Scaling—a Plenitude of Power Laws

Principles of Complex Systems | @pocsvox
CSYS/MATH 300, Fall, 2017

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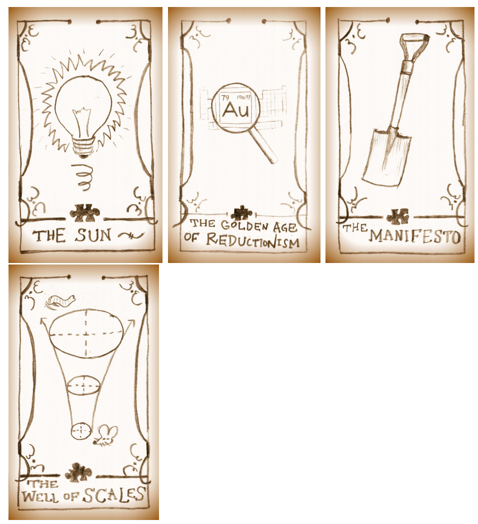
Outline



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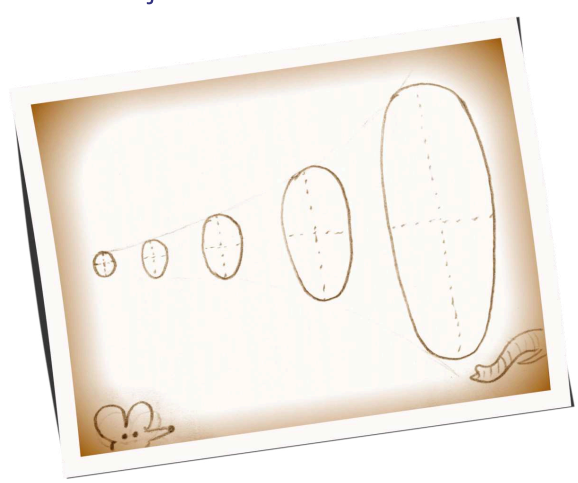


Scalingarama

General observation:

Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of scaling.

Outline—All about scaling:

- Basic definitions.
- Examples.

In CocoNuTs:

- Advances in measuring your power-law relationships.
- Scaling in blood and river networks.
- The Unsolved Allometry Theoricides.

Definitions

A power law relates two variables x and y as follows:

$$y = cx^{\alpha}$$

- α is the scaling exponent (or just exponent)
- α can be any number in principle but we will find various restrictions.
- c is the prefactor (which can be important!)

Definitions

- The prefactor c must balance dimensions.
- Imagine the height ℓ and volume v of a family of shapes are related as:

$$\ell = cv^{1/4}$$

- Using $[\cdot]$ to indicate dimension, then

$$[c] = [l]/[V^{1/4}] = L/L^{3/4} = L^{1/4}.$$

- More on this later with the Buckingham π theorem.

Looking at data

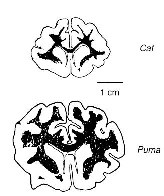
- Power-law relationships are linear in log-log space:

$$y = cx^{\alpha}$$

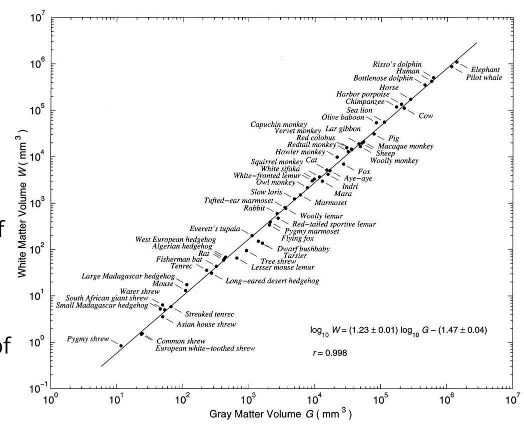
$$\Rightarrow \log_b y = \alpha \log_b x + \log_b c$$

  with slope equal to α, the scaling exponent.
- Much searching for straight lines on log-log or double-logarithmic plots.
- Good practice: Always, always, always use base 10.
- Talk only about orders of magnitude (powers of 10).

A beautiful, heart-warming example:



- G = volume of gray matter: 'computing elements'
- W = volume of white matter: 'wiring'
- $W \sim cG^{1.23}$



- from Zhang & Sejnowski, PNAS (2000) [35]

Why is $\alpha \simeq 1.23$?

Quantities (following Zhang and Sejnowski):

- G = Volume of gray matter (cortex/processors)
- W = Volume of white matter (wiring)
- T = Cortical thickness (wiring)
- S = Cortical surface area
- L = Average length of white matter fibers
- p = density of axons on white matter/cortex interface

A rough understanding:

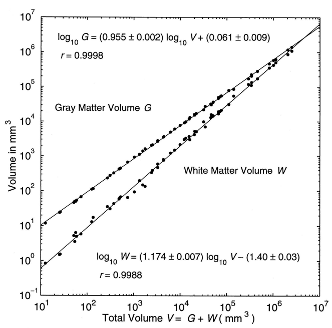
- $G \sim ST$ (convolutions are okay)
- $W \sim \frac{1}{2} pSL$
- $G \sim L^3 \leftarrow$ this is a little sketchy...
- Eliminate S and L to find $W \propto G^{4/3}/T$

Why is $\alpha \simeq 1.23$?

A rough understanding:

- We are here: $W \propto G^{4/3}/T$
- Observe weak scaling $T \propto G^{0.10 \pm 0.02}$.
- Implies $S \propto G^{0.9} \rightarrow$ convolutions fill space.
- $\Rightarrow W \propto G^{4/3}/T \propto G^{1.23 \pm 0.02}$

Tricksiness:



- With $V = G + W$, some power laws must be approximations.
- Measuring exponents is a hairy business...

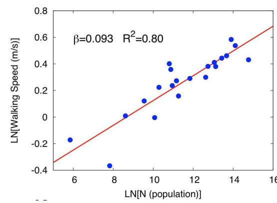
Good scaling:

General rules of thumb:

- *High quality:* scaling persists over three or more orders of magnitude for each variable.
- *Medium quality:* scaling persists over three or more orders of magnitude for only one variable and at least one for the other.
- *Very dubious:* scaling 'persists' over less than an order of magnitude for both variables.

Unconvincing scaling:

Average walking speed as a function of city population:



Two problems:

1. use of natural log, and
2. minute varation in dependent variable.

- from Bettencourt et al. (2007) [4]; otherwise totally great—see later.

Definitions

Power laws are the signature of scale invariance:

Scale invariant 'objects'
look the 'same'
when they are appropriately
rescaled.

- Objects = geometric shapes, time series, functions, relationships, distributions,...
- 'Same' might be 'statistically the same'
- To rescale means to change the units of measurement for the relevant variables

Scale invariance

Our friend $y = cx^\alpha$:

- If we rescale x as $x = rx'$ and y as $y = r^\alpha y'$,
- then

$$r^\alpha y' = c(rx')^\alpha$$

- $$\Rightarrow y' = c r^\alpha x'^\alpha r^{-\alpha}$$

- $$\Rightarrow y' = c x'^\alpha$$

Scale invariance

Compare with $y = ce^{-\lambda x}$:

- If we rescale x as $x = rx'$, then

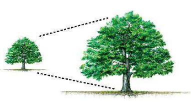
$$y = ce^{-\lambda r x'}$$

- Original form cannot be recovered.
- Scale matters for the exponential.

More on $y = ce^{-\lambda x}$:

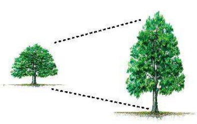
- Say $x_0 = 1/\lambda$ is the characteristic scale.
- For $x \gg x_0$, y is small, while for $x \ll x_0$, y is large.

Isometry:



- Dimensions scale linearly with each other.

Allometry:



- Dimensions scale nonlinearly.

Allometry:

- Refers to differential growth rates of the parts of a living organism's body part or process.
- First proposed by Huxley and Teissier, Nature, 1936 "Terminology of relative growth" [14, 31]

Definitions

Isometry versus Allometry:

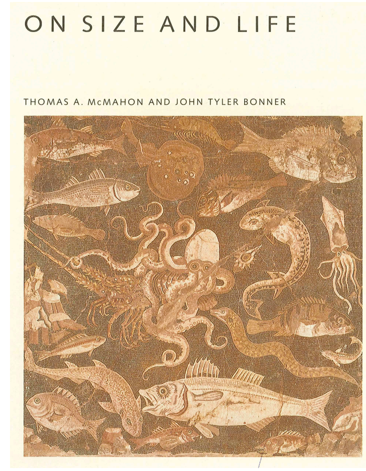
- Iso-metry = 'same measure'
- Allo-metry = 'other measure'

We use allometric scaling to refer to both:

1. Nonlinear scaling of a dependent variable on an independent one (e.g., $y \propto x^{1/3}$)
2. The relative scaling of correlated measures (e.g., white and gray matter).

An interesting, earlier treatise on scaling:

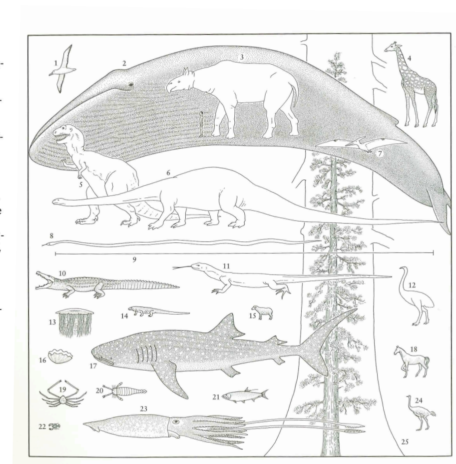
McMahon and Bonner, 1983 [24]



The many scales of life:

The biggest living things (*left*). All the organisms are drawn to the same scale. *1*, The largest flying bird (albatross); *2*, the largest known animal (the blue whale), *3*, the largest extinct land mammal (*Baluchitherium*) with a human figure shown for scale; *4*, the tallest living land animal (giraffe); *5*, *Tyrannosaurus*; *6*, *Diplodocus*; *7*, one of the largest flying reptiles (*Pteranodon*); *8*, the largest extinct snake; *9*, the length of the largest tapeworm found in man; *10*, the largest living reptile (West African crocodile); *11*, the largest extinct lizard; *12*, the largest extinct bird (*Aepyornis*); *13*, the largest jellyfish (*Cyanea*); *14*, the largest living lizard (Komodo dragon); *15*, sheep; *16*, the largest bivalve mollusc (*Tridacna*); *17*; the largest fish (whale shark); *18*, horse; *19*, the largest crustacean (Japanese spider crab); *20*, the largest sea scorpion (Eurypterid); *21*, large tarpon; *22*, the largest lobster; *23*, the largest mollusc (deep-water squid, *Architeuthis*); *24*, ostrich; *25*, the lower 105 feet of the largest organism (giant sequoia), with a 100-foot larch superposed.

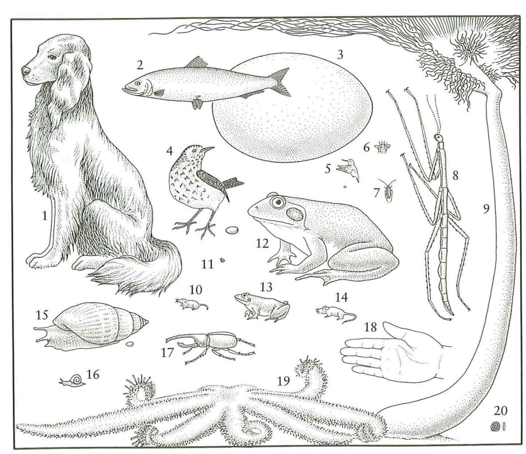
p. 2, McMahon and Bonner [24]



The many scales of life:

Medium-sized creatures (*above*). *1*, Dog; *2*, common herring; *3*, the largest egg (*Aepyornis*); *4*, song thrush with egg; *5*, the smallest bird (hummingbird) with egg; *6*, queen bee; *7*, common cockroach; *8*, the largest stick insect; *9*, the largest polyp (*Branchiocerianthus*); *10*, the smallest mammal (flying shrew); *11*, the smallest vertebrate (a tropical frog); *12*, the largest frog (goliath frog); *13*, common grass frog; *14*, house mouse; *15*, the largest land snail (*Achatina*) with egg; *16*, common snail; *17*, the largest beetle (goliath beetle); *18*, human hand; *19*, the largest starfish (*Luidia*); *20*, the largest free-moving protozoan (an extinct nummulite).

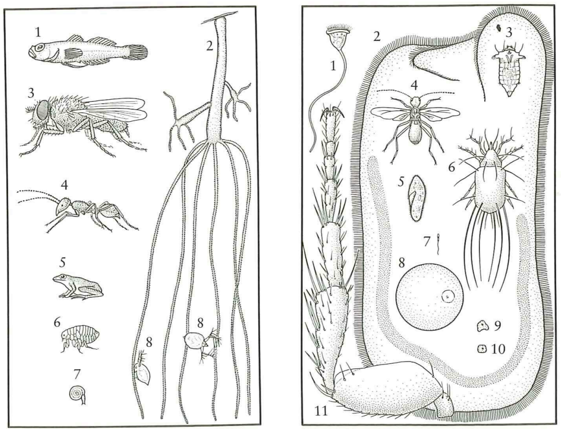
p. 3, McMahon and Bonner [24]
More on the Elephant Bird here.



The many scales of life:

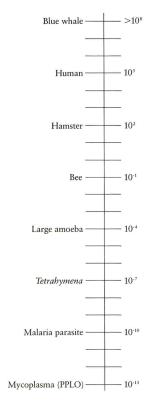
Small, "naked-eye" creatures (*lower left*). *1*, One of the smallest fishes (*Trimmatom nanus*); *2*, common brown hydra, expanded; *3*, housefly; *4*, medium-sized ant; *5*, the smallest vertebrate (a tropical frog, the same as the one numbered *11* in the figure *above*); *6*, flea (*Xenopsylla cheopis*); *7*, the smallest land snail; *8*, common water flea (*Daphnia*).

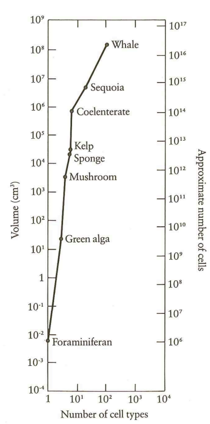
The smallest "naked-eye" creatures and some large microscopic animals and cells (*below right*). *1*, *Vorticella*, a ciliate; *2*, the largest ciliate protozoan (*Bursaria*); *3*, the smallest many-celled animal (a rotifer); *4*, smallest flying insect (*Elaphis*); *5*, another ciliate (*Paramecium*); *6*, cheese mite; *7*, human sperm; *8*, human ovum; *9*, dysentery amoeba; *10*, human liver cell; *11*, the foreleg of the flea (numbered *6* in the figure to the *left*).



3, McMahon and Bonner [24]

Size range (in grams) and cell differentiation:





10^{-13} to 10^{8} g, p. 3, McMahon and Bonner [24]

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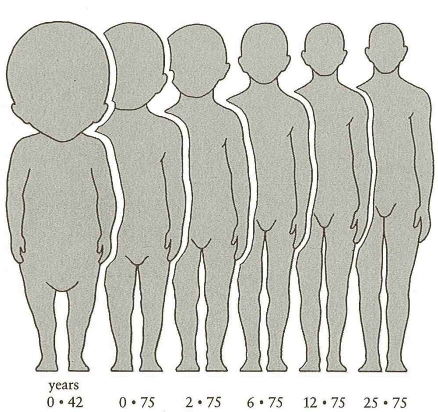
Non-uniform growth:



p. 32, McMahon and Bonner [24]

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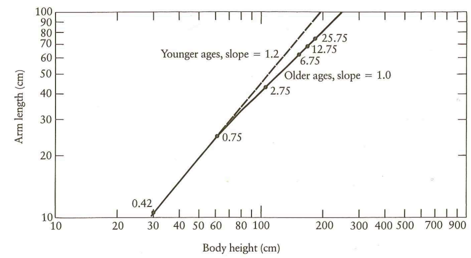
Non-uniform growth—arm length versus height:

Good example of a break in scaling:



A crossover in scaling occurs around a height of 1 metre.

p. 32, McMahon and Bonner [24]

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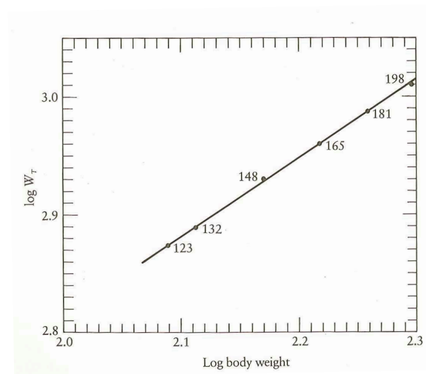
Weightlifting: $M_{\text{world record}} \propto M_{\text{lifter}}^{2/3}$



Idea: Power $\sim$ cross-sectional area of isometric lifters.

p. 53, McMahon and Bonner [24]

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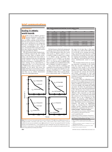
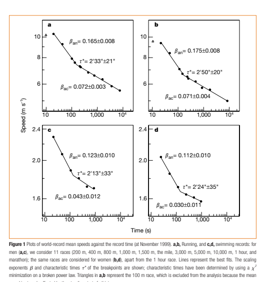


"Scaling in athletic world records"
Savaglio and Carbone,
Nature, **404**, 244, 2000. [30]



- Eek: Small scaling regimes
- Mean speed $\langle s \rangle$ decays with race time τ:

$$\langle s \rangle \sim \tau^{-\beta}$$

- Break in scaling at around $\tau \simeq 150$–170 seconds
- Anaerobic–aerobic transition
- Roughly 1 km running race
- Running decays faster than swimming

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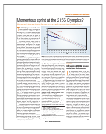


"Athletics: Momentous sprint at the 2156 Olympics?"
Tatem et al.,
Nature, **431**, 525–525, 2004. [32]

Linear extrapolation for the 100 metres:

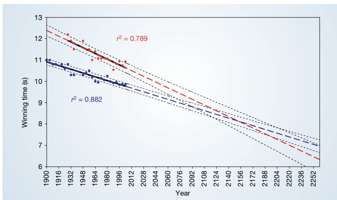
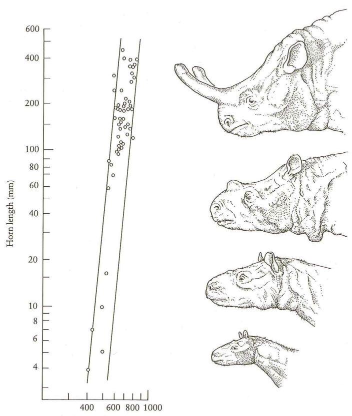


Figure 1 The winning Olympic 100-metre sprint times for men (blue points) and women (red points), with superimposed best-fit linear regression lines (solid black lines) and coefficients of determination. The regression lines are extrapolated (broken blue and red lines for men and women, respectively) and 95% confidence intervals (dotted black lines) based on the available points are superimposed. The projections intersect just before the 2156 Olympics, when the winning women's 100-metre sprint time of 8.079 s will be faster than the men's at 8.098 s.

Tatem: "If I'm wrong anyone is welcome to come and question me about the result after the 2156 Olympics."

Titanothere horns: $L_{\text{horn}} \sim L_{\text{skull}^4}$



p. 36, McMahon and Bonner [24]; a bit dubious.

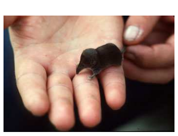
Animal power

Fundamental biological and ecological constraint:

$$\boxed{P = c\,M^{\alpha}}$$

$P =$ basal metabolic rate

$M =$ organismal body mass



$P = c\,M^{\alpha}$

Prefactor c depends on body plan and body temperature:

Birds	39–41 $^\circ C$
Eutherian Mammals	36–38 $^\circ C$
Marsupials	34–36 $^\circ C$
Monotremes	30–31 $^\circ C$


What one might expect:

$\alpha = 2/3$ because ...

- Dimensional analysis suggests an energy balance surface law:

$$P \propto S \propto V^{2/3} \propto M^{2/3}$$

- Assumes isometric scaling (not quite the spherical cow).
- Lognormal fluctuations:
  Gaussian fluctuations in $\log_{10} P$ around $\log_{10} cM^{\alpha}$.
- Stefan-Boltzmann law for radiated energy:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sigma\varepsilon S T^4 \propto S$$

The prevailing belief of the Church of Quarterology:

$$\boxed{\alpha = 3/4}$$

$$P \propto M^{3/4}$$

Huh?

The prevailing belief of the Church of Quarterology:

Most obvious concern:

$$3/4 - 2/3 = 1/12$$

- An exponent higher than 2/3 points suggests a fundamental inefficiency in biology.
- Organisms must somehow be running 'hotter' than they need to balance heat loss.

Related putative scalings:

Wait! There's more!:

- number of capillaries $\propto M^{3/4}$
- time to reproductive maturity $\propto M^{1/4}$
- heart rate $\propto M^{-1/4}$
- cross-sectional area of aorta $\propto M^{3/4}$
- population density $\propto M^{-3/4}$

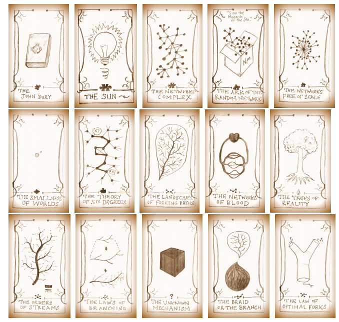
The great 'law' of heartbeats:

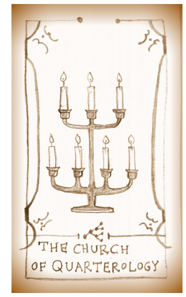
Assuming:

- Average lifespan $\propto M^{\beta}$
- Average heart rate $\propto M^{-\beta}$
- Irrelevant but perhaps $\beta = 1/4$.

Then:

- Average number of heart beats in a lifespan
  $\simeq$ (Average lifespan) $\times$ (Average heart rate)
  $\propto M^{\beta-\beta}$
  $\propto M^{0}$
- Number of heartbeats per life time is independent of organism size!
- $\approx$ 1.5 billion....





Stories—The Fraction Assassin:

Ecology—Species-area law:

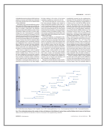
Allegedly (data is messy): [19, 17]

"An equilibrium theory of insular zoogeography"
MacArthur and Wilson,
Evolution, **17**, 373–387, 1963. [19]

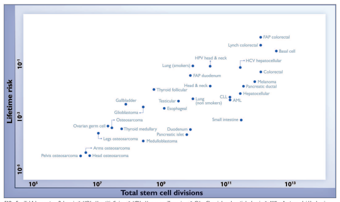
$$N_{\text{species}} \propto A^{\beta}$$

- According to physicists—on islands: $\beta \approx 1/4$.
- Also—on continuous land: $\beta \approx 1/8$.

Cancer:



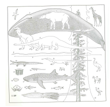
"Variation in cancer risk among tissues can be explained by the number of stem cell divisions"
Tomasetti and Vogelstein,
Science, **347**, 78–81, 2015. [33]



FAP = Familial Adenomatous Polyposis ◆ HCV = Hepatitis C virus ◆ HPV = Human papillomavirus ◆ CLL = Chronic lymphocytic leukemia ◆ AML = Acute myeloid leukemia

Fig. 1. The relationship between the number of stem cell divisions in the lifetime of a given tissue and the lifetime risk of cancer in that tissue. Values are from table S1, the derivation of which is discussed in the supplementary materials.

Roughly: $p \sim r^{2/3}$ where p = life time probability and r = rate of stem cell replication.



"A general scaling law reveals why the largest animals are not the fastest"
Hirt et al.,
Nature Ecology & Evolution, **1**, 1116, 2017. [11]

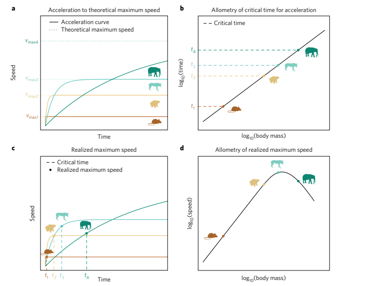
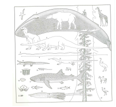


Figure 1 | Concept of time-dependent and mass-dependent realized maximum speed of animals. a, Acceleration of animals follows a saturation curve (solid lines) approaching the theoretical maximum speed (dotted lines) depending on body mass (colour code). **b**, The time available for acceleration increases with body mass following a power law. **c**,**d**, This critical time determines the realized maximum speed (**c**), yielding a hump-shaped increase of maximum speed with body mass (**d**).



"How fast do living organisms move: Maximum speeds from bacteria to elephants and whales"
Meyer-Vernet and Rospars,
American Journal of Physics, **83**, 719–722, 2015. [25]

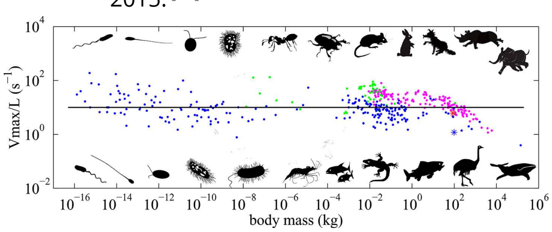
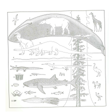


Fig. 1. Maximum relative speed versus body mass for 202 running species (157 mammals plotted in magenta and 45 non-mammals plotted in green), 127 swimming species and 91 micro-organisms (plotted in blue). The sources of the data are given in Ref. 16. The solid line is the maximum relative speed [Eq. (13)] estimated in Sec. III. The human world records are plotted as asterisks (upper for running and lower for swimming). Some examples of organisms of various masses are sketched in black (drawings by François Meyer).

Insert question from assignment 1



Theoretical story:

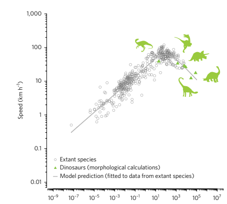
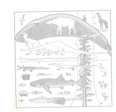


Figure 4 | Predicting the maximum speed of extinct species with the time-dependent model. The model prediction (grey line) is fitted to data of extant species (grey circles) and extended to higher body masses. Speed data for dinosaurs (green triangles) come from detailed morphological model calculations (values in Table 1) and were not used to obtain model parameters.

- Maximum speed increases with size: $v_{\text{max}} = aM^b$
- Takes a while to get going: $v(t) = v_{\text{max}}(1 - e^{-kt})$
- $k \sim F_{\text{max}}/M \sim cM^{d-1}$ Literature: $0.75 \lesssim d \lesssim 0.94$
- Acceleration time = depletion time for anaerobic energy: $\tau \sim fM^g$ Literature: $0.76 \lesssim g \lesssim 1.27$
- $v_{\text{max}} = aM^b\left(1 - e^{-hM^i}\right)$
- $i = d - 1 + g$ and $h = cf$

- Literature search for for maximum speeds of running, flying and swimming animals.
- Search terms: "maximum speed", "escape speed" and "sprint speed".

Note: [25] not cited.



"A general scaling law reveals why the largest animals are not the fastest"
Hirt et al.,
Nature Ecology & Evolution, **1**, 1116, 2017. [11]

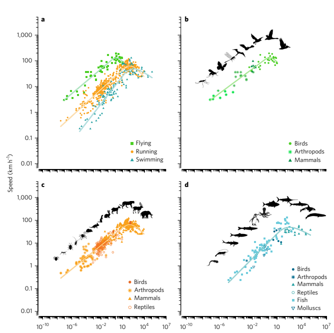
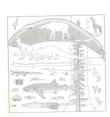
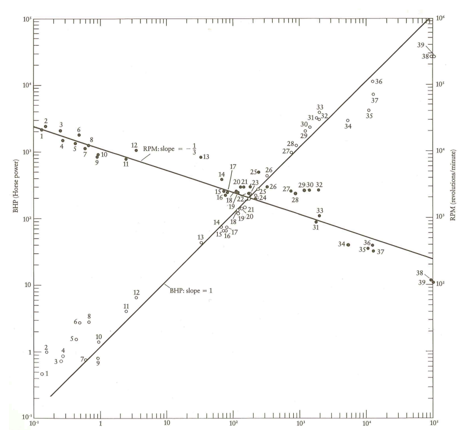


Figure 2 | Empirical data and time-dependent model fit for the allometric scaling of maximum speed. a, Comparison of scaling for the different locomotion modes (flying, running, swimming). **b–d**, Taxonomic differences are illustrated separately for flying (**b**; $n = 55$), running (**c**; $n = 458$) and swimming (**d**; $n = 109$) animals. Overall model fit: $R^2 = 0.893$. The residual variation does not exhibit a signature of taxonomy (only a weak effect of thermoregulation; see Methods).



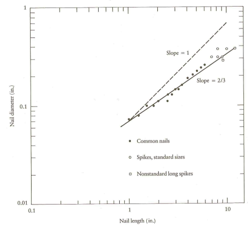
Engines:



BHP = brake horse power

The allometry of nails:

Observed: Diameter $\propto$ Length$^{2/3}$ or $d \propto \ell^{2/3}$.



Since $\ell d^2 \propto$ Volume v:

- Diameter $\propto$ Mass$^{2/7}$ or $d \propto v^{2/7}$.
- Length $\propto$ Mass$^{3/7}$ or $\ell \propto v^{3/7}$.
- Nails lengthen faster than they broaden (c.f. trees).

p. 58–59, McMahon and Bonner [24]

The allometry of nails:

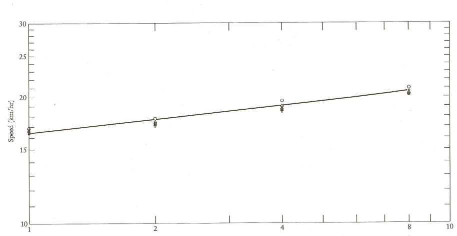
A buckling instability?:

- Physics/Engineering result: Columns buckle under a load which depends on d^4/ℓ^2.
- To drive nails in, posit resistive force $\propto$ nail circumference = πd.
- Match forces independent of nail size: $d^4/\ell^2 \propto d$.
- Leads to $d \propto \ell^{2/3}$.
- Argument made by Galileo [10] in 1638 in "Discourses on Two New Sciences." Also, see here.
- Another smart person's contribution: Euler, 1757
- Also see McMahon, "Size and Shape in Biology," Science, 1973. [23]

Rowing: Speed $\propto$ (number of rowers)$^{1/9}$

Shell dimensions and performances.

No. of oarsmen	Modifying description	Length, l (m)	Beam, b (m)	l/b	Boat mass per oarsman (kg)	Time for 2000 m (min) I	II	III	IV
8	Heavyweight	18.28	0.610	30.0	14.7	5.87	5.92	5.82	5.73
8	Lightweight	18.28	0.598	30.6	14.7				
4	With coxswain	12.80	0.574	22.3	18.1				
4	Without coxswain	11.75	0.574	21.0	18.1	6.33	6.42	6.48	6.13
2	Double scull	9.76	0.381	25.6	13.6				
2	Pair-oared shell	9.76	0.356	27.4	13.6	6.87	6.92	6.95	6.77
1	Single scull	7.93	0.293	27.0	16.3	7.16	7.25	7.28	7.17



- Very weak scaling and size variation but it's theoretically explainable ...

Physics:

Scaling in elementary laws of physics:

- Inverse-square law of gravity and Coulomb's law:

$$F \propto \frac{m_1 m_2}{r^2} \quad \text{and} \quad F \propto \frac{q_1 q_2}{r^2}.$$

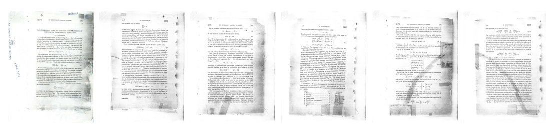
- Force is diminished by expansion of space away from source.
- The square is $d - 1 = 3 - 1 = 2$, the dimension of a sphere's surface.
- We'll see a gravity law applies for a range of human phenomena.

Dimensional Analysis:

The Buckingham π theorem:[1]

"On Physically Similar Systems: Illustrations of the Use of Dimensional Equations"
E. Buckingham,
Phys. Rev., **4**, 345–376, 1914. [7]

As captured in the 1990s in the MIT physics library:



[1] Stigler's Law of Eponymy applies. See here. More later.

Dimensional Analysis:[2]

Fundamental equations cannot depend on units:

- System involves n related quantities with some unknown equation $f(q_1, q_2, \ldots, q_n) = 0$.
- Geometric ex.: area of a square, side length ℓ: $A = \ell^2$ where $[A] = L^2$ and $[\ell] = L$.
- Rewrite as a relation of $p \leq n$ independent dimensionless parameters where p is the number of independent dimensions (mass, length, time, luminous intensity ...):

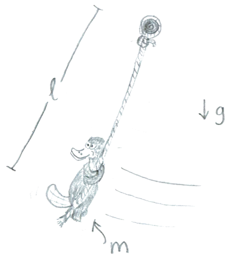
$$F(\pi_1, \pi_2, \ldots, \pi_p) = 0$$

- e.g., $A/\ell^2 - 1 = 0$ where $\pi_1 = A/\ell^2$.
- Another example: $F = ma \Rightarrow F/ma - 1 = 0$.
- Plan: solve problems using only backs of envelopes.

[2] Length is a dimension, furlongs and smoots are units

Example:

Simple pendulum:

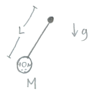


- Idealized mass/platypus swinging forever.
- Four quantities:
  1. Length ℓ,
  2. mass m,
  3. gravitational acceleration g, and
  4. pendulum's period τ.

- Variable dimensions: $[\ell] = L$, $[m] = M$, $[g] = LT^{-2}$, and $[\tau] = T$.
- Turn over your envelopes and find some π's.

A little formalism:

- Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out p (which must be $\leq n$).
- Consider $\pi_i = q_1^{x_1} q_2^{x_2} \cdots q_n^{x_n}$.
- We (desperately) want to find all sets of powers x_j that create dimensionless quantities.
- Dimensions: want $[\pi_i] = [q_1]^{x_1}[q_2]^{x_2}\cdots[q_n]^{x_n} = 1$.
- For the platypus pendulum we have $[q_1] = L$, $[q_2] = M$, $[q_3] = LT^{-2}$, and $[q_4] = T$, with dimensions $d_1 = L$, $d_2 = M$, and $d_3 = T$.
- So: $[\pi_i] = L^{x_1} M^{x_2} (LT^{-2})^{x_3} T^{x_4}$.
- We regroup: $[\pi_i] = L^{x_1+x_3} M^{x_2} T^{-2x_3+x_4}$.
- We now need: $x_1 + x_3 = 0$, $x_2 = 0$, and $-2x_3 + x_4$.
- Time for matrixology ...



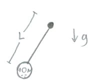
Well, of course there are matrices:

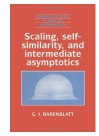
- Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.
- Number of dimensionless parameters = Dimension of null space = $n - r$ where n is the number of columns of **A** and r is the rank of **A**.
- Here: $n = 4$ and $r = 3 \rightarrow F(\pi_1) = 0 \rightarrow \pi_1$ = const.
- In general: Create a matrix **A** where ijth entry is the power of dimension i in the jth variable, and solve by row reduction to find basis null vectors.
- We (you) find: $\pi_1 = \ell / g\tau^2$ = const. Upshot: $\tau \propto \sqrt{\ell}$.

Insert question from assignment 1



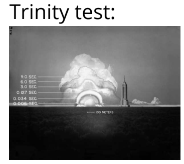


"Scaling, self-similarity, and intermediate asymptotics" by G. I. Barenblatt (1996). [2]

G. I. Taylor, magazines, and classified secrets:

Self-similar blast wave:

1945
New Mexico
Trinity test:

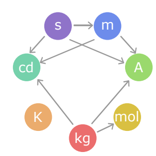


- Radius: $[R] = L$,
  Time: $[t] = T$,
  Density of air: $[\rho] = M/L^3$,
  Energy: $[E] = ML^2/T^2$.
- Four variables, three dimensions.
- One dimensionless variable: E = constant $\times \rho R^5/t^2$.
- Scaling: Speed decays as $1/R^{3/2}$.

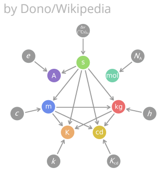
Related: Radiolab's Elements on the Cold War, the Bomb Pulse, and the dating of cell age (33:30).

We're still sorting out units:

Proposed 2018 revision of SI base units:



by Dono/Wikipedia

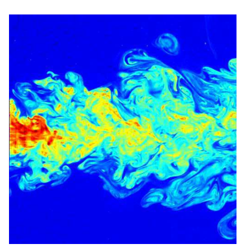


by Wikipetzi/Wikipedia

- Now: kilogram is an artifact in Sèvres, France.
- Future: Defined by fixing Planck's constant as $6.62606X \times 10^{-34}$ s^{-1}·m^2·kg.[3]
- Metre chosen to fix speed of light at 299792458 m·s^{-1}.
- Radiolab piece: $\leq$ kg

[3] X = still arguing ...

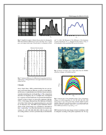
Turbulence:



Big whirls have little whirls
That heed on their velocity,
And little whirls have littler whirls
And so on to viscosity.

— Lewis Fry Richardson

- Image from here.
- Jonathan Swift (1733): "Big fleas have little fleas upon their backs to bite 'em, And little fleas have lesser fleas, and so, ad infinitum." The Siphonaptera.



"Turbulent luminance in impassioned van Gogh paintings"
Aragón et al.,
J. Math. Imaging Vis., **30**, 275–283, 2008. [1]

- Examined the probability pixels a distance R apart share the same luminance.
- "Van Gogh painted perfect turbulence" by Phillip Ball, July 2006.
- Apparently not observed in other famous painter's works or when van Gogh was stable.
- Oops: Small ranges and natural log used.

Advances in turbulence:

In 1941, Kolmogorov, armed only with dimensional analysis and an envelope figures this out: [?]

$$E(k) = C\epsilon^{2/3}k^{-5/3}$$

- $E(k)$ = energy spectrum function.
- ϵ = rate of energy dissipation.
- $k = 2\pi/\lambda$ = wavenumber.

- Energy is distributed across all modes, decaying with wave number.
- No internal characteristic scale to turbulence.
- Stands up well experimentally and there has been no other advance of similar magnitude.

"The Geometry of Nature": Fractals

- "Anomalous" scaling of lengths, areas, volumes relative to each other.
- The enduring question: how do self-similar geometries form?

- Robert E. Horton: Self-similarity of river (branching) networks (1945). [12]
- Harold Hurst—Roughness of time series (1951). [13]
- Lewis Fry Richardson—Coastlines (1961).
- Benoît B. Mandelbrot—Introduced the term "Fractals" and explored them everywhere, 1960s on. [20, 21, 22]

[d] Note to self: Make millions with the "Fractal Diet"

Scaling in Cities:

"Growth, innovation, scaling, and the pace of life in cities"
Bettencourt et al.,
Proc. Natl. Acad. Sci., **104**, 7301–7306, 2007. [4]

- Quantified levels of
  - Infrastructure
  - Wealth
  - Crime levels
  - Disease
  - Energy consumption

  as a function of city size N (population).

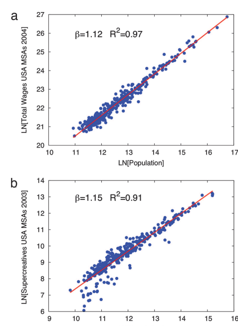


Fig. 1. Examples of scaling relationships. (*a*) Total wages per MSA in 2004 for the U.S. (blue points) vs. metropolitan population. (*b*) Supercreative employment per MSA in 2003, for the U.S. (blue points) vs. metropolitan population. Best-fit scaling relations are shown as solid lines.

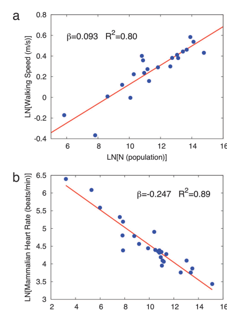


Fig. 2. The pace of urban life increases with city size in contrast to the pace of biological life, which decreases with organism size. (*a*) Scaling of walking speed vs. population for cities around the world. (*b*) Heart rate vs. the size (mass) of organisms.

Scaling in Cities:

Table 1. Scaling exponents for urban indicators vs. city size

Y	β	95% CI	Adj-R^2	Observations	Country–year
New patents	1.27	[1.25,1.29]	0.72	331	U.S. 2001
Inventors	1.25	[1.22,1.27]	0.76	331	U.S. 2001
Private R&D employment	1.34	[1.29,1.39]	0.92	266	U.S. 2002
"Supercreative" employment	1.15	[1.11,1.18]	0.89	287	U.S. 2003
R&D establishments	1.19	[1.14,1.22]	0.77	287	U.S. 1997
R&D employment	1.26	[1.18,1.43]	0.93	295	China 2002
Total wages	1.12	[1.09,1.13]	0.96	361	U.S. 2002
Total bank deposits	1.08	[1.03,1.11]	0.91	267	U.S. 1996
GDP	1.15	[1.06,1.23]	0.96	295	China 2002
GDP	1.26	[1.09,1.46]	0.64	196	EU 1999–2003
GDP	1.13	[1.03,1.23]	0.94	37	Germany 2003
Total electrical consumption	1.07	[1.03,1.11]	0.88	392	Germany 2002
New AIDS cases	1.23	[1.18,1.29]	0.76	93	U.S. 2002–2003
Serious crimes	1.16	[1.11, 1.18]	0.89	287	U.S. 2003
Total housing	1.00	[0.99,1.01]	0.99	316	U.S. 1990
Total employment	1.01	[0.99,1.02]	0.98	331	U.S. 2001
Household electrical consumption	1.00	[0.94,1.06]	0.88	377	Germany 2002
Household electrical consumption	1.05	[0.89,1.22]	0.91	295	China 2002
Household water consumption	1.01	[0.89,1.11]	0.96	295	China 2002
Gasoline stations	0.77	[0.74,0.81]	0.93	318	U.S. 2001
Gasoline sales	0.79	[0.73,0.80]	0.94	318	U.S. 2001
Length of electrical cables	0.87	[0.82,0.92]	0.75	380	Germany 2002
Road surface	0.83	[0.74,0.92]	0.87	29	Germany 2002

Data sources are shown in *SI Text*. CI, confidence interval; Adj-R^2, adjusted R^2; GDP, gross domestic product.

Scaling in Cities:

Intriguing findings:

- Global supply costs scale sublinearly with N ($\beta < 1$).
  - Returns to scale for infrastructure.
- Total individual costs scale linearly with N ($\beta = 1$)
  - Individuals consume similar amounts independent of city size.
- Social quantities scale superlinearly with N ($\beta > 1$)
  - Creativity (# patents), wealth, disease, crime, ...

Density doesn't seem to matter...

- Surprising given that across the world, we observe two orders of magnitude variation in area covered by agglomerations of fixed populations.

"Urban scaling and its deviations: Revealing the structure of wealth, innovation and crime across cities"
Bettencourt et al.,
PLoS ONE, **5**, e13541, 2010. [5]

Comparing city features across populations:

- Cities = Metropolitan Statistical Areas (MSAs)
- Story: Fit scaling law and examine residuals
- Does a city have more or less crime than expected when normalized for population?

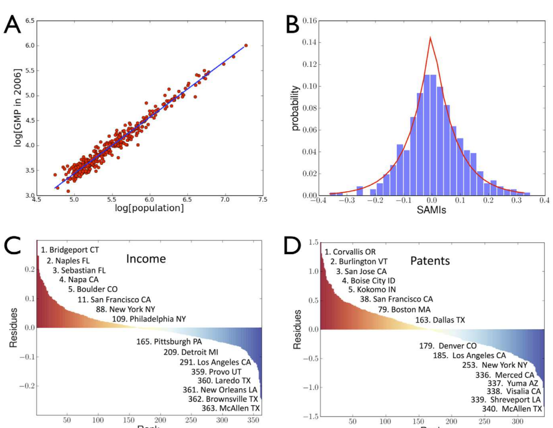
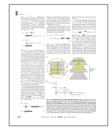


Figure 1. Urban Agglomeration effects result in per capita nonlinear scaling of urban metrics. Subtracting these effects produces a truly local measure of urban dynamics and a reference scale for ranking cities. a) A typical superlinear scaling law (solid line): Gross Metropolitan Product of US MSAs in 2006 (red dots) vs. population; the slope of the solid line has exponent, $\beta = 1.126$ (95% CI [1.101,1.149]). b) Histogram showing frequency of residuals, (SAMIs, see Eq. (2)); the statistics of residuals is well described by a Laplace distribution (red line). Scale independent ranking (SAMIs) for US MSAs by c) personal income and d) patenting (red denotes above average performance, blue below). For more details see Text S1, Table S1 and Figure S1.
doi:10.1371/journal.pone.0013541.g001

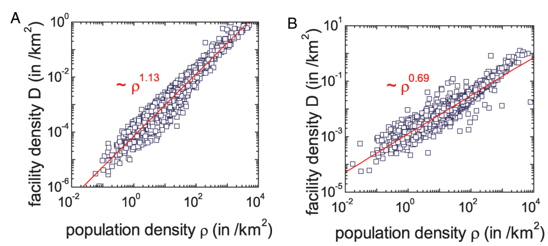
A possible theoretical explanation?



"The origins of scaling in cities"
Luís M. A. Bettencourt,
Science, **340**, 1438–1441, 2013. [3]

#sixthology

Density of public and private facilities:



$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{\alpha}$$

- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.

"Pattern in escalations in insurgent and terrorist activity"
Johnson et al.,
Science Magazine, **333**, 81–84, 2011. [15]

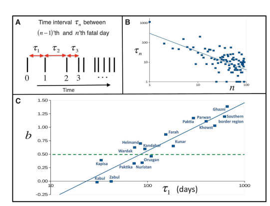
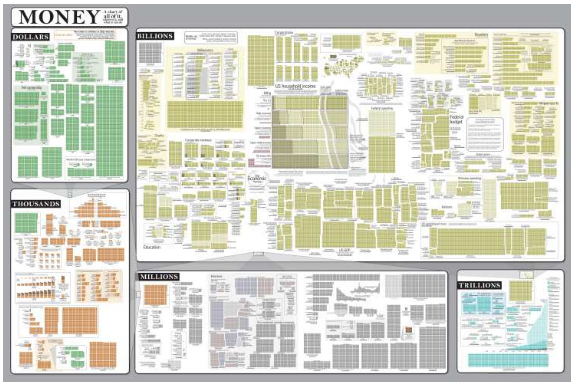


Fig. 1. (**A**) Schematic timeline of successive fatal days shown as vertical bars. τ_1 is the time interval between the first two fatal days, labeled 0 and 1. (**B**) Successive time intervals τ_n between days with IED fatalities in the Afghanistan province of Kandahar (squares). On this log-log plot, the best-fit power-law progress curve is by definition a straight (blue) line with slope $-b$ (b is an escalation rate). (**C**) The solid blue line shows best linear fit through progress-curve parameter values τ_1 and b for individual Afghanistan provinces (blue squares) for all hostile fatalities (all coalition military fatalities attributed to insurgent activity). The green dashed line shows value $b = 0.5$, which is the situation in which there are no correlations. The subset of fatalities recorded in icasualties as "southern Afghanistan" is shown as a separate region because of their likely connection to operations near the Pakistan border.

- Escalation: $\tau_n \sim \tau_1 n^{-b}$
- b = scaling exponent (escalation rate)
- Interevent time τ_n between fatal attacks $n-1$ and n (binned by days)
- Learning curves organizations [34]
- More later on size distributions [9, 16, 6]



Explore the original zoomable and interactive version here: http://xkcd.com/980/.

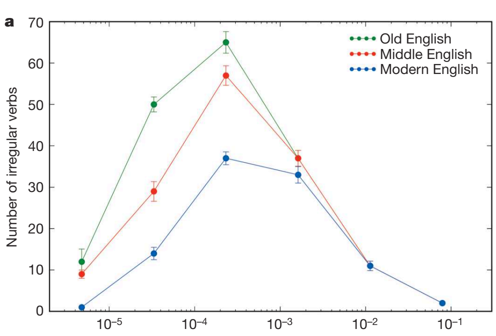
Irregular verbs

Cleaning up the code that is English:

"Quantifying the evolutionary dynamics of language"
Lieberman et al.,
Nature, **449**, 713–716, 2007. [18]

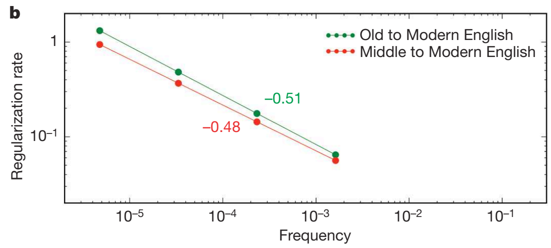
- Exploration of how verbs with irregular conjugation gradually become regular over time.
- Comparison of verb behavior in Old, Middle, and Modern English.

Irregular verbs



- Universal tendency towards regular conjugation
- Rare verbs tend to be regular in the first place

Irregular verbs



- Rates are relative.
- The more common a verb is, the more resilient it is to change.

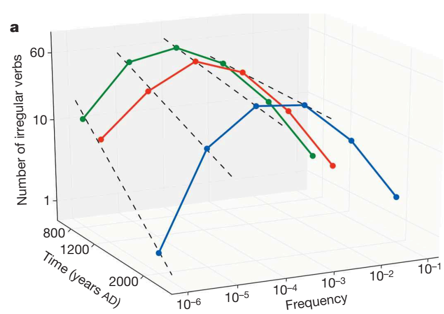
Irregular verbs

Table 1 | The 177 irregular verbs studied

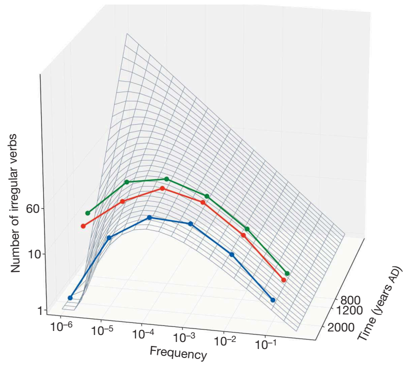
Frequency	Verbs	Regularization (%)	Half-life (yr)
10^{-1}–1	be, have	0	38,800
10^{-2}–10^{-1}	come, do, find, get, give, go, know, say, see, take, think	0	14,400
10^{-3}–10^{-2}	begin, break, bring, buy, choose, draw, drink, drive, eat, fall, fight, forget, grow, hang, help, hold, leave, let, lie, lose, reach, rise, run, seek, set, shake, sit, sleep, speak, stand, teach, throw, understand, walk, win, work, write	10	5,400
10^{-4}–10^{-3}	arise, bake, bear, beat, bind, bite, blow, bow, burn, burst, carve, chew, climb, cling, creep, dare, dig, drag, flee, float, flow, fly, fold, freeze, grind, leap, lend, lock, melt, reckon, ride, rush, shape, shine, shoot, shrink, sigh, sing, sink, slide, slip, smoke, spin, spring, starve, steal, step, stretch, strike, stroke, suck, swallow, swear, sweep, swim, swing, tear, wake, wash, weave, weep, weigh, wind, yell, yield	43	2,000
10^{-5}–10^{-4}	bark, bellow, bid, blend, braid, brew, cleave, cringe, crow, dive, drip, fare, fret, glide, gnaw, grip, heave, knead, low, milk, mourn, mow, prescribe, redden, reek, row, scrape, seethe, shear, shed, shove, slay, slit, smite, sow, span, spurn, sting, stink, strew, stride, swell, tread, uproot, wade, warp, wax, wield, wring, writhe	72	700
10^{-6}–10^{-5}	bide, chide, delve, flay, hew, rue, shrive, slink, snip, spew, sup, wreak	91	300

177 Old English irregular verbs were compiled for this study. These are arranged according to frequency bin, and in alphabetical order within each bin. Also shown is the percentage of verbs in each bin that have regularized. The half-life is shown in years. Verbs that have regularized are indicated in red. As we move down the list, an increasingly large fraction of the verbs are red; the frequency-dependent regularization of irregular verbs becomes immediately apparent.

- Red = regularized
- Estimates of half-life for regularization ($\propto f^{1/2}$)

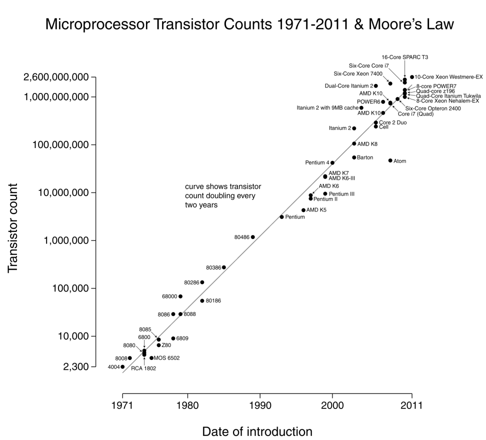


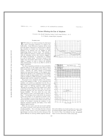
- 'Wed' is next to go.
- -ed is the winning rule...
- But 'snuck' is sneaking up on sneaked. [26]



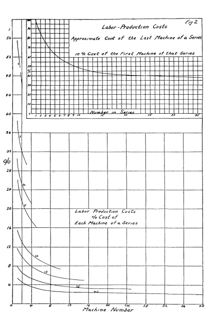
- Projecting back in time to proto-Zipf story of many tools.

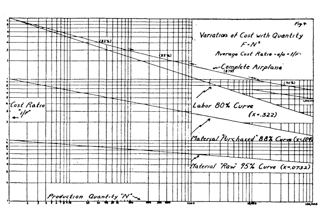
Moore's Law:





"Factors affecting the costs of airplanes"
T. P. Wright,
Journal of Aeronautical Sciences, **10**, 302–328, 1936. [34]





- Power law decay of cost with number of planes produced.
- "The present writer started his studies of the variation of cost with quantity in 1922."

Scaling laws for technology production:

- "Statistical Basis for Predicting Technological Progress [28]" Nagy et al., PLoS ONE, 2013.
- y_t = stuff unit cost; x_t = total amount of stuff made.
- Wright's Law, cost decreases as a power of total stuff made: [34]

$$y_t \propto x_t^{-w}.$$

- Moore's Law, framed as cost decrease connected with doubling of transistor density every two years: [27]

$$y_t \propto e^{-mt}.$$

- Sahal's observation that Moore's law gives rise to Wright's law if stuff production grows exponentially: [29]

$$x_t \propto e^{gt}.$$

- Sahal + Moore gives Wright with $w = m/g$.

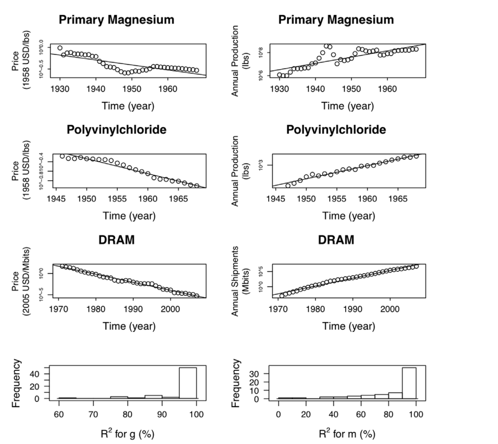


Figure 3. Three examples showing the logarithm of price as a function of time in the left column and the logarithm of production as a function of time in the right column, based on industry-wide data. We have chosen these examples to be representative: The top row contains an example with one of the worst fits, the second row an example with an intermediate goodness of fit, and the third row one of the best examples. The fourth row of the figure shows histograms of R^2 values for fitting g and m for the 62 datasets.
doi:10.1371/journal.pone.0052669.g003

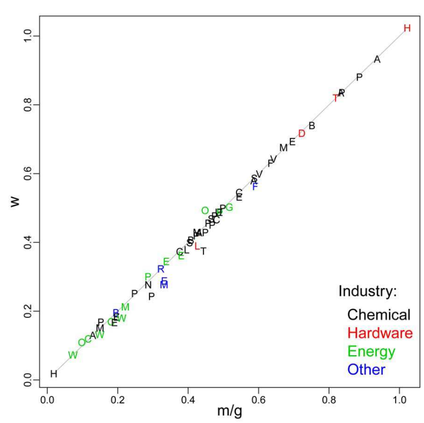


Figure 4. An illustration that the combination of exponentially increasing production and exponentially decreasing cost are equivalent to Wright's law. The value of the Wright parameter w is plotted against the prediction m/g based on the Sahal formula, where m is the exponent of cost reduction and g the exponent of the increase in cumulative production.
doi:10.1371/journal.pone.0052669.g004

Scaling of Specialization:

"Scaling of Differentiation in Networks: Nervous Systems, Organisms, Ant Colonies, Ecosystems, Businesses, Universities, Cities, Electronic Circuits, and Legos"
M. A. Changizi, M. A. McDannald and D. Widders [8]
J. Theor. Biol., 2002.

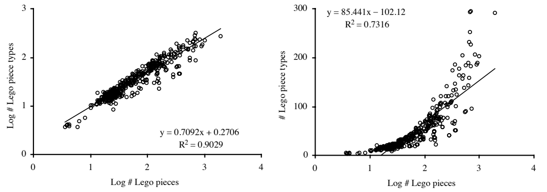


FIG. 3. Log–log (base 10) (left) and semi-log (right) plots of the number of Lego piece types vs. the total number of parts in Lego structures ($n = 391$). To help to distinguish the data points, logarithmic values were perturbed by adding a random number in the interval [−0.05, 0.05], and non-logarithmic values were perturbed by adding a random number in the interval [−1, 1].

- Nice 2012 wired.com write-up

$C \sim N^{1/d}$, $d \geq 1$:

- C = network differentiation = # node types.
- N = network size = # nodes.
- d = combinatorial degree.
- Low d: strongly specialized parts.
- High d: strongly combinatorial in nature, parts are reused.
- Claim: Natural selection produces high d systems.
- Claim: Engineering/brains produces low d systems.

TABLE 1
*Summary of results**

Network	Node	No. data points	Range of log N	Log–log R^2	Semi-log R^2	p_{power}/p_{log}	Relationship between C and N	Comb. degree	Exponent v for type-net scaling	Figure in text
Selected networks										
Electronic circuits	Component	373	2.12	0.747	0.602	0.05/4e−5	Power law	2.29	0.92	2
Legos™	Piece	391	2.65	0.903	0.732	0.09/1e−7	Power law	1.41	—	3
Businesses										
military vessels	Employee	13	1.88	0.971	0.832	0.05/3e−3	Power law	1.60	—	4
military offices	Employee	8	1.59	0.964	0.789	0.16/0.16	Increasing	1.13	—	4
universities	Employee	9	1.55	0.786	0.749	0.27/0.27	Increasing	1.37	—	4
insurance co.	Employee	52	2.30	0.748	0.685	0.11/0.10	Increasing	3.04	—	4
Universities										
across schools	Faculty	112	2.72	0.695	0.549	0.09/0.01	Power law	1.81	—	5
history of Duke	Faculty	46	0.94	0.921	0.892	0.09/0.05	Increasing	2.07	—	5
Ant colonies										
caste = type	Ant	46	6.00	0.481	0.454	0.11/0.04	Power law	8.16	—	6
size range = type	Ant	22	5.24	0.658	0.548	0.17/0.04	Power law	8.00	—	6
Organisms	Cell	134	12.40	0.249	0.165	0.08/0.02	Power law	17.73	—	7
Neocortex	Neuron	10	0.85	0.520	0.584	0.16/0.16	Increasing	4.56	—	9
Competitive networks										
Biotas	Organism	—	—	—	—	—	Power law	≈3	0.3 to 1.0	—
Cities	Business	82	2.44	0.985	0.832	0.08/8e-8	Power law	1.56	—	10

*(1) The kind of network, (2) what the nodes are within that kind of network, (3) the number of data points, (4) the logarithmic range of network sizes N (i.e. $\log(N_{max}/N_{min})$), (5) the log–log correlation, (6) the semi-log correlation, (7) the serial-dependence probabilities under, respectively, power-law and logarithmic models, (8) the empirically determined best-fit relationship between differentiation C and organization size N (if one of the two models can be refuted with $p < 0.05$; otherwise we just write "increasing" to denote that neither model can be rejected), (9) the combinatorial degree (i.e. the inverse of the best-fit slope of a log–log plot of C versus N), (10) the scaling exponent for how quickly the edge-degree δ scales with type-network size C (in those places for which data exist), (11) figure in this text where the plots are presented. Values for biotas represent the broad trend from the literature.

Shell of the nut:

- Scaling is a fundamental feature of complex systems.
- Basic distinction between isometric and allometric scaling.
- Powerful envelope-based approach: Dimensional analysis.
- "Oh yeah, well that's just dimensional analysis" said the [insert your own adjective] physicist.
- Tricksiness: A wide variety of mechanisms give rise to scalings, both normal and unusual.

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