Mechanisms for Generating Power-Law Size Distributions, Part 1

Principles of Complex Systems | @pocsvox
CSYS/MATH 300, Fall, 2017

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Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

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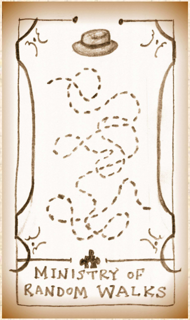
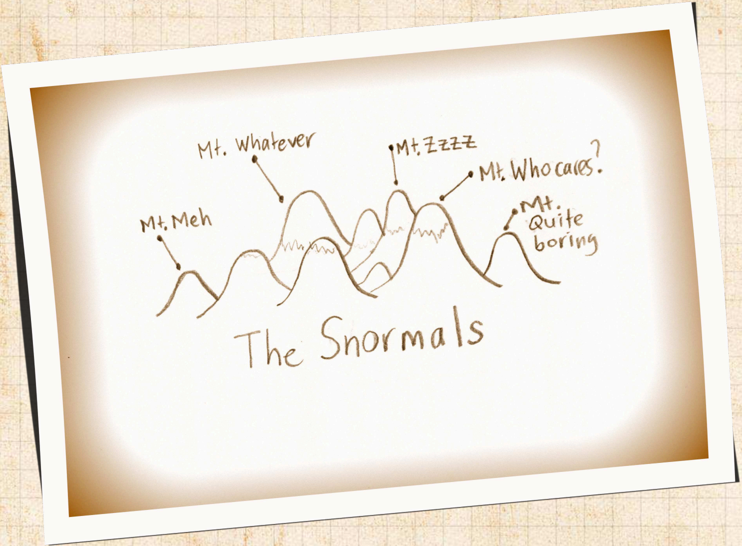
Outline


THE SUN
Au
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE MANY TALES OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
THE DOORS THAT SHOULD NOT BE OPENED
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE ARK OF THE RANDOM NETWORK
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED PHOENIX BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE RATIO OF GAIN
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE MECHANICS OF FAME
THE STORIES CONTAGIOUS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE JOHN DORY
THE STORYTELLERS
THE ADJACENT NARRATIVES
THE END


MINISTRY OF RANDOM WALKS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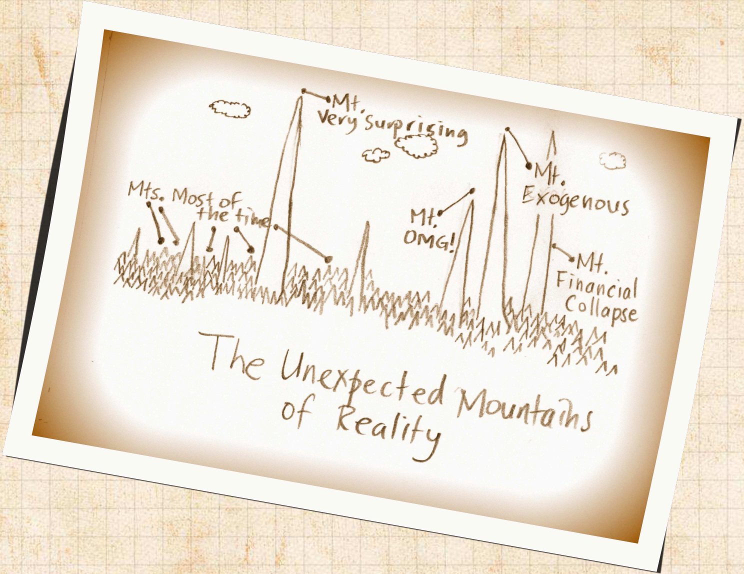
Mt. Zzzz
Mt. Who cares?
Mt. Meh
Mt. Quite boring
The Snormals


MINISTRY OF RANDOM WALKS


Mt. Very Surprising
Mts. Most of the time
Mt. OMG!
Mt. Exogenous
Mt. Financial Collapse
The Unexpected Mountains of Reality

Mechanisms:

A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks.

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

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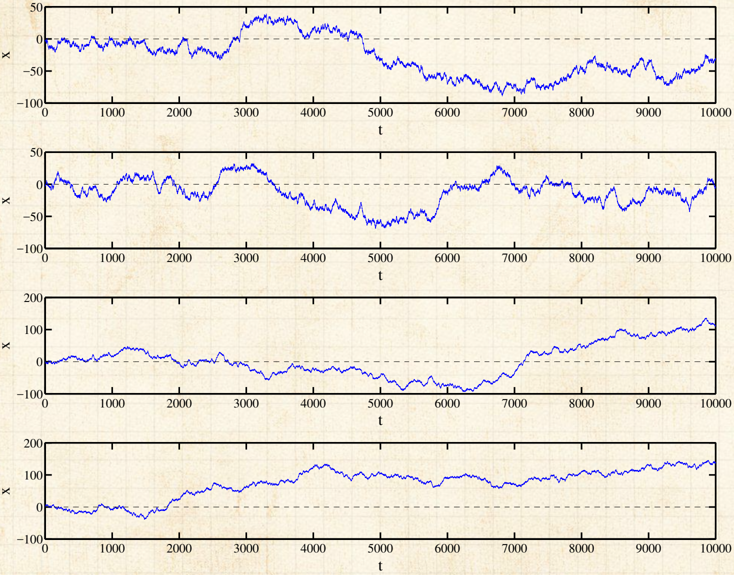
A few random random walks:



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Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?

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Variances sum:*

$$\mathsf{Var}(x_t) = \mathsf{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \mathsf{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

A non-trivial scaling law arises out of additive aggregation or accumulation.

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Stock Market randomness:

Also known as the bean machine, the quincunx (simulation), and the Galton box.

Great moments in Televised Random Walks:

Plinko! from the Price is Right.

Random walk basics:

Counting random walks:

- Each specific random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert question from assignment 3

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

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How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \ldots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

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- The whole is different from the parts. #nutritious
- See also: Stable Distributions

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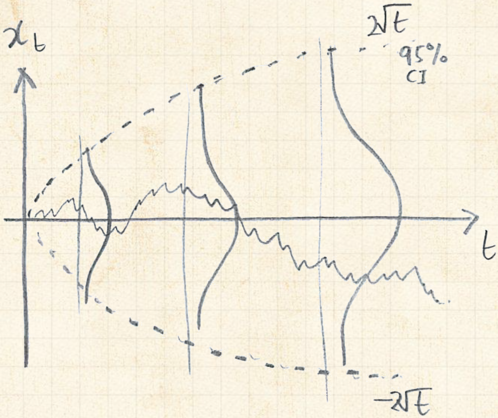
Universality is also not left-handed:



- This is Diffusion: the most essential kind of spreading (more later).
- View as Random Additive Growth Mechanism.


$t^{1/2}$
t
The Snormals
The Random Road
through the Forests of Forgettable Events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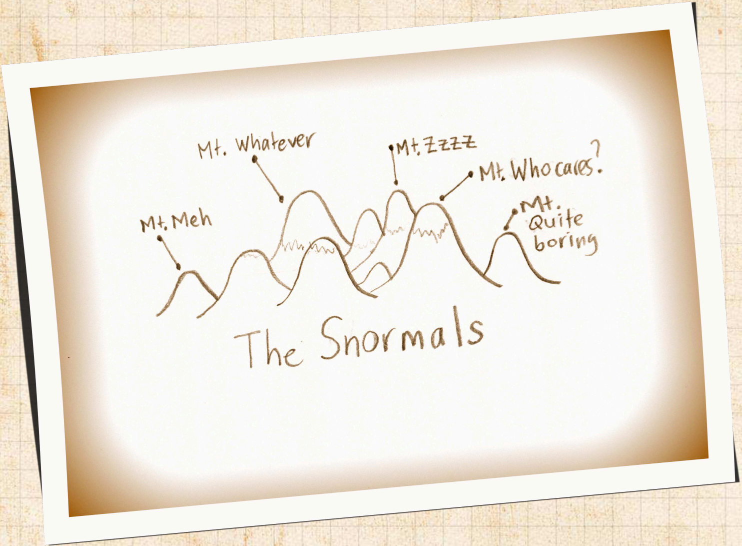
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- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞

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- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
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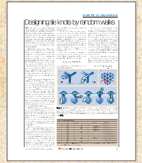
See Feller, Intro to Probability Theory, Volume I [3]

Applied knot theory:



"Designing tie knots by random walks"
Fink and Mao,
Nature, **398**, 31–32, 1999. [4]

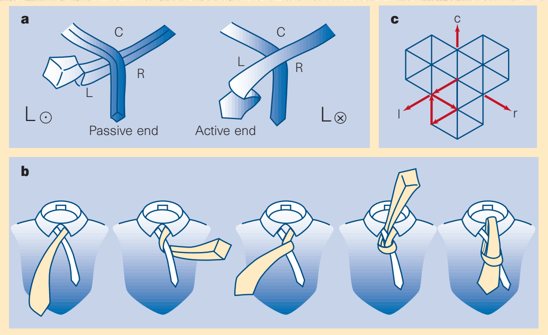


Figure 1 All diagrams are drawn in the frame of reference of the mirror image of the actual tie. **a,** The two ways of beginning a knot, $L_{\odot}$ and $L_{\otimes}$. For knots beginning with $L_{\odot}$, the tie must begin inside-out. **b,** The four-in-hand, denoted by the sequence $L_{\otimes} R_{\odot} L_{\otimes} C_{\odot} T$. **c,** A knot may be represented by a persistent random walk on a triangular lattice. The example shown is the four-in-hand, indicated by the walk $\hat{l}\,\hat{r}\,\hat{l}\,\hat{c}$.

Applied knot theory:

Table 1 **Aesthetic tie knots**

h	γ	γ/h	$K(h, \gamma)$	s	b	Name	Sequence
3	1	0.33	1	0	0		$L_{\odot}R_{\otimes}C_{\odot}T$
4	1	0.25	1	−1	1	Four-in-hand	$L_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$
5	2	0.40	2	−1	0	Pratt knot	$L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$
6	2	0.33	4	0	0	Half-Windsor	$L_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$
7	2	0.29	6	−1	1		$L_{\odot}R_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$
7	3	0.43	4	0	1		$L_{\odot}C_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$
8	2	0.25	8	0	2		$L_{\otimes}R_{\odot}L_{\otimes}C_{\odot}R_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$
8	3	0.38	12	−1	0	Windsor	$L_{\otimes}C_{\odot}R_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$
9	3	0.33	24	0	0		$L_{\odot}R_{\otimes}C_{\odot}L_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$
9	4	0.44	8	−1	2		$L_{\odot}C_{\otimes}R_{\odot}C_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$

Knots are characterized by half-winding number h, centre number γ, centre fraction γ/h, knots per class $K(h, \gamma)$, symmetry s, balance b, name and sequence.

- h = number of moves
- γ = number of center moves
- $K(h,\gamma) = 2^{\gamma-1}\binom{h-\gamma-2}{\gamma-1}$
- $s = \sum_{i=1}^{h} x_i$ where x = -1 for L and +1 for R.
- $b = \frac{1}{2}\sum_{i=2}^{h-1} |\omega_i + \omega_{i-1}|$ where $\omega = \pm 1$ represents winding direction.

Outline

Random walks #crazytownbananapants

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

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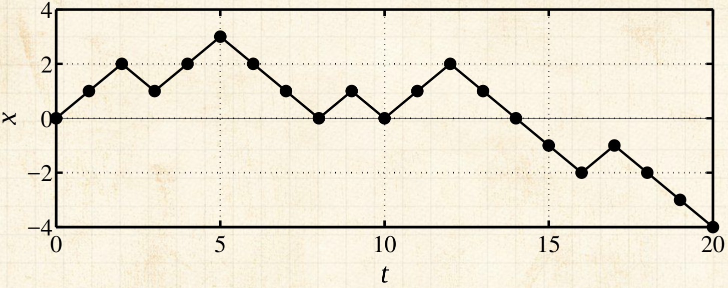
For random walks in 1-d:



- A return to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8$, 10, and 14.
- Call $P_{\mathrm{fr}(2n)}$ the probability of first return at $t = 2n$.
- Probability calculation ≡ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.

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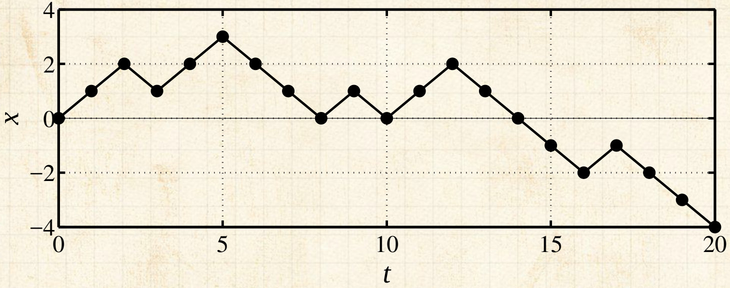

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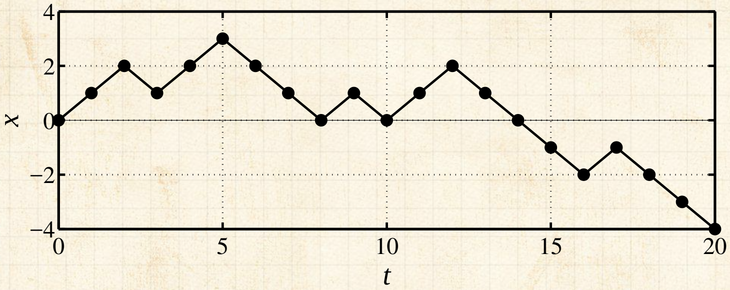

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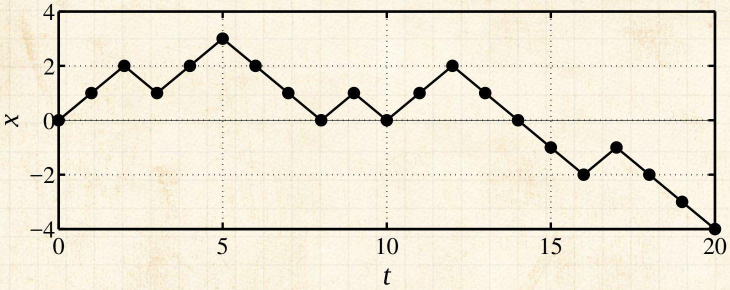

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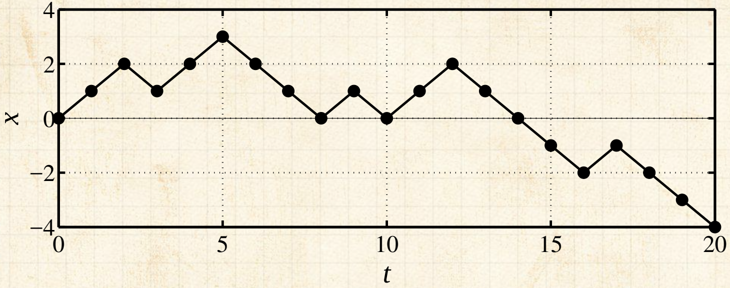

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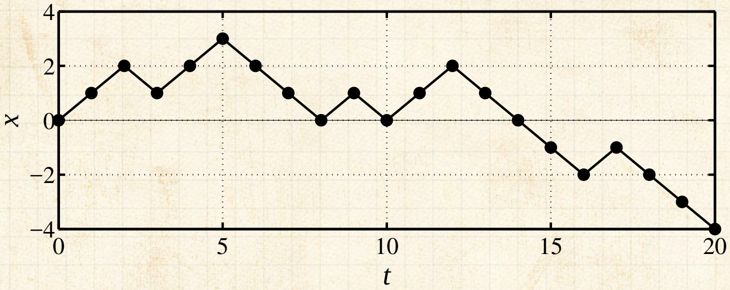

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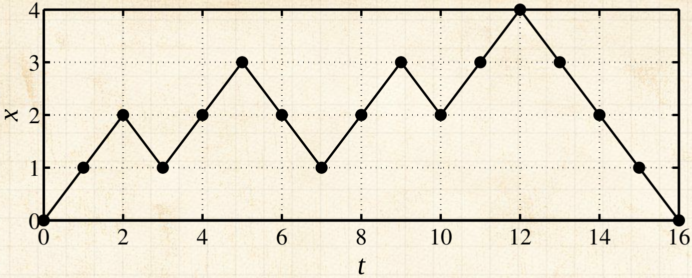
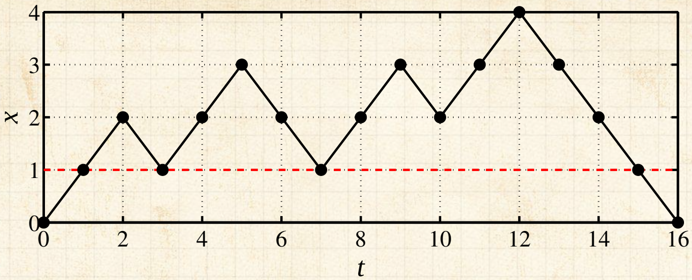


- Can assume drunkard first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \le t \le 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) =$
  $2 \cdot \frac{1}{2} Pr(x_t \ge 1, 1 \le t \le 2n-1,$ and $x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for drunkards that first lurch to $x = -1$.



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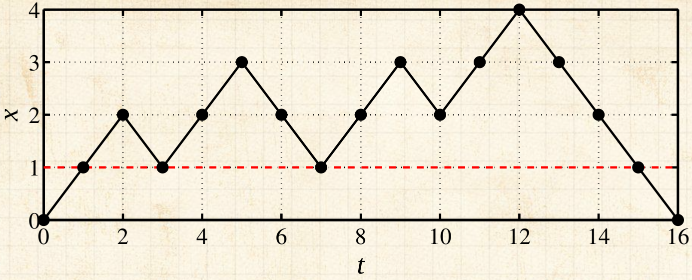

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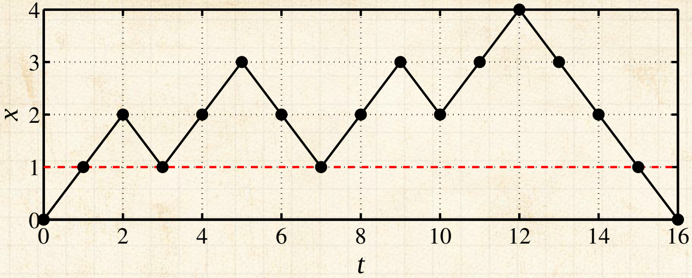

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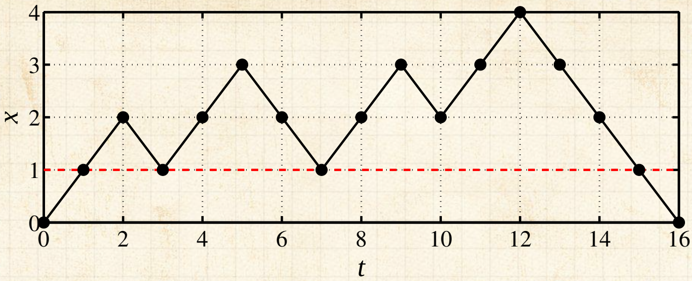

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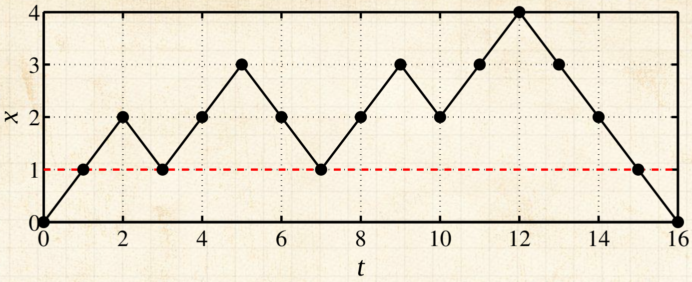

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Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
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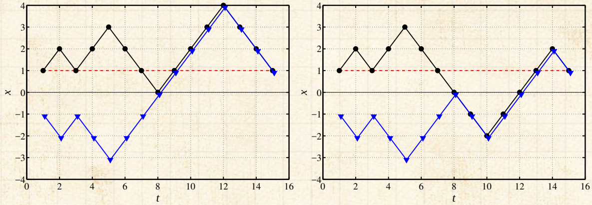

Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
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- # of t-step paths starting and ending at $x=1$ and hitting $x=0$ at least once
- So $N_{\text{first return}}(2n) = N(1,1,2n-2) - N(-1,1,2n-2)$

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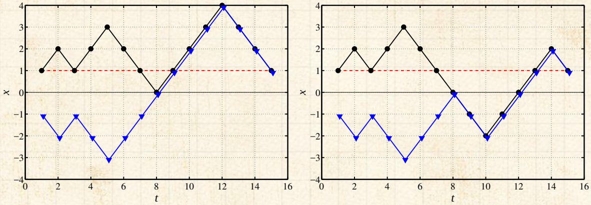

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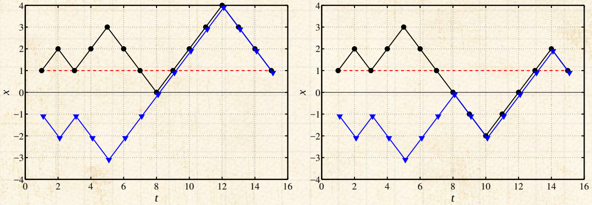

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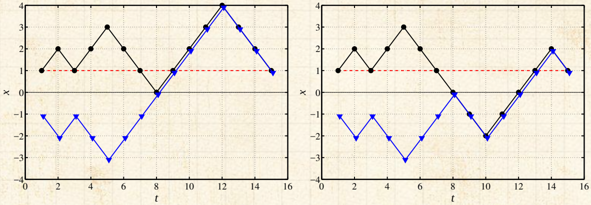

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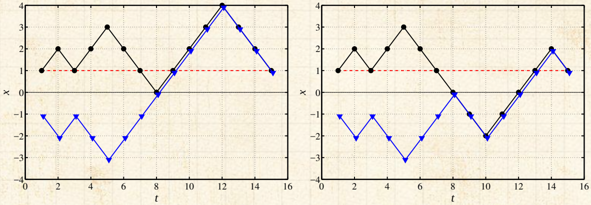

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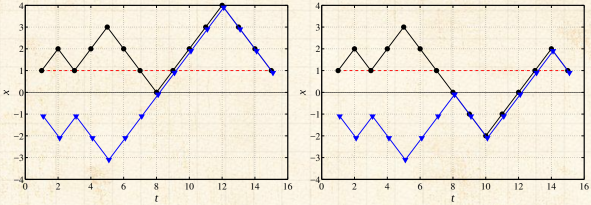
- So $N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$

Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
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Insert question from assignment 3 :

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- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
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Random walks

On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
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Outline

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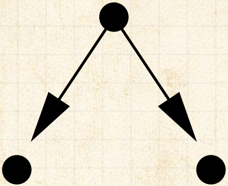

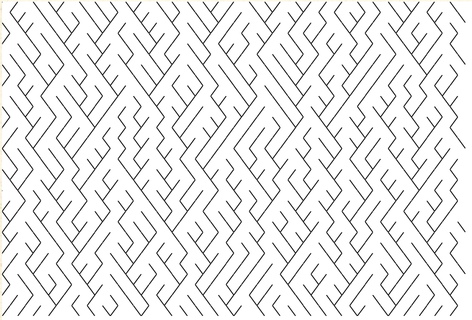

- Random directed network on triangular lattice.
- Toy model of real networks.
- ‘Flow’ is southeast or southwest with equal probability.

Scheidegger networks

- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

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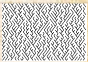

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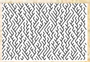

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- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
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Generalize relationship between area and length:

- Hack's law[5]:

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- Find τ in terms of γ and h.
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- A very simple model of failure/death: [11]
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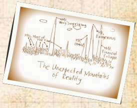
- Start with $x_0 > 0$.
- Entity fails when x hits 0.

Streams

- Dispersion of suspended sediments in streams.
- Long times for clearing.



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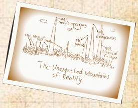

More than randomness

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- See Montroll and Shlesinger for example: [6]
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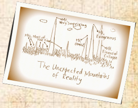

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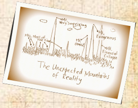

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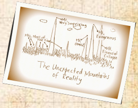

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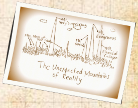

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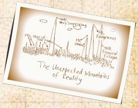
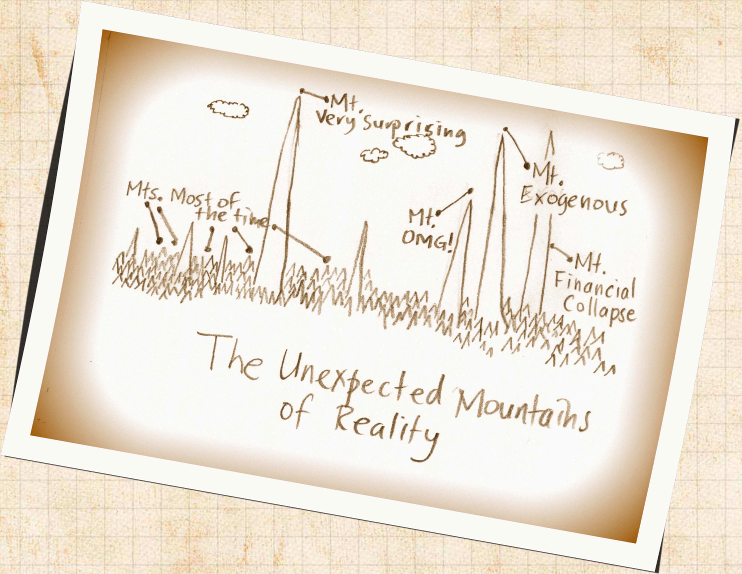
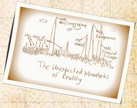
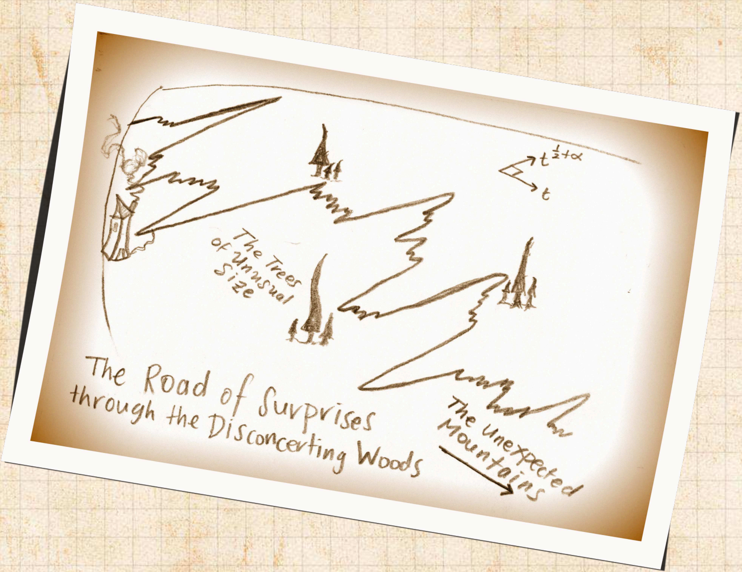
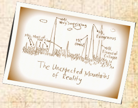



Mt. Very Surprising
Mts. Most of the time
Mt. OMG!
Mt. Exogenous
Mt. Financial Collapse
The Unexpected Mountains of Reality



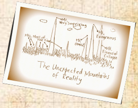

t^(1/2+α)
t
The Trees of Unusual Size
The Road of Surprises through the Disconcerting Woods
The Unexpected Mountains


The Unexpected Mountains of Reality

Neural reboot (NR):

Desert rain frog/Squeaky toy:

https://www.youtube.com/v/cBkWhkAZ9ds?rel=0




THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
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THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
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It was the best of times,
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CLASSIFIED
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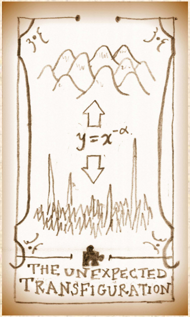
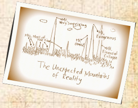
THE STORIES CONTAGIOUS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE JOHN DORY
THE STORYTELLERS
THE ADJACENT NARRATIVES
THE END


y = x^-α
THE UNEXPECTED TRANSFIGURATION

Outline



Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
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- $P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y))\frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$
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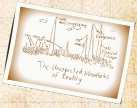

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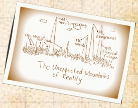

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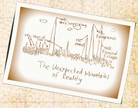

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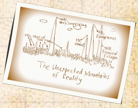

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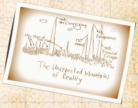

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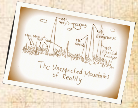

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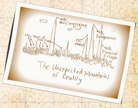

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

- $P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y))\frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$
- Often easier to do by hand...

General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
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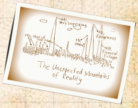
$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$



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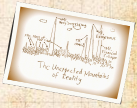

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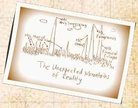

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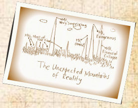

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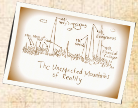

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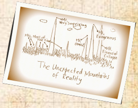

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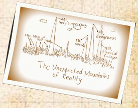

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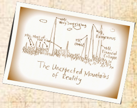

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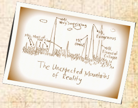

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$$P_y(y)\mathsf{d}y = P_x(x)\mathsf{d}x$$

- If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_x(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

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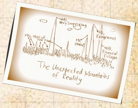

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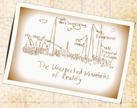

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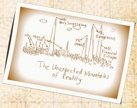

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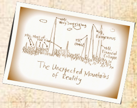


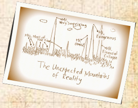
Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness (easy)...
- More later when we cover robustness.



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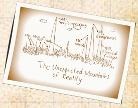

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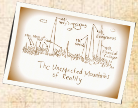

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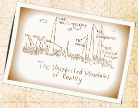

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- Select a random point in the universe $\vec{x}$
- Measure the force of gravity $F(\vec{x})$
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PoCS
Principles of Complex Systems
@pocsvox
What's the Story?

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Matter is concentrated in stars: [10]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-\frac{1}{2}}$$

- Connect differentials: $\mathrm{d}r \propto \mathrm{d}F^{-\frac{1}{2}} \propto F^{-\frac{3}{2}}\mathrm{d}F$

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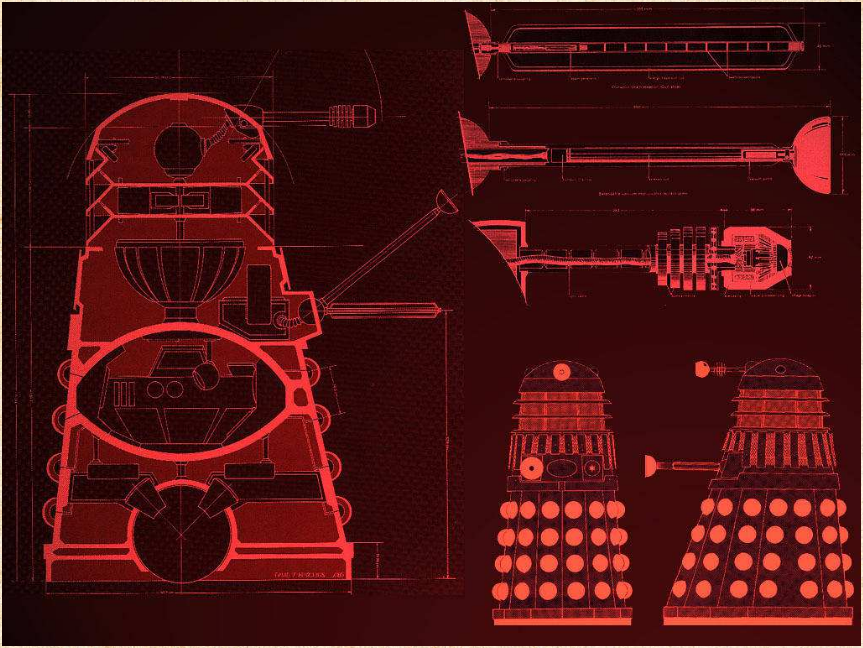
Doctorin' the Tardis

☐ Todo: Build Dalek army.



Outline

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument ...
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Neural reboot (NR):

Zoomage in slow motion

https://www.youtube.com/v/axrTxEVQqN4?rel=0

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