack's law: [6]

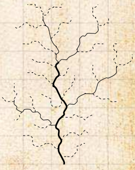
- $$\ell \propto a^h$$
- Typically observed that $0.5 \lesssim h \lesssim 0.7$.
- Use Horton laws to connect h to Horton ratios:

$$\bar{\ell}_\omega \propto R_s^\omega \text{ and } \bar{a}_\omega \propto R_n^\omega$$

- Observe:

$$\bar{\ell}_\omega \propto e^{\omega \ln R_s} \propto \left(e^{\omega \ln R_n}\right)^{\ln R_s / \ln R_n}$$

$$\propto \left(R_n^\omega\right)^{\ln R_s / \ln R_n} \propto \bar{a}_\omega^{\ln R_s / \ln R_n} \Rightarrow \boxed{h = \ln R_s / \ln R_n}$$



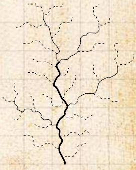
We mentioned there were a good number of ‘laws’: [2]

Relation:	Name or description:
$T_k = T_1 (R_T)^{k-1}$	Tokunaga’s law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega / n_{\omega+1} = R_n$	Horton’s law of stream numbers
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	Horton’s law of main stream lengths
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	Horton’s law of basin areas
$\bar{s}_{\omega+1} / \bar{s}_\omega = R_s$	Horton’s law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack’s law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein’s law
$\lambda \sim L^\varphi$	variation of Langbein’s law

Connecting exponents

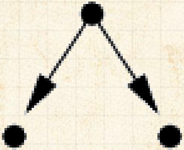
Only 3 parameters are independent:
e.g., take d, R_n, and R_s

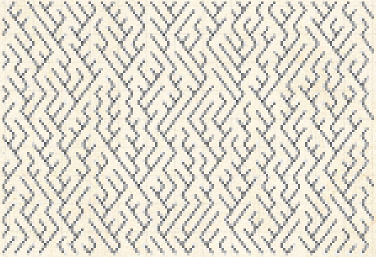
relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d
$T_k = T_1 (R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_\omega / n_{\omega+1} = R_n$	R_n
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	$R_a = R_n$
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \ln R_s / \ln R_n$
$a \sim L^D$	$D = d/h$
$L_\perp \sim L^H$	$H = d/h - 1$
$P(a) \sim a^{-\tau}$	$\tau = 2 - h$
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim a^\beta$	$\beta = 1 + h$
$\lambda \sim L^\varphi$	$\varphi = d$



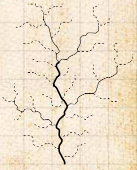
Scheidegger's model

Directed random networks [11, 12]

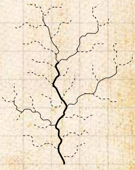




- $$P(\searrow) = P(\swarrow) = 1/2$$
- Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]
- Useful and interesting test case

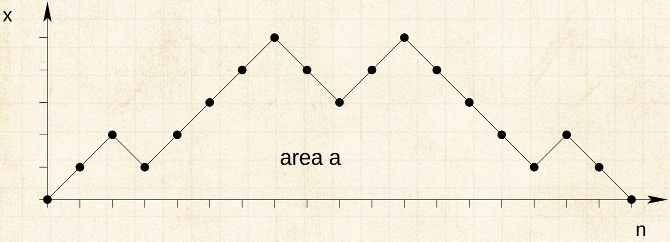


A toy model—Scheidegger's model

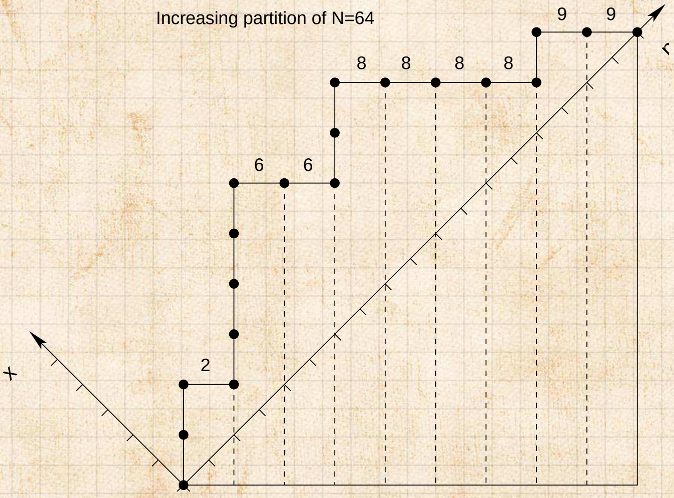


Random walk basins:

- Boundaries of basins are random walks



Scheidegger's model



Scheidegger's model

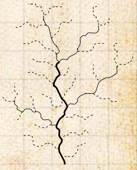
Prob for first return of a random walk in (1+1) dimensions (from CSYS/MATH 300):

- $$P(n) \sim \frac{1}{2\sqrt{\pi}}\, n^{-3/2}.$$

  and so $P(\ell) \propto \ell^{-3/2}$.
- Typical area for a walk of length n is $\propto n^{3/2}$:

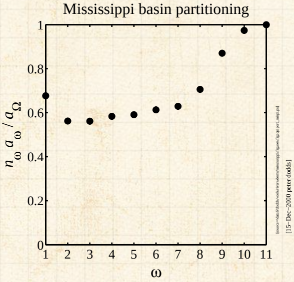
  $$\ell \propto a^{2/3}.$$

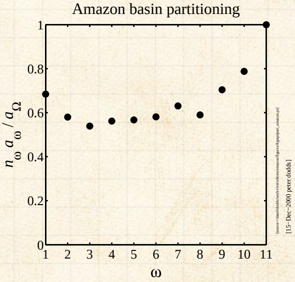
- Find $\tau = 4/3$, $h = 2/3$, $\gamma = 3/2$, $d = 1$.
- Note $\tau = 2 - h$ and $\gamma = 1/h$.
- R_n and R_ℓ have not been derived analytically.

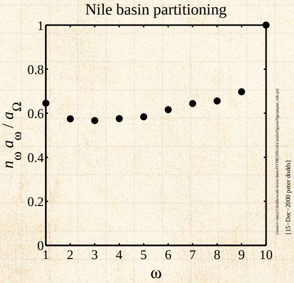


Equipartitioning reexamined:

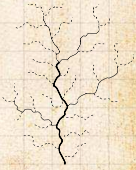
Recall this story:





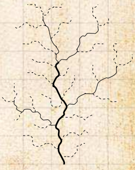


Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations
Models
Nutshell
References



Equipartitioning

- What about
$$P(a) \sim a^{-\tau} \qquad ?$$
- Since $\tau > 1$, suggests no equipartitioning:
$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$
- $P(a)$ overcounts basins within basins ...
- while stream ordering separates basins ...



Hard neural reboot (sound matters):

https://twitter.com/round_boys/status/951873765964681216

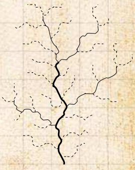
Fluctuations

Moving beyond the mean:

- Both Horton's laws and Tokunaga's law relate average properties, e.g.,

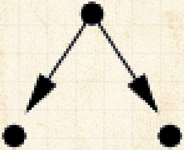
$$\bar{s}_{\omega}/\bar{s}_{\omega-1} = R_s$$

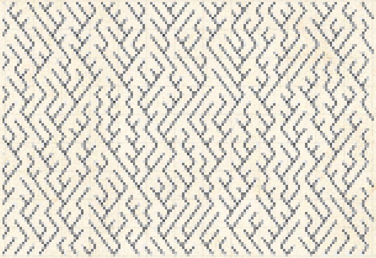
- Natural generalization to consider relationships between probability distributions
- Yields rich and full description of branching network structure
- See into the heart of randomness ...



A toy model—Scheidegger's model

Directed random networks [11, 12]

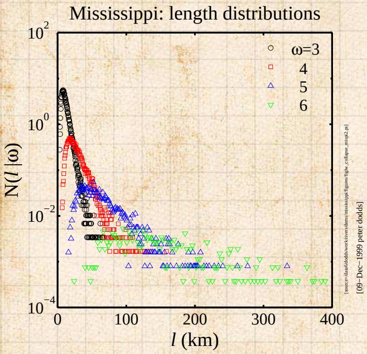


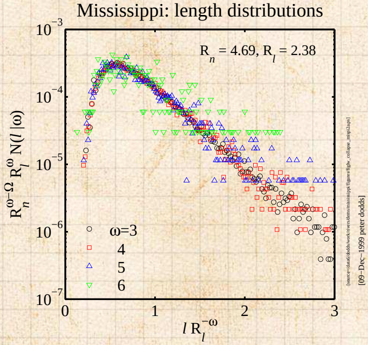


- $$P(\searrow) = P(\swarrow) = 1/2$$
- Flow is directed downwards

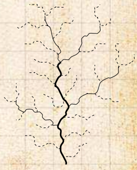
Generalizing Horton's laws

- $\bar{\ell}_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell|\omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell/R_\ell^\omega)$
- $\bar{a}_\omega \propto (R_a)^\omega \Rightarrow N(a|\omega) = (R_n^2)^{-\omega} F_a(a/R_n^\omega)$



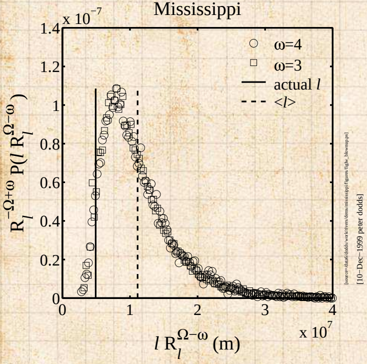


- Scaling collapse works well for intermediate orders
- All moments grow exponentially with order

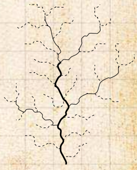


Generalizing Horton's laws

- How well does overall basin fit internal pattern?



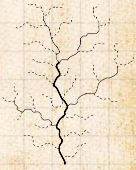
- Actual length = 4920 km (at 1 km res)
- Predicted Mean length = 11100 km
- Predicted Std dev = 5600 km
- Actual length/Mean length = 44 %
- Okay.



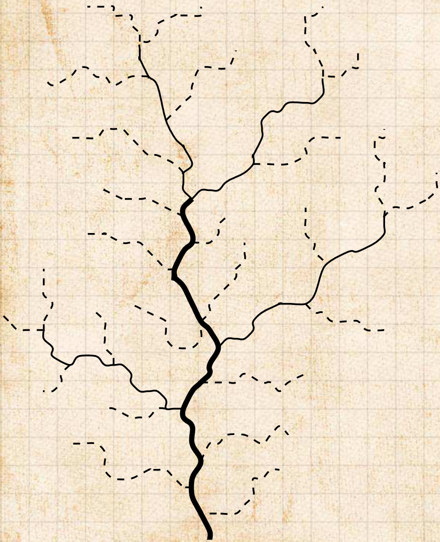
Generalizing Horton's laws

Comparison of predicted versus measured main stream lengths for large scale river networks (in 10^3 km):

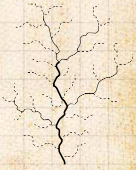
basin:	ℓ_Ω	$\bar{\ell}_\Omega$	σ_ℓ	$\ell_\Omega/\bar{\ell}_\Omega$	$\sigma_\ell/\bar{\ell}_\Omega$
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73
	a_Ω	$\bar{a}_\Omega$	σ_a	$a_\Omega/\bar{a}_\Omega$	$\sigma_a/\bar{a}_\Omega$
Mississippi	2.74	7.55	5.58	0.36	0.74
Amazon	5.40	9.07	8.04	0.60	0.89
Nile	3.08	0.96	0.79	3.19	0.82
Congo	3.70	10.09	8.28	0.37	0.82
Kansas	0.14	0.49	0.42	0.28	0.86



Combining stream segments distributions:



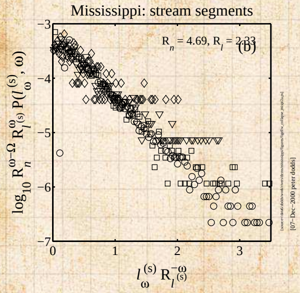
- Stream segments sum to give main stream lengths
- $$\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$$
- $P(\ell_\omega)$ is a convolution of distributions for the s_ω



Generalizing Horton's laws

- Sum of variables $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$ leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \cdots * N(s|\omega)$$

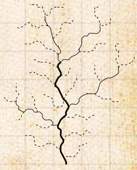


$$N(s|\omega) = \frac{1}{R_n^\omega R_\ell^\omega} F\left(s/R_\ell^\omega\right)$$

$$F(x) = e^{-x/\xi}$$

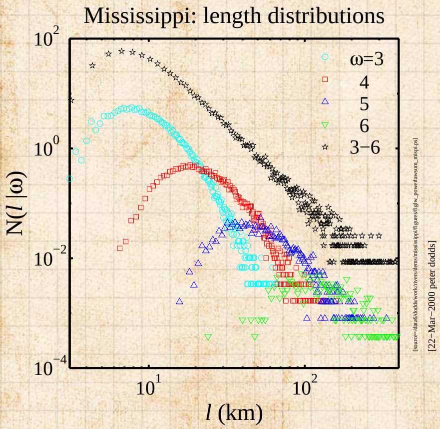
Mississippi: $\xi \simeq 900$ m.

Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations
Models
Nutshell
References

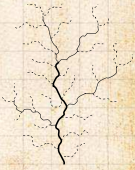


Generalizing Horton's laws

- Next level up: Main stream length distributions must combine to give overall distribution for stream length

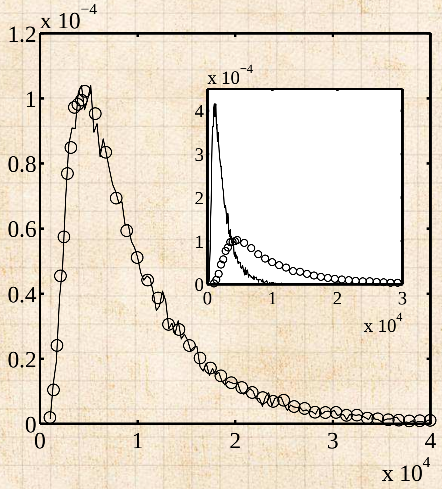


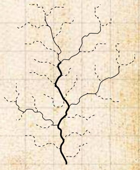
- $P(\ell) \sim \ell^{-\gamma}$
- Another round of convolutions [3]
- Interesting ...



Generalizing Horton’s laws

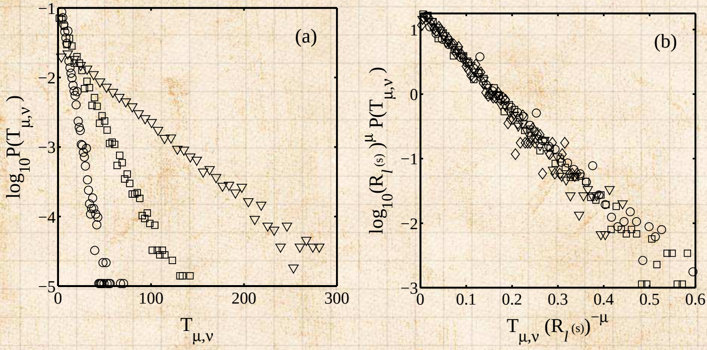
- Number and area distributions for the Scheidegger model [3]
- $P(n_{1,6})$ versus $P(a_6)$ for a randomly selected $\omega = 6$ basin.



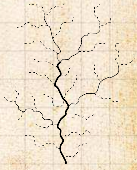


Generalizing Tokunaga's law

Scheidegger:

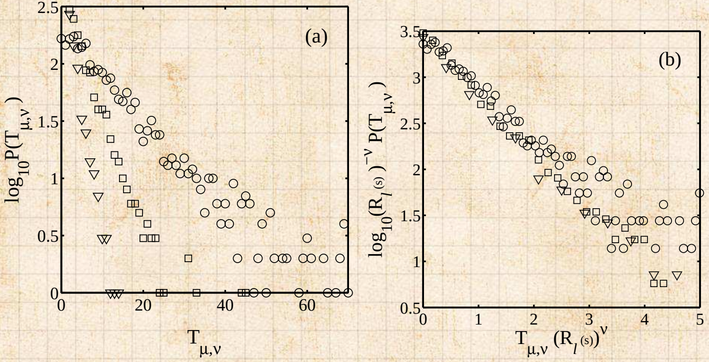


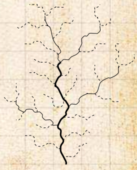
- Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using R_s



Generalizing Tokunaga's law

Mississippi:





- Same data collapse for Mississippi ...

Generalizing Tokunaga's law

So

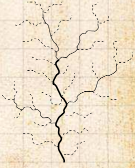
$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t \left[T_{\mu,\nu} / (R_s)^{\mu-\nu-1} \right]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

$$\boxed{P(s_\mu) \Leftrightarrow P(T_{\mu,\nu})}$$

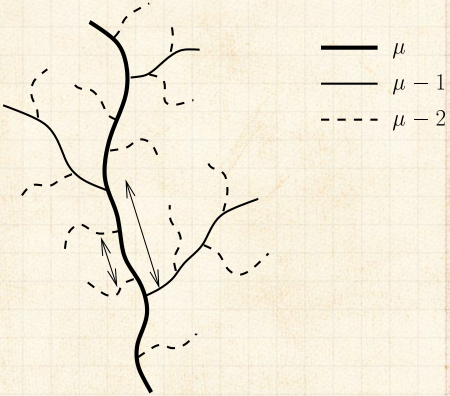
- Exponentials arise from randomness.
- Look at joint probability $P(s_\mu, T_{\mu,\nu})$.

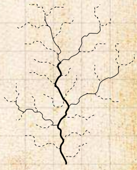


Generalizing Tokunaga's law

Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments



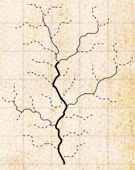


Generalizing Tokunaga's law

- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is constant:

$$\tilde{p}_\mu \simeq 1/(R_s)^{\mu-1}\xi_s$$

- Probability decays exponentially with stream order
- Inter-tributary lengths exponentially distributed
- $\Rightarrow$ random spatial distribution of stream segments

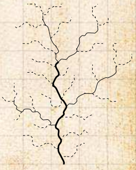


Generalizing Tokunaga's law

- Joint distribution for generalized version of Tokunaga's law:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

  where
  - p_ν = probability of absorbing an order ν side stream
  - $\tilde{p}_\mu$ = probability of an order μ stream terminating
- Approximation: depends on distance units of s_μ
- In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.



Generalizing Tokunaga's law

- Now deal with this thing:

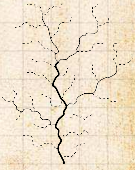
$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

- Set $(x, y) = (s_\mu, T_{\mu,\nu})$ and $q = 1 - p_\nu - \tilde{p}_\mu$, approximate liberally.
- Obtain

$$P(x, y) = N x^{-1/2} \left[F(y/x)\right]^x$$

where

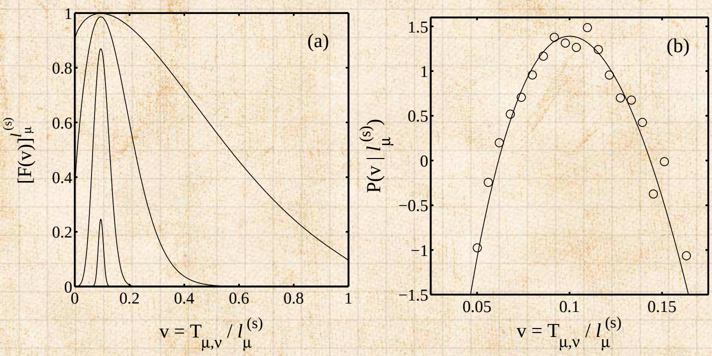
$$F(v) = \left(\frac{1-v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$

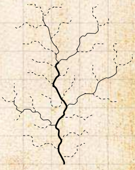


Generalizing Tokunaga's law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Scheidegger:





Generalizing Tokunaga's law

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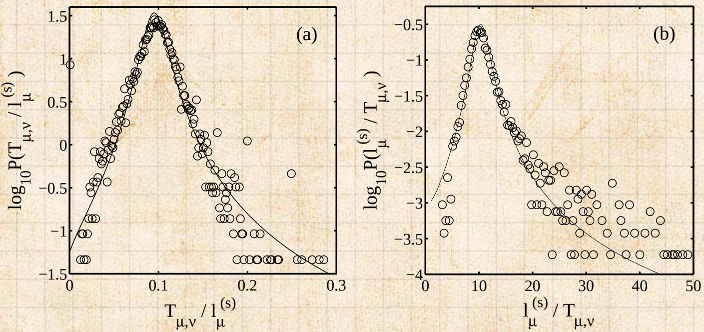
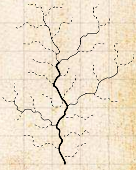
Scheidegger:





Generalizing Tokunaga's law

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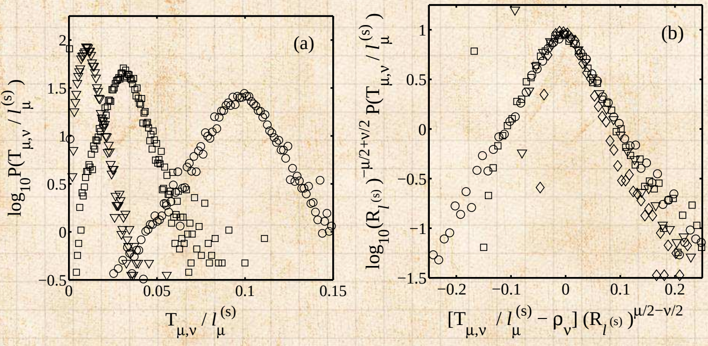
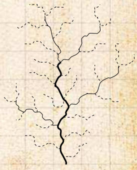
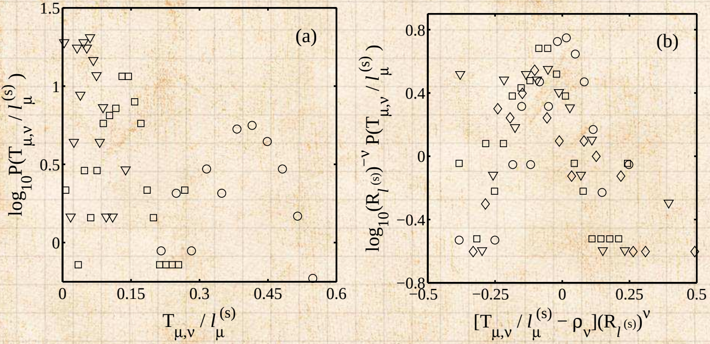


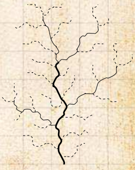


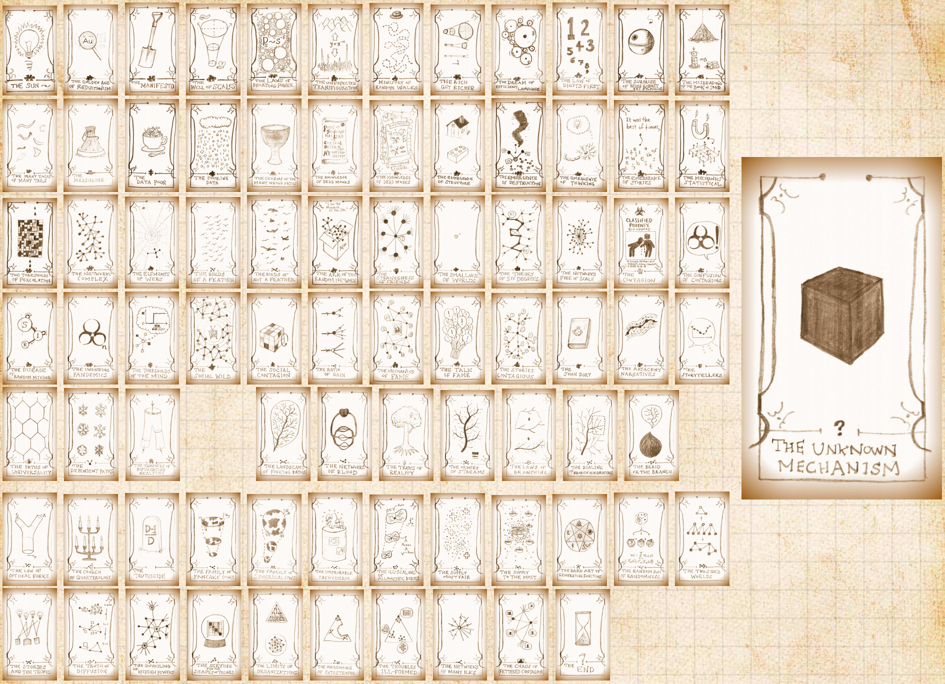
Generalizing Tokunaga's law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Mississippi:

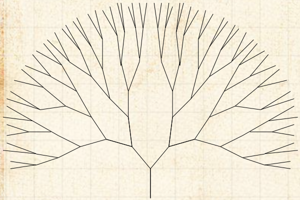




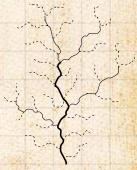

THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE MISREADING OF THE BOOK OF SAND
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE SCALING OF BROKEN HIERARCHIES
THE BRAID OR THE BRANCH
THE UNKNOWN MECHANISM
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END

Models

Random subnetworks on a Bethe lattice [13]

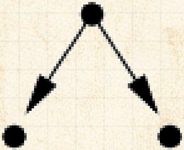


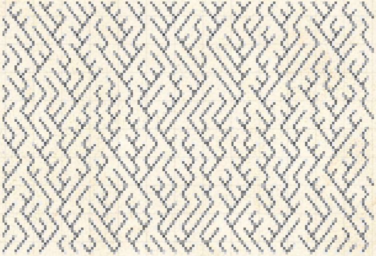
- Dominant theoretical concept for several decades.
- Bethe lattices are fun and tractable.
- Led to idea of "Statistical inevitability" of river network statistics [7]
- But Bethe lattices unconnected with surfaces.
- In fact, Bethe lattices $\simeq$ infinite dimensional spaces (oops).
- So let's move on ...



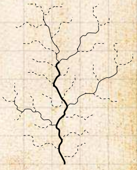
Scheidegger's model

Directed random networks [11, 12]





- $$P(\searrow) = P(\swarrow) = 1/2$$
- Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]



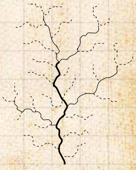
Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. [10]

- Landscapes $h(\vec{x})$ evolve such that energy dissipation $\dot{\varepsilon}$ is minimized, where

$$\dot{\varepsilon} \propto \int \mathrm{d}\vec{r}\ (\text{flux}) \times (\text{force}) \sim \sum_i a_i \nabla h_i \sim \sum_i a_i^{\gamma}$$

- Landscapes obtained numerically give exponents near that of real networks.
- But: numerical method used matters.
- And: Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network [8]



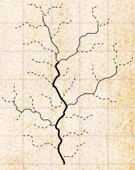
Theoretical networks

Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

$h \Rightarrow \ell \propto a^h$ (Hack's law).

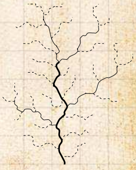
$d \Rightarrow \ell \propto L_\parallel^d$ (stream self-affinity).



Nutshell

Branching networks II Key Points:

- Horton's laws and Tokunaga law all fit together.
- For 2-d networks, these laws are 'planform' laws and ignore slope.
- Abundant scaling relations can be derived.
- Can take R_n, R_ℓ, and d as three independent parameters necessary to describe all 2-d branching networks.
- For scaling laws, only $h = \ln R_\ell / \ln R_n$ and d are needed.
- Laws can be extended nicely to laws of distributions.
- Numerous models of branching network evolution exist: nothing rock solid yet.



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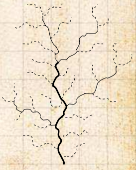

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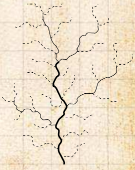

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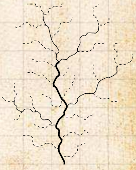

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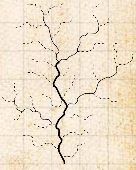

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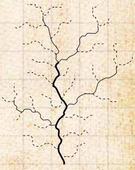

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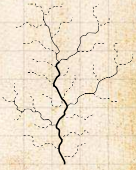

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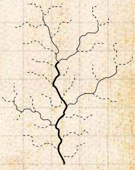

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