Principles of Complex Systems | @pocsvox CSYS/MATH 300, Fall, 2017

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Scale-free networks

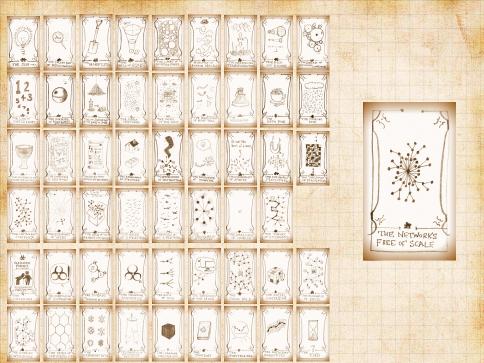
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Networks with power-law degree distributions have become known as scale-free networks.

Scale-free refers specifically to the degree distribution having a power-law decay in its tai

One of the seminal works in complex networks



Barabási and Albert, Science, **286**, 509–511, 1999. (as of October 8, 2015) what misleading nomenclature...

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Scale-free networks

Main story Model details

Scale-free networks are not fractal in any sense.

Usually talking about networks whose links are abstract, relational, informational, ...(non-physical Primary example: hyperlink network of the Web Much arguing about whether or networks are 'scale-free' or not... Model details Analysis A more plausible mechanism Robustness Krapivsky & Redhers model Generalized model Analysis Universality? Sublinear attachment kernels Superlinear attachment kernels

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### Some real data (we are feeling brave):

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Scale-free networks

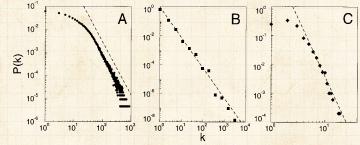
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### From Barabási and Albert's original paper<sup>[2]</sup>:



**Fig. 1.** The distribution function of connectivities for various large networks. **(A)** Actor collaboration graph with N = 212,250 vertices and average connectivity  $\langle k \rangle = 28.78$ . **(B)** WWW, N = 325,729,  $\langle k \rangle = 5.46$  (6). **(C)** Power grid data, N = 4941,  $\langle k \rangle = 2.67$ . The dashed lines have slopes (A)  $\gamma_{actor} = 2.3$ , (B)  $\gamma_{www} = 2.1$  and (C)  $\gamma_{power} = 4$ .

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### Random networks: largest components





 $\gamma = 2.5$ 

 $\langle k \rangle = 2.05333$ 



 $\gamma = 2.5$ 

(k) = 1.66667

ANA

 $\gamma$  = 2.5  $\langle k \rangle$  = 1.92



 $\gamma = 2.5$  $\langle k \rangle = 1.6$ 

 $\gamma = 2.5$ 

 $\langle k \rangle = 1.8$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.50667$   $\gamma = 2.5$  $\langle k \rangle = 1.62667$ 

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 $\gamma = 2.5$  $\langle k \rangle = 1.8$ 

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### The big deal:

We move beyond describing networks to finding mechanisms for why certain networks are the way they are. PoCS | @pocsvox

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### The big deal:

We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

### A big deal for scale-free networks:

How does the exponent γ depend on the mechanism?

Do the mechanism details matter?

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### Barabási-Albert model = BA model.

/ Ingredients: with and Preferential Attachment (PA). b 1: start with modisconnected nodes.

In essence, we have a rich-gets-richer scheme. Yes, we've seen this all before in Simon's mode Model details Analysis A more plausible mechanism Robustness Krapivsky & Redners model Generalized model Analysis Universality? Sublinear attachment kernels Superlingar attachment kernels

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Each new node makes *m* links to nodes alread present.

Preferential attachment—Probability of connecting to *i*th node is  $\infty k_i$ .

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# Befinition: $A_k$ is the attachment kernel for a node with degree k.

Definition  $P_{\text{attach}}(k,t)$  is the attachment probability.

For the original model

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Befinition:  $A_k$  is the attachment kernel for a node with degree k.

For the original model:

$$A_k = k$$

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For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

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where  $N(t) = m_0 + t$  is # nodes at time t

Scale-free networks

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## **BA** model

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where  $N(t) = m_0 + t$  is # nodes at time t and  $N_k(t)$  is # degree k nodes at time t.

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When (N + 1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1}-k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

Assumes probability of being connected to is small. Dispense with Expectation by assuming (hopir that over longer time frames, degree growth v be smooth and stable.

Approximate  $k_{i,N+1} - k_{i,N}$  with  $\frac{d}{dt}k_{i,t}$ 

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Approximate  $k_{i,N+1} - k_{i,N}$  with  $\frac{d}{dt}k_{i,t}$ :

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where  $t = N(t) - m_0$ .

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# Deal with denominator: each added node brings *m* new edges.

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$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

The node degree equation now simplifies

Rearrange and solve

 $rac{{\mathrm{d}} k_i(t)}{k_i(t)}=rac{{\mathrm{d}} t}{2t}$ 

Next find c;

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$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt}$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

## Rearrange and solve:

$$\frac{\mathsf{d}k_i(t)}{k_i(t)} = \frac{\mathsf{d}t}{2t}$$

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### Rearrange and solve:

$$\frac{\mathsf{d}k_i(t)}{k_i(t)} = \frac{\mathsf{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i \, t^{1/2}.}$$

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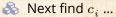
$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

### Rearrange and solve:

$$\frac{\mathsf{d}k_i(t)}{k_i(t)} = \frac{\mathsf{d}t}{2t} \Rightarrow \fbox{k_i(t) = c_i t^{1/2}.}$$



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Know ith node appears at time

$$t_{i,\text{start}} = \left\{ \begin{array}{ll} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{array} \right.$$

So for  $i > m_0$  (exclude initial nodes), we must have

First-mover advantage: Early nodes do be Clearly, a serie condenation

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## We are already at the Zipf distribution:

Degree of node i is the size of the ith ranked node:

$$k_i(t) = m \left( \frac{t}{t_{i,\text{start}}} \right)^{1/2} \text{ for } t \geq t_{i,\text{start}}.$$

Our connection  $lpha=1/(\gamma-1)$  or  $\gamma=1+1/lpha$  ther

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Solution  $\alpha = 1/(\gamma - 1)$  or  $\gamma = 1 + 1/\alpha$  then gives

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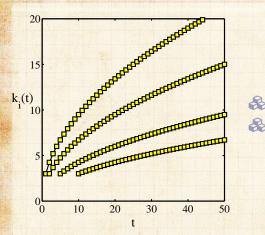
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## $\bigotimes m = 3$

$$t_{i,\text{start}} = 1, 2, 5, \text{ and } 10.$$

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## So what's the degree distribution at time t?

Use fact that birth time for added nodes is distributed uniformly between time 0 and t:

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Fransform variables—Jacobian:

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Transform variables—Jacobian:

$$\frac{\mathrm{d}t_{i,\mathrm{start}}}{\mathrm{d}k_i} = -2\frac{m^2t}{k_i(t)^3}.$$

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$$\Pr(k_i) dk_i = \Pr(t_{i,\text{start}}) dt_{i,\text{start}}$$

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$$\Pr(k_i) dk_i = \Pr(t_{i, \text{start}}) dt_{i, \text{start}}$$

$$= \mathbf{Pr}(t_{i,\text{start}}) \mathsf{d}k_i \left| \frac{\mathsf{d}t_{i,\text{start}}}{\mathsf{d}k_i} \right|$$

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$$= 2 \frac{m^2}{k_i(t)^3} \mathsf{d} k_i$$

$$\propto k_i^{-3} \mathrm{d}k_i$$

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# Solution We thus have a very specific prediction of $\Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$ .

Range true more generally for events with size distributions that have power-law tails. 2 finite mean and 'infinite' variance in practice,  $\gamma < 3$  means variance is governed bupper cutoff.

3: finite mean and variance

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We thus have a very specific prediction of **Pr**(k) ~ k<sup>-γ</sup> with γ = 3.
Typical for real networks: 2 < γ < 3.</li>

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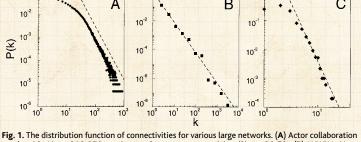
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### Back to that real data:

#### 101 10<sup>0</sup> 10<sup>°</sup> В C A 10-2 10-2 10-1 (k) H(k) 104 10-2 10.6 10-3 10-5 104 10-8 10<sup>-6</sup> 103 10<sup>0</sup> 10<sup>1</sup> 10<sup>2</sup> 10 101 10 10<sup>3</sup> 10<sup>4</sup> 100 10<sup>1</sup>

graph with N = 212,250 vertices and average connectivity  $\langle k \rangle = 28.78$ . (B) WWW, N = 325,729,  $\langle k \rangle = 5.46$  (6). (C) Power grid data, N = 4941,  $\langle k \rangle = 2.67$ . The dashed lines have slopes (A)  $\gamma_{actor} = 2.3$ , (B)  $\gamma_{www} = 2.1$  and (C)  $\gamma_{power} = 4$ .

### From Barabási and Albert's original paper<sup>[2]</sup>:



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### Examples

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 $\begin{array}{ll} \mbox{Web} & \gamma\simeq 2.1 \mbox{ for in-degree} \\ \mbox{Web} & \gamma\simeq 2.45 \mbox{ for out-degree} \\ \mbox{Movie actors} & \gamma\simeq 2.3 \\ \mbox{Words (synonyms)} & \gamma\simeq 2.8 \end{array}$ 

The Internets is a different business



### Examples

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The Internets is a different business...

🚳 Vary attachment kernel. A Vary mechanisms: 1. Add edge deletion Add node deletion 3. Add edge rewiring Deal with directed versus undirected networks.

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#### Scale-free networks

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# Let's look at preferential attachment (PA) a little more closely.

PA implies arriving nodes have **conclete fnowledge** of the existing network's degree distribution.

For example: If  $P_{\text{attach}}(k) \propto k$ , we need to determine the constant of proportionality. We need to know what everyone's degree i PA is : an outrageous assumption of node capability.

But a very simple mechanism saves the day...

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- 😤 Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- Sor example: If  $P_{\text{attach}}(k) \propto k$ , we need to determine the constant of proportionality.
- 🛞 We need to know what everyone's degree is...
- A is .: an outrageous assumption of node capability.
- But a very simple mechanism saves the day...

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# Preferential attachment through randomness

lnstead of attaching preferentially, allow new nodes to attach randomly.

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# Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
  - Assuming the existing network is random, we know probability of a random friend having degree *k* is

So nch gets richer scheme can now be seen to work in a natural way.

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$$Q_k \propto k P_k$$

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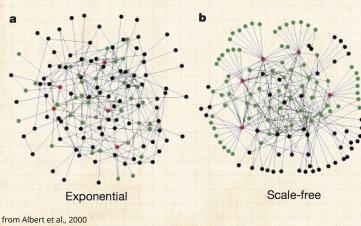
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- Albert et al., Nature, 2000: "Error and attack tolerance of complex networks"<sup>[1]</sup>
- Standard random networks (Erdős-Rényi) versus Scale-free networks:



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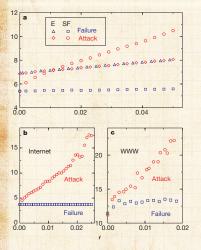
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from Albert et al., 2000

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 removed
 Erdős-Rényi versus scale-free networks
 blue symbols = random removal
 red symbols = targeted removal (most connected first)

Plots of network

diameter as a function

of fraction of nodes

2

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Scale-free networks are thus robust to random

failures yet fragile to targeted ones.

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Scale-free networks are thus robust to random failures yet fragile to targeted ones.

🚳 All very reasonable: Hubs are a big deal.

Representing all webpages as the same size noc is obviously a stretch (e.g., google vs. a random person's webpage) Most connected nodes are either:

Need to explore cost of various targeting schem

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 Physically larger nodes that may be harder to 'target'
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#### Not a robust paper:



"The "Robust yet Fragile" nature of the Internet" Doyle et al., Proc. Natl. Acad. Sci., **2005**, 14497–14502, 2005. <sup>[3]</sup>

- lot networks versus scale-free networks
- Same degree distributions, different arrangements.
- look at the actual Internet.
- 🚳 Excellent project material.

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### Fooling with the mechanism:

2001: Krapivsky & Redner (KR)<sup>[4]</sup> explored the general attachment kernel:

Pr(attach to node *i*)  $\propto A_k = k_i^{\nu}$ where  $A_k$  is the attachment kernel and  $\nu >$ KR also looked at changing the details of the attachment kernel. KR model will be fully studied in CoNKS.

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🛞 We'll follow KR's approach using rate equations 🗹.

#### where $N_k$ is the number of nodes of degree k

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We'll follow KR's approach using rate equations C.
 Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[ A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

 One node with one link is added per unit time.
 The first term corresponds to degree k - 1 node becoming degree k nodes.
 The second term corresponds to degree k node becoming degree k - 1 nodes.
 A is the correct normalization (coming up).
 Seed with some initial network.

Detail:  $A_0 = 0$ 

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In general, probability of attaching to a specific node of degree k at time t is

Detail: we are ignoring initial seed network's edges.

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ln general, probability of attaching to a specific node of degree k at time t is

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**Pr**(attach to node *i*) =  $\frac{A_k}{A(t)}$ 

where 
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.  
E.g., for BA model,  $A_k = k$  and  $A$ .  
For  $A_k = k$ , we have

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since one edge is being added per unit time.

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🔏 So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[ A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t} \left[ (k-1)N_{k-1} - kN_k \right] + \delta_{k1}$$

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🙈 So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[ A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

As for BA method, look for steady-state growing solution:  $N_{\mu} = n_{\mu}t$ .

 $\bigotimes$  We replace  $dN_k/dt$  with  $dn_kt/dt = n_k$ .

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As for BA method, look for steady-state growing solution:  $N_k = n_k t$ .

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$$n_{k} = \frac{1}{2t} \left[ (k-1)n_{k-1}t - kn_{k}t \right] + \delta_{k1}$$

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# Outline

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# Universality?

As expected, we have the same result as for the BA model:

 $N_k(t) = n_k(t)t \propto k^{-3}t$  for large k.

Now: what happens if we start playing around with the attachment kernel  $A_k$ ? Again, we're asking if the result  $\gamma = 3$ KR's natural modification:  $A_k = k^{\nu}$  with  $\nu \neq 1$ . But we'll first explore a more subtle modification of  $A_k$  made by Krapivsky/Redner Keep  $A_k$  linear in k but tweak details. The Relax from  $A_k = k$  to  $A_k \sim k$  as  $k \to \infty$ .

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🚳 Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

We assume that  $A = \mu t$ We'll find  $\mu$  later and make sure that our assumption is consistent. As before, also assume  $N_k(t) = n_k t$ .

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So For  $A_k = k$  we had

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For 
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$$n_{k} = \frac{1}{2} \left[ (k-1)n_{k-1} - kn_{k} \right] + \delta_{k1}$$

### 🚳 This now becomes

$$n_{k} = \frac{1}{\mu} \left[ A_{k-1} n_{k-1} - A_{k} n_{k} \right] + \delta_{k1}$$

#### Again two cases

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$$n_{k} = \frac{1}{\mu} \left[ A_{k-1} n_{k-1} - A_{k} n_{k} \right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

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$$k=1:n_1=\frac{\mu}{\mu+A_1};$$

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For 
$$A_k = k$$
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### 🚳 Again two cases:

$$k = 1: n_1 = \frac{\mu}{\mu + A_1}; \qquad k > 1: n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

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Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \rightarrow k$  as  $k \rightarrow \infty$ .

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Solution Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ . For large k, we find:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \propto \frac{k^{-\mu - 1}}{k}$$

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Time for pure excitement: Find asymptotic behavior of n<sub>k</sub> given A<sub>k</sub> → k as k → ∞.
For large k, we find:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \propto k^{-\mu - 1}$$

 $\mathfrak{S}$  Since  $\mu$  depends on  $A_k$ , details matter...

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### 3 Now we need to find $\mu$ .

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Now we need to find  $\mu$ . Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t)A_k$ Since  $N_k = n_k t$ , we have the simplification

Now subsitute in our expression for n

Closed form expression for  $\mu$ . We can solve for  $\mu$  in some cases. Our assumption that  $A = \mu t$  looks to be not too horrible.

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We can solve for µ in some cases.
Our assumption that A = µt looks to be not too horrible.

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### $\Im$ Consider tunable $A_1 = \alpha$ and $A_k = k$ for $k \ge 2$ .

Again, we can find  $\gamma = \mu + 1$  by finding , Closed form expression for  $\mu$ :

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 $\bigotimes$  Consider tunable  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ . Again, we can find  $\gamma = \mu + 1$  by finding  $\mu$ . Solution Closed form expression for  $\mu$ :

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

#mathisfun



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 $\bigotimes$  Consider tunable  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ . Again, we can find  $\gamma = \mu + 1$  by finding  $\mu$ .  $\bigotimes$  Closed form expression for  $\mu$ :

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

### #mathisfun

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$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}.$$

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#### #mathisfun

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$$\mu(\mu-1) = 2\alpha \Rightarrow \mu = \frac{1+\sqrt{1+8\alpha}}{2}$$

Since  $\gamma = \mu + 1$ , we have

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

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Solution Consider tunable  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ . Solution Again, we can find  $\gamma = \mu + 1$  by finding  $\mu$ . Solution Closed form expression for  $\mu$ :

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

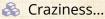
### #mathisfun

R

$$\mu(\mu-1) = 2\alpha \Rightarrow \mu = \frac{1+\sqrt{1+8\alpha}}{2}$$

Since  $\gamma = \mu + 1$ , we have

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$



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Rich-get-somewhat-richer:

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Bistribution of degree is universal providing  $\nu < 1$ .

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Details:

Solve  $1/2 < \nu < 1$ :

$$n_k \sim k^{-\nu} e^{-\mu \left(\frac{k^{1-\nu}-2^{1-\nu}}{1-\nu}\right)}$$

And for  $1/(r+1) < \nu < 1/r$ , we have r pieces i exponential.

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For 
$$1/3 < \nu < 1/2$$
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$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

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### 🙈 Rich-get-much-richer:

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Now a semicle take-all mechanism. One single node ends up being connected to almost all other nodes. For  $\nu > 2$ , all but a finite # of nodes connect to node.

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### Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
  - But focus on dynamics is more of a physics/stat-mech/comp-sci flavor. Two main areas of focus:
  - Description: Characterizing very large network
     Explanation: Micro story ⇒ Macro features
  - Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure.... Still much work to be done, especially with respect

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## Neural reboot (NR):

### Turning the corner:

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