Random Networks

Principles of Complex Systems | @pocsvox
CSYS/MATH 300, Fall, 2017

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

PoCS
Principles of Complex Systems
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What's the Story?

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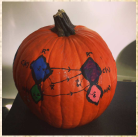

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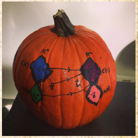
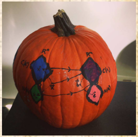


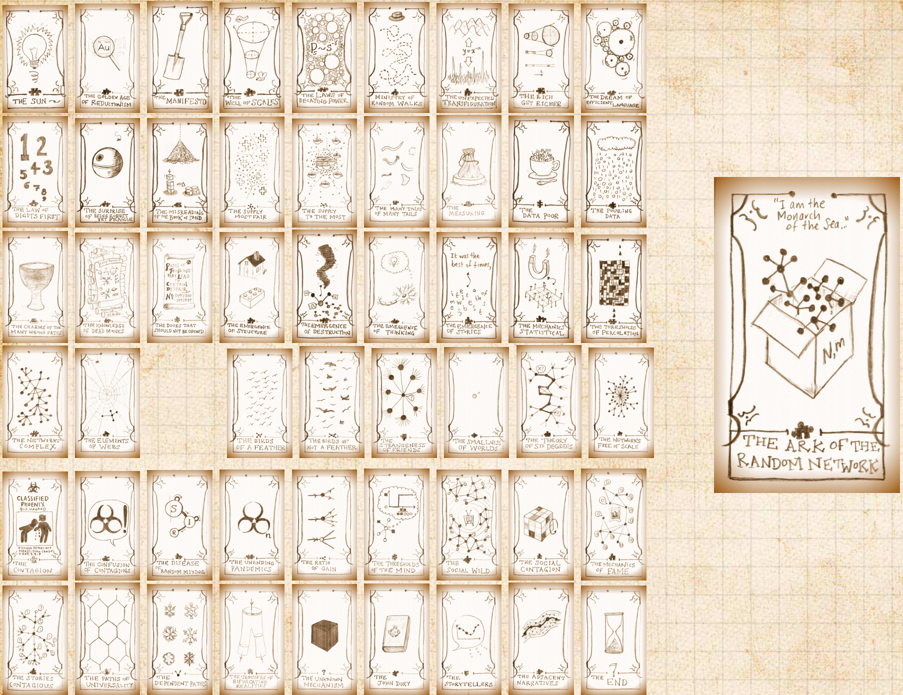
Outline

Models

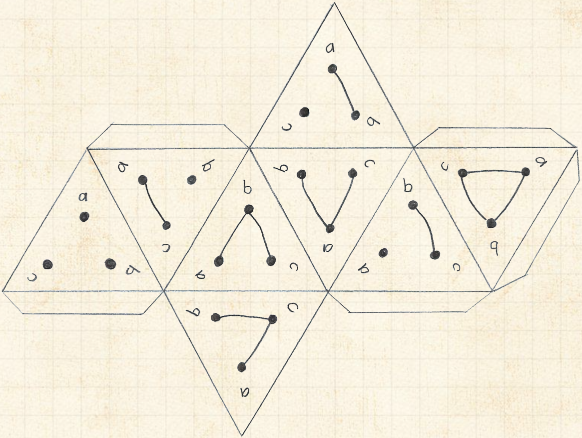
Some important models:

1. Generalized random networks;
2. Small-world networks;
3. Generalized affiliation networks;
4. Scale-free networks;
5. Statistical generative models (p^*).




THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
MINISTRY OF RANDOM WALKS
THE UNEXPECTED TRANSFIGURATION
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
THE DOORS THAT SHOULD NOT BE OPENED
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE RATIO OF GAIN
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE MECHANICS OF FAME
THE STORIES CONTAGIOUS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE JOHN DORY
THE STORYTELLERS
THE ADJACENT NARRATIVES
THE END
"I am the Monarch of the Sea."
N,m
THE ARK OF THE RANDOM NETWORK

Random network generator for $N = 3$:



- Get your own exciting generator here.
- As $N \nearrow$, polyhedral die rapidly becomes a ball...

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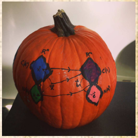
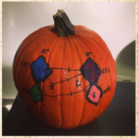


Random networks

Pure, abstract random networks:

- Consider set of all networks with N labelled nodes and m edges.
- Standard random network = one randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or ER graphs.



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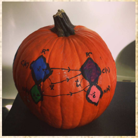

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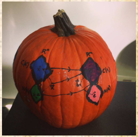

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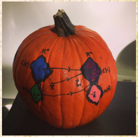

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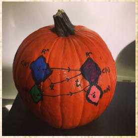

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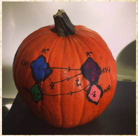


Random networks—basic features:

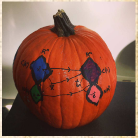
- Number of possible edges:

$$0 \le m \le \binom{N}{2} = \frac{N(N-1)}{2}$$

- Limit of $m = 0$: empty graph.
- Limit of $m = \binom{N}{2}$: complete or fully-connected graph.
- Number of possible networks with N labelled nodes:

$$2^{\binom{N}{2}} \sim e^{\frac{\ln 2}{2} N^2}.$$

- Given m edges, there are $\binom{\binom{N}{2}}{m}$ different possible networks.
- Crazy factorial explosion for $1 \ll m \ll \binom{N}{2}$.
- Real world: links are usually costly so real networks are almost always sparse.



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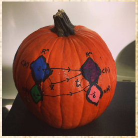

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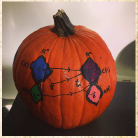

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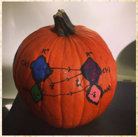

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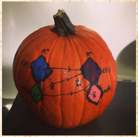

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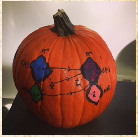

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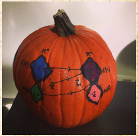

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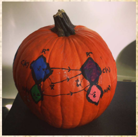
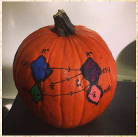


Random networks

How to build standard random networks:

- Given N and m.
- Two probablistic methods (we'll see a third later on)

1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability p.
    - Useful for theoretical work
2. Take N nodes and add exactly m links by selecting edges without replacement.
    - Algorithm: Randomly choose a pair of nodes i and j, $i \neq j$, and connect if unconnected; repeat until all m edges are allocated.
    - Best for adding relatively small numbers of links (most cases).
    - 1 and 2 are effectively equivalent for large N.



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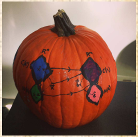

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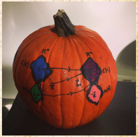

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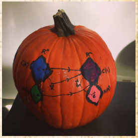

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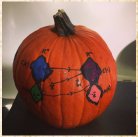

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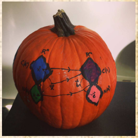

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Random networks

A few more things:

- For method 1, # links is probablistic:

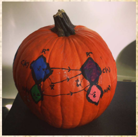
$$\langle m \rangle = p\binom{N}{2} = p\frac{1}{2}N(N-1)$$

- So the expected or average degree is

$$\langle k \rangle = \frac{2\langle m \rangle}{N}$$

$$= \frac{2}{N}p\frac{1}{2}N(N-1) = \frac{2}{N}p\frac{1}{2}N(N-1) = p(N-1).$$

- Which is what it should be...
- If we keep $\langle k \rangle$ constant then $p \propto 1/N \to 0$ as $N \to \infty$.



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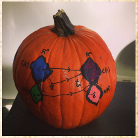

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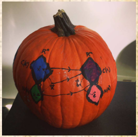

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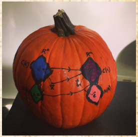

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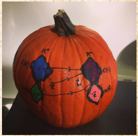

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- So the expected or average degree is

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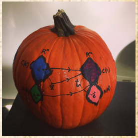

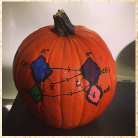

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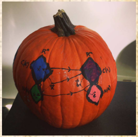
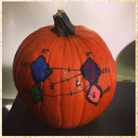


Random networks: examples

Next slides:

Example realizations of random networks

- $N = 500$
- Vary m, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.



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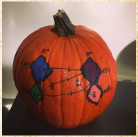

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Random networks: examples for N=500

m = 100
$\langle k \rangle$ = 0.4

m = 200
$\langle k \rangle$ = 0.8

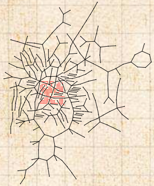
m = 230
$\langle k \rangle$ = 0.92

m = 240
$\langle k \rangle$ = 0.96

m = 250
$\langle k \rangle$ = 1

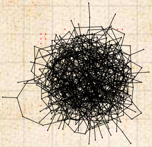
m = 260
$\langle k \rangle$ = 1.04

m = 280
$\langle k \rangle$ = 1.12

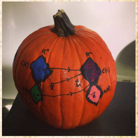


m = 300
$\langle k \rangle$ = 1.2

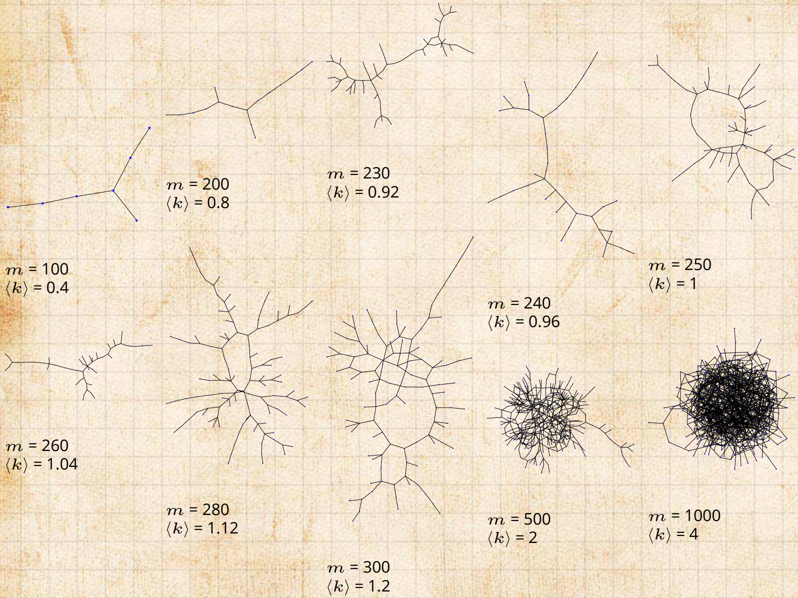
m = 500
$\langle k \rangle$ = 2



m = 1000
$\langle k \rangle$ = 4



Random networks: largest components



Random networks: examples for N=500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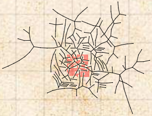
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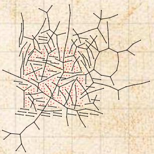
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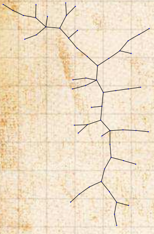

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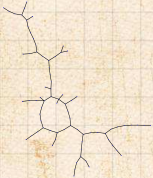

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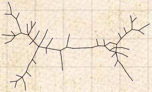

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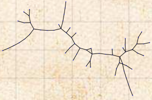

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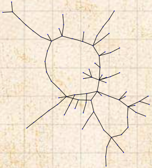

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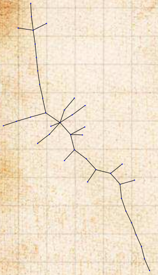

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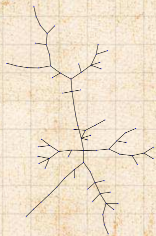

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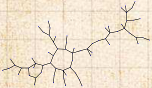

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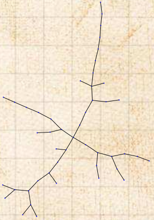

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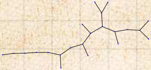

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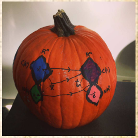
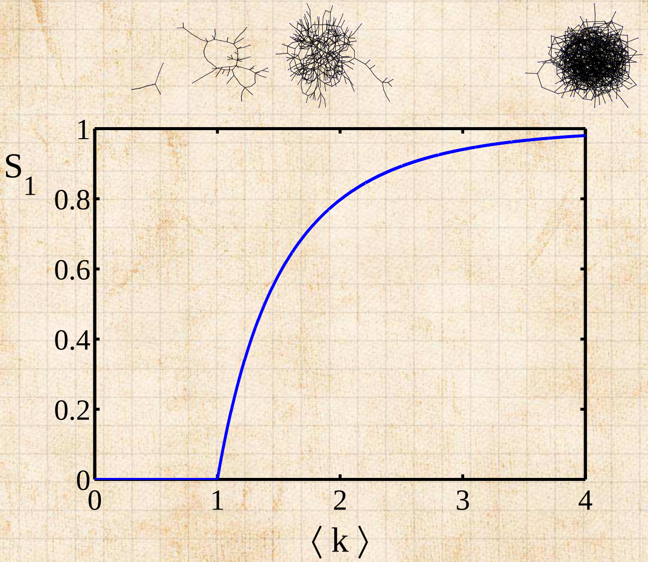


Giant component



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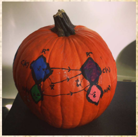

Clustering in random networks:

- For construction method 1, what is the clustering coefficient for a finite network?
- Consider triangle/triple clustering coefficient: [?]

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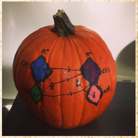

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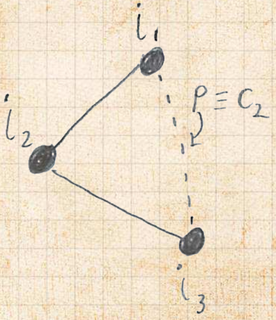
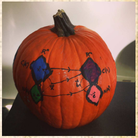


- Recall: C_2 = probability that two friends of a node are also friends.
- Or: C_2 = probability that a triple is part of a triangle.
- For standard random networks, we have simply that

$$C_2 = p.$$



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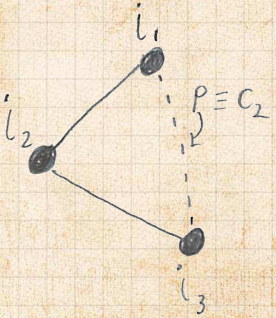

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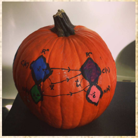

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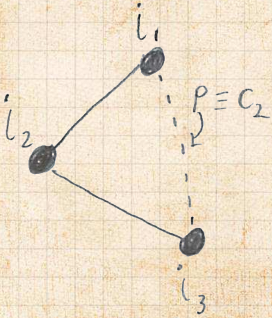

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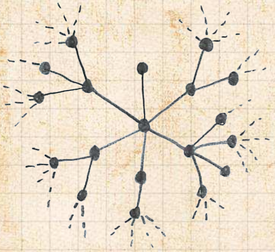

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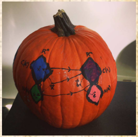
Degree distribution:

- Recall P_k = probability that a randomly selected node has degree k.
- Consider method 1 for constructing random networks: each possible link is realized with probability p.
- Now consider one node: there are '$N-1$ choose k' ways the node can be connected to k of the other $N-1$ nodes.
- Each connection occurs with probability p, each non-connection with probability $(1-p)$.
- Therefore have a binomial distribution:

$$P(k;p,N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$



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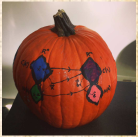

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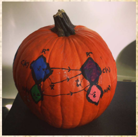

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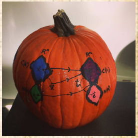

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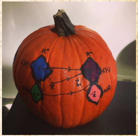

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- Our degree distribution: $P(k;p,N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$.
- What happens as $N \to \infty$?
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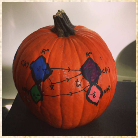

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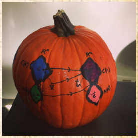

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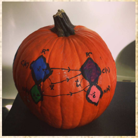

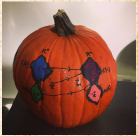

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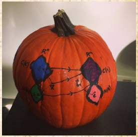

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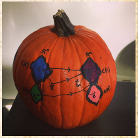

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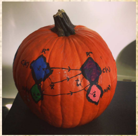
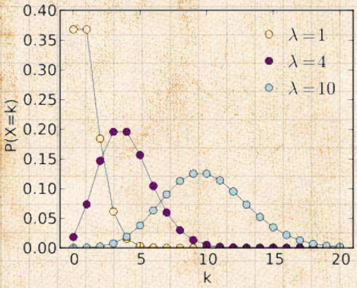
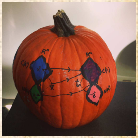


Poisson basics:

$$P(k;\lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$$



- $\lambda > 0$
- $k = 0, 1, 2, 3, \ldots$
- Classic use: probability that an event occurs k times in a given time period, given an average rate of occurrence.
- e.g.: phone calls/minute, horse-kick deaths.
- 'Law of small numbers'



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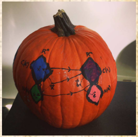

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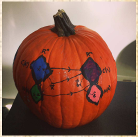

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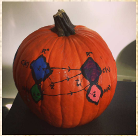

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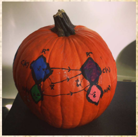

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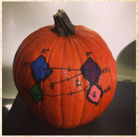

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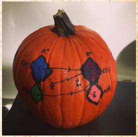

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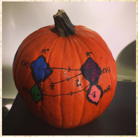

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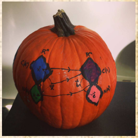

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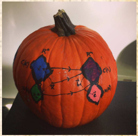

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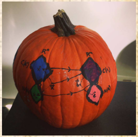

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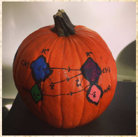

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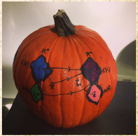

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- The variance of degree distributions for random networks turns out to be very important.
- Using calculation similar to one for finding $\langle k \rangle$ we find the second moment to be:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

- Variance is then

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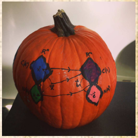

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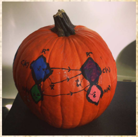

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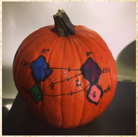

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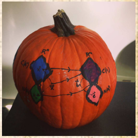
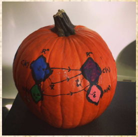


Neural reboot (NR):

Unrelated: Feline elevation



Outline

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- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution P_k.
- Also known as the configuration model.[?]
- Can generalize construction method from ER random networks.
- Assign each node a weight w from some distribution P_w and form links with probability

$$P(\text{link between } i \text{ and } j) \propto w_i w_j.$$

- But we'll be more interested in
  1. Randomly wiring up (and rewiring) already existing nodes with fixed degrees.
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Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

- $N = 1000$.
- $P_k \propto k^{-\gamma}$ for $k \geq 1$.
- Set $P_0 = 0$ (no isolated nodes).
- Vary exponent γ between 2.10 and 2.91.
- Again, look at full network plus the largest component.
- Apart from degree distribution, wiring is random.

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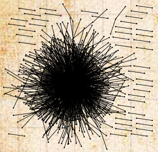
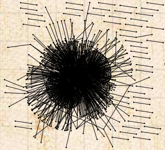


γ = 2.1
$\langle k \rangle$ = 3.448



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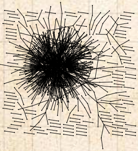

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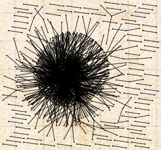

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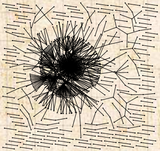

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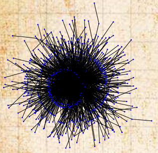
Random networks: largest components



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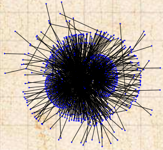

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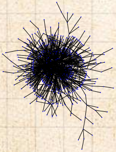

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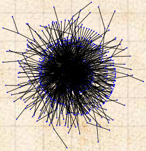

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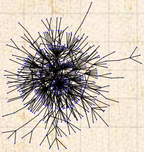

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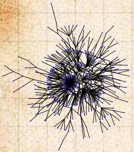

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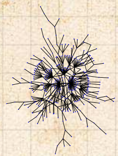

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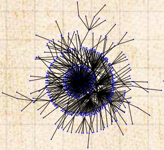

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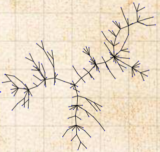

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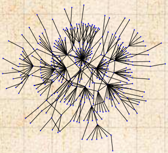

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Outline

Models

Generalized random networks:

- Arbitrary degree distribution P_k.
- Create (unconnected) nodes with degrees sampled from P_k.
- Wire nodes together randomly.
- Create ensemble to test deviations from randomness.

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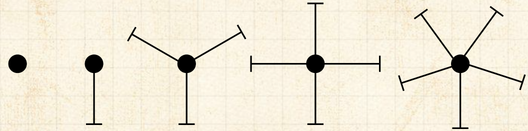
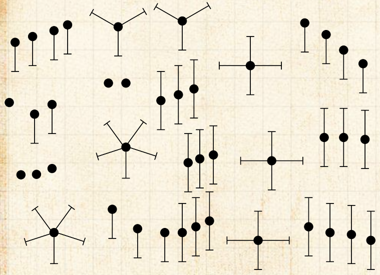
Building random networks: Stubs

Phase 1:

- **Idea:** start with a soup of unconnected nodes with stubs (half-edges):





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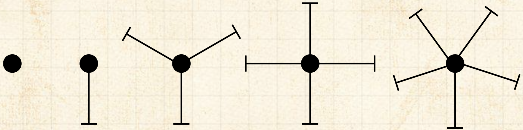

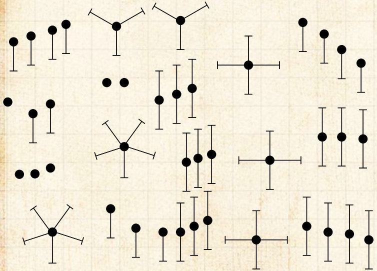

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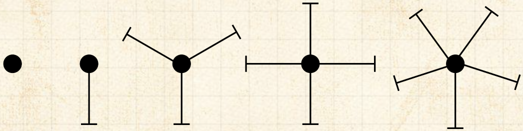

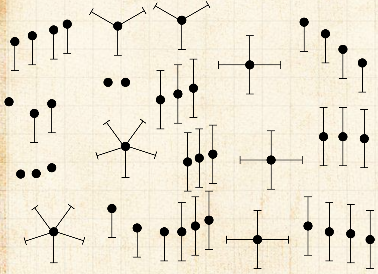

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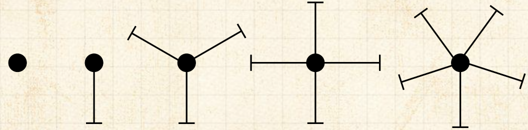

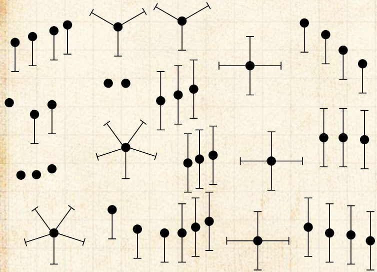

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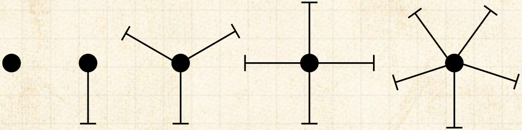

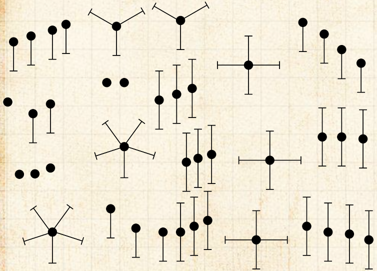

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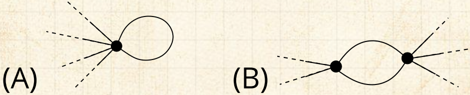
Phase 2:

- Now find any (A) self-loops and (B) repeat edges and randomly rewire them.



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- Simplest solution: randomly rewire two edges at a time.

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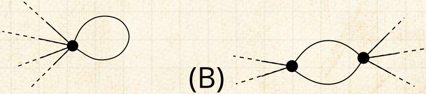

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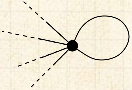
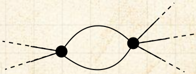
(A) 

(B) 

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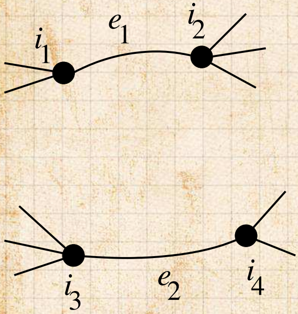

- Randomly choose two edges. (Or choose problem edge and a random edge)
- Check to make sure edges are disjoint.
- Rewire one end of each edge.
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- Same as finding on/off/on/off 4-cycles, and rotating them.

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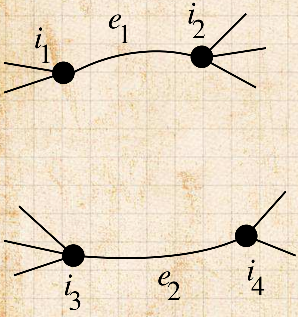

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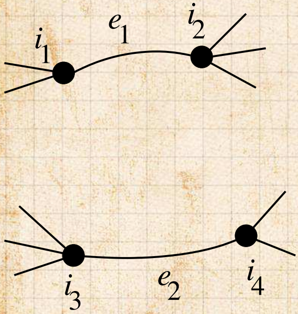

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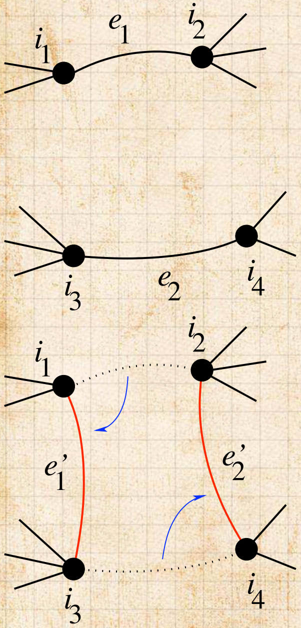

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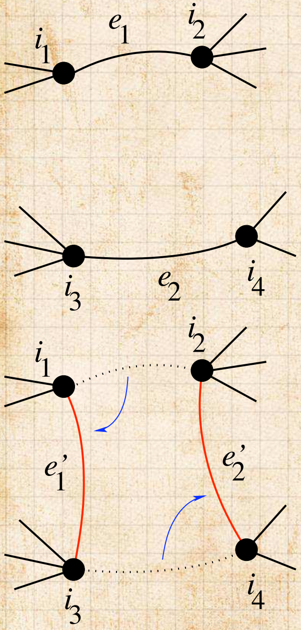

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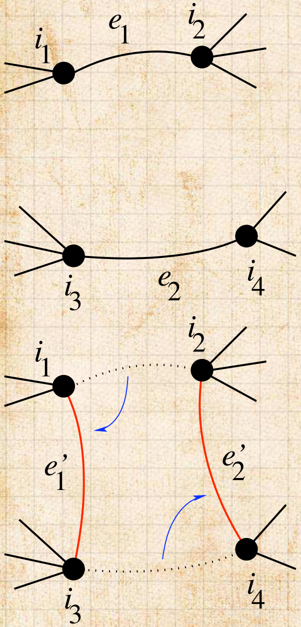

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Phase 2:

- Use rewiring algorithm to remove all self and repeat loops.

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- Randomize network wiring by applying rewiring algorithm liberally.
- Rule of thumb: # Rewirings $\simeq 10 \times$ # edges [5].

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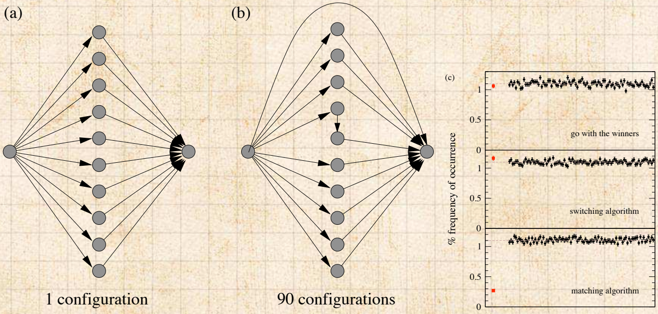

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Network motifs

- Idea of motifs [8] introduced by Shen-Orr, Alon et al. in 2002.
- Looked at gene expression within full context of transcriptional regulation networks.
- Specific example of Escherichia coli.
- Directed network with 577 interactions (edges) and 424 operons (nodes).
- Used network randomization to produce ensemble of alternate networks with same degree frequency N_k.
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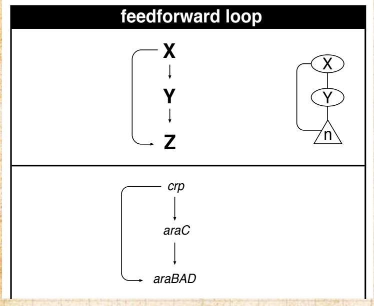
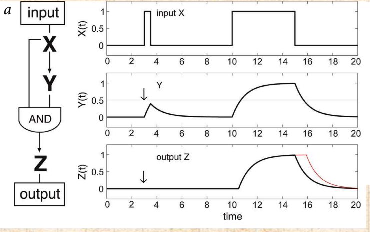
- Idea of motifs [8] introduced by Shen-Orr, Alon et al. in 2002.
- Looked at gene expression within full context of transcriptional regulation networks.
- Specific example of Escherichia coli.
- Directed network with 577 interactions (edges) and 424 operons (nodes).
- Used network randomization to produce ensemble of alternate networks with same degree frequency N_k.
- Looked for certain subnetworks (motifs) that appeared more or less often than expected

Network motifs





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- Turning off X rapidly turns off Z.
- Analogy to elevator doors.

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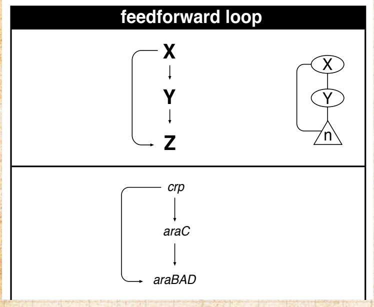

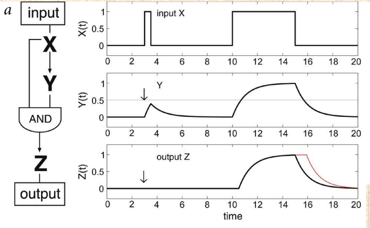

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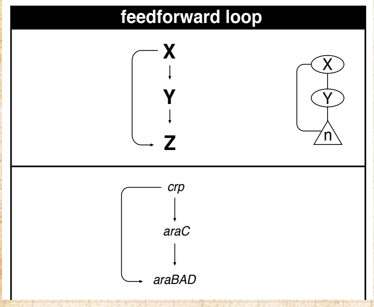

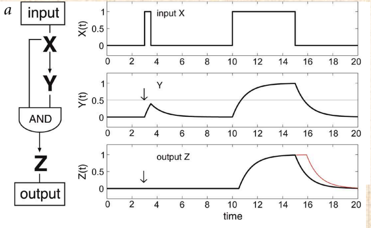

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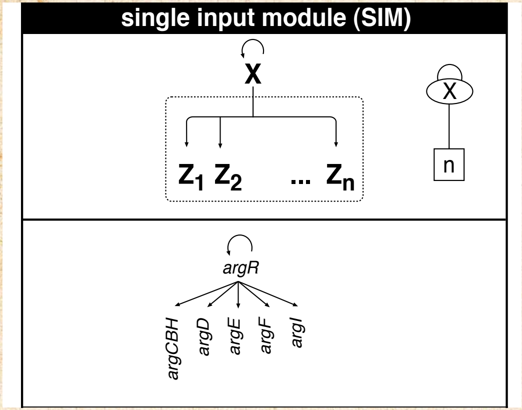
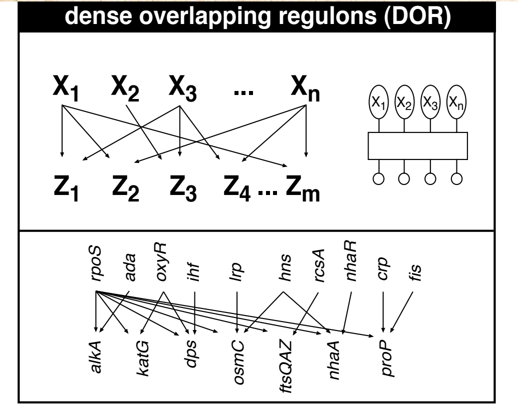


- Master switch.

Network motifs



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- For more, see work carried out by Wiggins *et al.* at Columbia.

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Outline

THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
MINISTRY OF RANDOM WALKS
THE UNEXPECTED TRANSFIGURATION
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
PATHS OF THINKING THAT LEAD TO CERTAIN DESPAIR
THE DOORS THAT SHOULD NOT BE OPENED
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
"I am the Monarch of the Sea"
THE ARK OF THE RANDOM NETWORK
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
THE STRANGENESS OF FRIENDS
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE RATIO OF GAIN
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE MECHANICS OF FAME
THE STORIES CONTAGIOUS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE JOHN DORY
THE STORYTELLERS
THE ADJACENT NARRATIVES
THE END

The edge-degree distribution:

- The degree distribution P_k is fundamental for our description of many complex networks
- Again: P_k is the degree of randomly chosen node.
- A second very important distribution arises from choosing randomly on edges rather than on nodes.
- Define Q_k to be the probability the node at a random end of a randomly chosen edge has degree k.
- Now choosing nodes based on their degree (i.e., size):

$$Q_k \propto k P_k$$

- Normalized form:

$$Q_k = \frac{k P_k}{\sum_{k'=0}^{\infty} k' P_{k'}} = \frac{k P_k}{\langle k \rangle}.$$

- Big deal: Rich-get-richer mechanism is built into this selection process.

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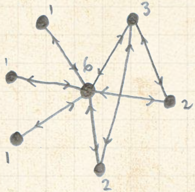

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- Probability of landing on a node of degree k after randomly selecting an edge and then randomly choosing one direction to travel: $Q_1 = 3/16$, $Q_2 = 4/16$, $Q_3 = 3/16$, $Q_6 = 6/16$.
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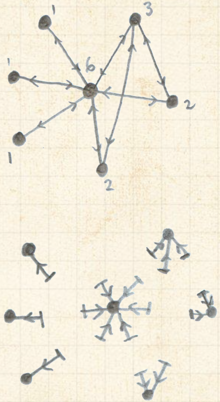

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- For random networks, Q_k is also the probability that a friend (neighbor) of a random node has k friends.
- Useful variant on Q_k:

  R_k = probability that a friend of a random node has k other friends.

- $$R_k = \frac{(k+1)P_{k+1}}{\sum_{k'=0}(k'+1)P_{k'+1}} = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$
- Equivalent to friend having degree $k+1$.
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$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle \frac{\langle k^2 \rangle}{\langle k \rangle^2} = \langle k \rangle \frac{\sigma^2 + \langle k \rangle^2}{\langle k \rangle^2} = \langle k \rangle \left(1 + \frac{\sigma^2}{\langle k \rangle^2}\right) \geq \langle k \rangle$$

- So only if everyone has the same degree (variance$= \sigma^2 = 0$) can a node be the same as its friends.
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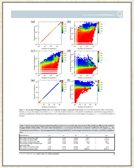

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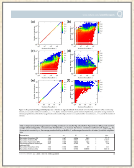

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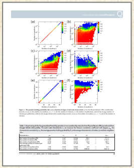

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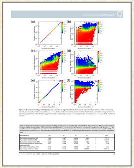

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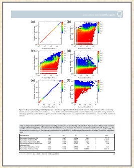

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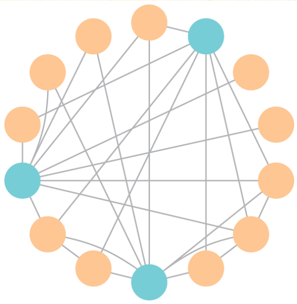

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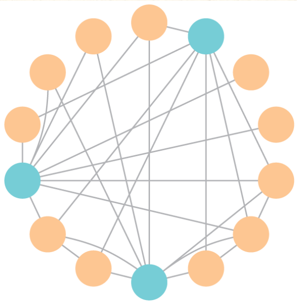

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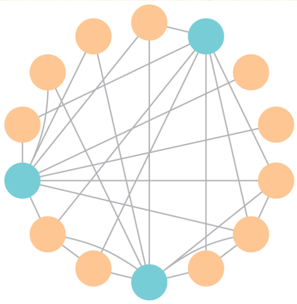

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- Again: thinking in edge space changes everything.

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Two reasons why this matters

(Big) Reason #2:

- $\langle k \rangle_R$ is key to understanding how well random networks are connected together.
- e.g., we'd like to know what's the size of the largest component within a network.
- As $N \to \infty$, does our network have a giant component?
- Defn: Component = connected subnetwork of nodes such that $\exists$ path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.
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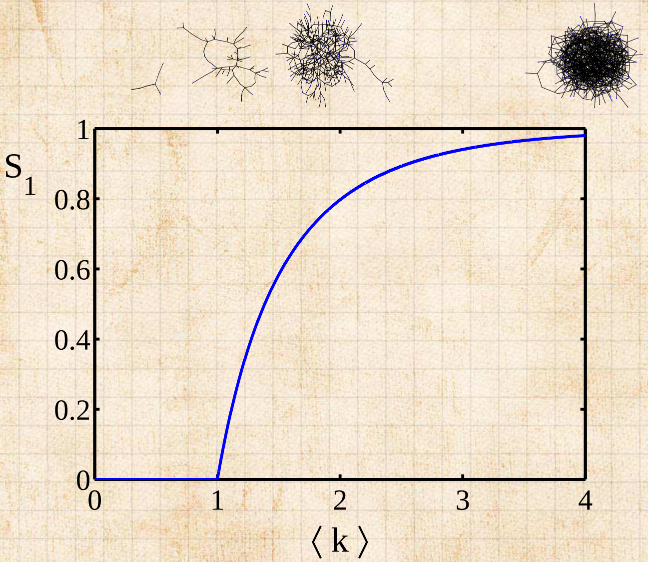
Outline

Giant component



Structure of random networks

Giant component:

- A giant component exists if when we follow a random edge, we are likely to hit a node with at least 1 other outgoing edge.
- Equivalently, expect exponential growth in node number as we move out from a random node.
- All of this is the same as requiring $\langle k \rangle_R > 1$.
- Giant component condition (or percolation condition):

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$$

- Again, see that the second moment is an essential part of the story.
- Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$

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Spreading on Random Networks

- For random networks, we know local structure is pure branching.
- Successful spreading is ∴ contingent on single edges infecting nodes.
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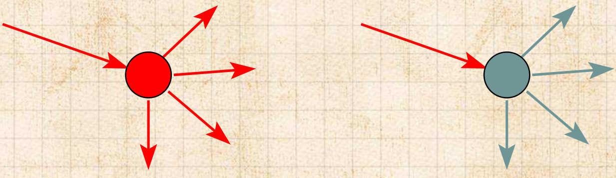

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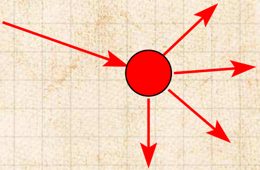
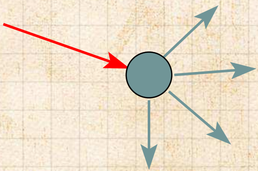


Failure:



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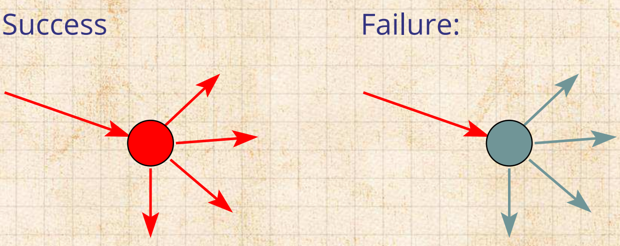

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- We need to find: [2]
  R = the average # of infected edges that one random infected edge brings about.
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- A fraction (1-β) of edges do not transmit infection.
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- Recall $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.
- Determine condition for giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle - \langle k \rangle}{\langle k \rangle} = \langle k \rangle$$

- Therefore when $\langle k \rangle > 1$, standard random networks have a giant component.
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Random networks with skewed P_k:

- e.g, if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$, $k \geq 1$, then

$$\langle k^2 \rangle = c \sum_{k=1}^{\infty} k^2 k^{-\gamma}$$

$$\sim \int_{x=1}^{\infty} x^{2-\gamma} \mathrm{d}x$$

$$\propto x^{3-\gamma} \Big|_{x=1}^{\infty} = \infty \quad (\gg \langle k \rangle).$$

- So giant component always exists for these kinds of networks.
- Cutoff scaling is k^{-3}: if $\gamma > 3$ then we have to look harder at $\langle k \rangle_R$.
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And how big is the largest component?

- Define S_1 as the size of the largest component.
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- We can figure out some limits and details for $S_1 = 1 - e^{-\langle k \rangle S_1}$.
- First, we can write $\langle k \rangle$ in terms of S_1:

$$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}.$$

- As $\langle k \rangle \to 0$, $S_1 \to 0$.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$.
- Notice that at $\langle k \rangle = 1$, the critical point, $S_1 = 0$.
- Only solvable for $S_1 > 0$ when $\langle k \rangle > 1$.
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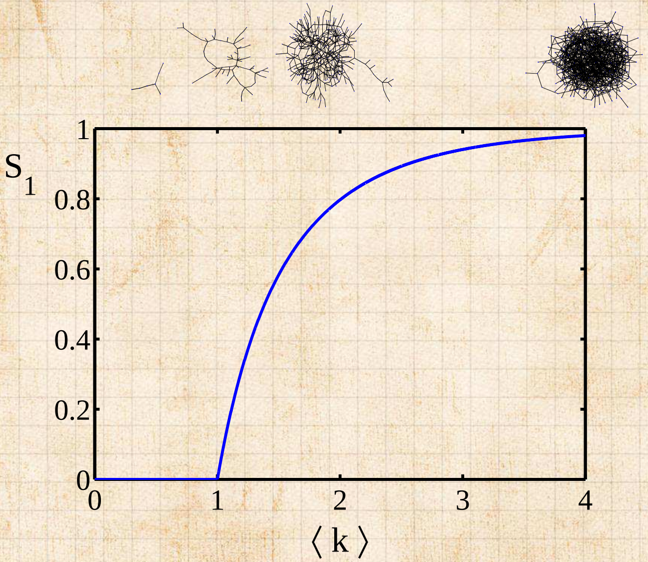

Turns out we were lucky...

- Our dirty trick only works for ER random networks.
- The problem: We assumed that neighbors have the same probability δ of belonging to the largest component.
- But we know our friends are different from us...
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Neural reboot (NR):

Falling maple leaf

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