Lognormals and friends

Principles of Complex Systems | @pocsvox
CSYS/MATH 300, Fall, 2017

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

PoCS
Principles of Complex Systems
@pocsvox
What's the Story?

PoCS
Principles of Complex Systems
@pocsvox
What's the Story?

These slides are brought to you by:

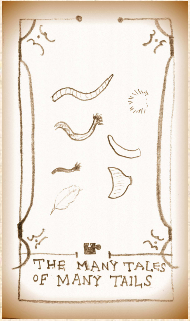
These slides are also brought to you by:

Special Guest Executive Producer: Pratchett

On Instagram at pratchett_the_cat

Outline


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
MINISTRY OF RANDOM WALKS
THE UNEXPECTED TRANSFIGURATION
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
THE DOORS THAT SHOULD NOT BE OPENED
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE ARK OF THE RANDOM NETWORK
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE RATIO OF GAIN
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE MECHANICS OF FAME
THE STORIES CONTAGIOUS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE JOHN DORY
THE STORYTELLERS
THE ADJACENT NARRATIVES
THE END


THE MANY TALES OF MANY TAILS

Outline

Alternative distributions

There are other 'heavy-tailed' distributions:

1. The Log-normal distribution

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions

$$P(x)\mathrm{d}x = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}}\mathrm{d}x$$

CCDF = stretched exponential.

3. Gamma distributions, and more.

Alternative distributions

There are other 'heavy-tailed' distributions:

1. The Log-normal distribution

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right)$$

2. Weibull distributions

$$P(x)\mathrm{d}x = \frac{k}{\lambda} \left(\frac{x}{\lambda} \right)^{\mu-1} e^{-(x/\lambda)^{\mu}} \mathrm{d}x$$

CCDF = stretched exponential.

3. Gamma distributions, and more.

Alternative distributions

There are other 'heavy-tailed' distributions:

1. The Log-normal distribution

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions

$$P(x)\mathrm{d}x = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}}\mathrm{d}x$$

   CCDF = stretched exponential.

3. Gamma distributions, and more.

lognormals

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- $\ln x$ is distributed according to a normal distribution with mean μ and variance σ.
- Appears in economics and biology where growth increments are distributed normally.

lognormals

- Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu + \frac{1}{2}\sigma^2}, \qquad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu - \sigma^2}.$$

- All moments of lognormals are finite.

lognormals

- Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu + \frac{1}{2}\sigma^2}, \qquad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu - \sigma^2}.$$

- All moments of lognormals are finite.

lognormals

- Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu + \frac{1}{2}\sigma^2}, \qquad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu-\sigma^2}.$$

- All moments of lognormals are finite.

Derivation from a normal distribution

Take Y as distributed normally:

$$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\mathrm{d}y$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

Derivation from a normal distribution

Take Y as distributed normally:

- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\mathrm{d}y$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

Derivation from a normal distribution

Take Y as distributed normally:

$$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\mathrm{d}y$$

Set $Y = \ln X$:

Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$

$$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

Derivation from a normal distribution

Take Y as distributed normally:

- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\mathrm{d}y$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

Derivation from a normal distribution

Take Y as distributed normally:

- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\mathrm{d}y$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

Derivation from a normal distribution

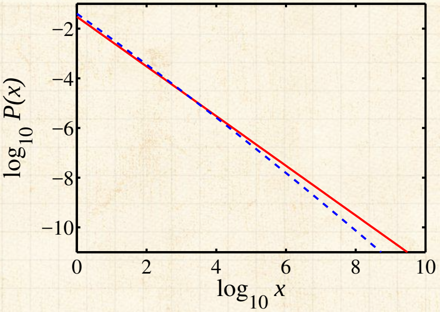
Take Y as distributed normally:

- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\mathrm{d}y$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude!

- For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- For power law (red), $\gamma = 1$ and $c = 0.03$.

Confusion

What's happening:

$$\ln P(x) = \ln\left\{\frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\right\}$$

$$= -\ln x - \ln\sqrt{2\pi}\sigma - \frac{(\ln x-\mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2}-1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\boxed{\ln P(x) \sim -\left(1-\frac{\mu}{\sigma^2}\right)\ln x + \text{const.}}$$

Confusion

What's happening:

$$\ln P(x) = \ln\left\{\frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\right\}$$

$$= -\ln x - \ln\sqrt{2\pi}\sigma - \frac{(\ln x-\mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2}-1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\boxed{\ln P(x) \sim -\left(1-\frac{\mu}{\sigma^2}\right)\ln x + \text{const.}}$$

Confusion

What's happening:

$$\ln P(x) = \ln\left\{\frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\right\}$$

$$= -\ln x - \ln\sqrt{2\pi}\sigma - \frac{(\ln x-\mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2}-1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\boxed{\ln P(x) \sim -\left(1-\frac{\mu}{\sigma^2}\right)\ln x + \text{const.}}$$

Confusion

What's happening:

$$\ln P(x) = \ln\left\{\frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\right\}$$

$$= -\ln x - \ln\sqrt{2\pi}\sigma - \frac{(\ln x-\mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2}-1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\ln P(x) \sim -\left(1-\frac{\mu}{\sigma^2}\right)\ln x + \text{const.}$$

Confusion

What's happening:

$$\ln P(x) = \ln\left\{\frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\right\}$$

$$= -\ln x - \ln\sqrt{2\pi}\sigma - \frac{(\ln x-\mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2}-1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\boxed{\ln P(x) \sim -\left(1-\frac{\mu}{\sigma^2}\right)\ln x + \text{const.}} \Rightarrow \boxed{\gamma = 1-\frac{\mu}{\sigma^2}}$$

PoCS
Principles of Complex Systems
@pocsvox
What's the Story?

Confusion

What's happening:

$$\ln P(x) = \ln\left\{\frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\right\}$$

$$= -\ln x - \ln\sqrt{2\pi}\sigma - \frac{(\ln x-\mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2}-1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\boxed{\ln P(x) \sim -\left(1-\frac{\mu}{\sigma^2}\right)\ln x + \text{const.}} \Rightarrow \boxed{\gamma = 1-\frac{\mu}{\sigma^2}}$$

Confusion

- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ,

$$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2}-1\right)\ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2-\mu)\log_{10} e$$

- $\Rightarrow$ If you find a -1 exponent,
  you may have a lognormal distribution...

Confusion

- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ,

$$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2} - 1\right)\ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu)\log_{10} e$$

- $\Rightarrow$ If you find a -1 exponent,
  you may have a lognormal distribution...

Confusion

- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ,

$$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2} - 1\right)\ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu)\log_{10} e \simeq 0.05(\sigma^2 - \mu)$$

- $\Rightarrow$ If you find a -1 exponent,
  you may have a lognormal distribution...

Confusion

- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ,

$$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1\right) \ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu)\log_{10} e \simeq 0.05(\sigma^2 - \mu)$$

- $\Rightarrow$ If you find a -1 exponent,
  you may have a lognormal distribution...

Confusion

- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ,

$$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

  $-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2} - 1\right)\ln x$

  $\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu)\log_{10} e \simeq 0.05(\sigma^2 - \mu)$

- $\Rightarrow$ If you find a -1 exponent,
  you may have a lognormal distribution...

Confusion

- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ,

$$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2} - 1\right)\ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu)\log_{10} e \simeq 0.05(\sigma^2 - \mu)$$

- $\Rightarrow$ If you find a -1 exponent,
  you may have a lognormal distribution...

Confusion

- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ,

$$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2} - 1\right)\ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu)\log_{10} e \simeq 0.05(\sigma^2 - \mu)$$

- $\Rightarrow$ If you find a -1 exponent, you may have a lognormal distribution...

Confusion

- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ,

$$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1\right) \ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu)\log_{10} e \simeq 0.05(\sigma^2 - \mu)$$

- $\Rightarrow$ If you find a -1 exponent,
  you may have a lognormal distribution...

PoCS
Principles of Complex Systems
@pocsvox
What's the Story?

Outline

Generating lognormals:

Random multiplicative growth:

- $$x_{n+1} = r x_n$$

  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:

  $$\ln x_{n+1} = \ln r + \ln x_n$$
- $\Rightarrow \ln x_n$ is normally distributed
- $\Rightarrow x_n$ is lognormally distributed

PoCS
Principles of Complex Systems
@pocsvox
What's the Story?

Generating lognormals:

Random multiplicative growth:

- $$x_{n+1} = r x_n$$

  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:

  $$\ln x_{n+1} = \ln r + \ln x_n$$

- $\Rightarrow \ln x_n$ is normally distributed
- $\Rightarrow x_n$ is lognormally distributed

Generating lognormals:

Random multiplicative growth:

- $$x_{n+1} = r x_n$$

  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:

  $$\ln x_{n+1} = \ln r + \ln x_n$$

- $\Rightarrow \ln x_n$ is normally distributed
- $\Rightarrow x_n$ is lognormally distributed

Generating lognormals:

Random multiplicative growth:

- $$x_{n+1} = r x_n$$
  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:
  $$\ln x_{n+1} = \ln r + \ln x_n$$
- $\Rightarrow \ln x_n$ is normally distributed
- $\Rightarrow x_n$ is lognormally distributed

Generating lognormals:

Random multiplicative growth:

- $$x_{n+1} = r x_n$$

  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:

  $$\ln x_{n+1} = \ln r + \ln x_n$$

- $\Rightarrow \ln x_n$ is normally distributed
- $\Rightarrow x_n$ is lognormally distributed

Lognormals or power laws?

- Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).
- One piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size. [1].

Lognormals or power laws?

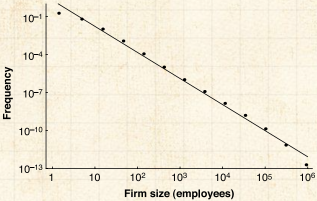
- Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).
- One piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size. [1].

Lognormals or power laws?

- Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).
- One piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size. [1].

Lognormals or power laws?

- Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).

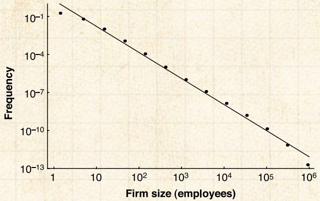


$$\text{Freq} \propto (\text{size})^{-\gamma}$$
$$\gamma \simeq 2$$

- One piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size. [1].

Lognormals or power laws?

- Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).



$\text{Freq} \propto (\text{size})^{-\gamma}$
$\gamma \simeq 2$

- One piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size. [1].

An explanation

- Axtel cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 2$
- The set up: N entities with size $x_i(t)$
- Generally:

$$x_i(t+1) = r x_i(t)$$

where r is drawn from some happy distribution
- Same as for lognormal but one extra piece.
- Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(r x_i(t), c\langle x_i \rangle)$$

An explanation

- Axtel cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 2$
- The set up: N entities with size $x_i(t)$
- Generally:

$$x_i(t+1) = r x_i(t)$$

where r is drawn from some happy distribution
- Same as for lognormal but one extra piece.
- Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(r x_i(t), c \langle x_i \rangle)$$

An explanation

- Axtel cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 2$
- The set up: N entities with size $x_i(t)$
- Generally:

$$x_i(t+1) = r x_i(t)$$

where r is drawn from some happy distribution
- Same as for lognormal but one extra piece.
- Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(r x_i(t), c \langle x_i \rangle)$$

An explanation

- Axtel cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 2$
- The set up: N entities with size $x_i(t)$
- Generally:

$$x_i(t+1) = r x_i(t)$$

where r is drawn from some happy distribution

- Same as for lognormal but one extra piece.
- Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(r x_i(t), c\langle x_i \rangle)$$

An explanation

- Axtel cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 2$
- The set up: N entities with size $x_i(t)$
- Generally:

$$x_i(t+1) = r x_i(t)$$

  where r is drawn from some happy distribution
- Same as for lognormal but one extra piece.
- Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(r x_i(t), c \langle x_i \rangle)$$

Some math later...

Insert question from assignment 7

- $$\text{Find} \quad P(x) \sim x^{-\gamma}$$

- where γ is implicitly given by

$$N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{(c/N)^{\gamma-1}-1}{(c/N)^{\gamma-1}-(c/N)}\right]$$

  N = total number of firms.

- $$\text{Now, if } c/N \ll 1 \text{ and } \gamma > 2 \quad N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{-1}{-(c/N)}\right]$$

- $$\text{Which gives} \quad \gamma \sim 1 + \frac{1}{1-c}$$

- Groovy... c small $\rightarrow \gamma \simeq 2$

Some math later...

Insert question from assignment 7

Find $P(x) \sim x^{-\gamma}$

where γ is implicitly given by

$$N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{(c/N)^{\gamma-1}-1}{(c/N)^{\gamma-1}-(c/N)}\right]$$

N = total number of firms.

Now, if $c/N \ll 1$ and $\gamma > 2$ $\quad N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{-1}{-(c/N)}\right]$

Which gives $\gamma \sim 1 + \frac{1}{1-c}$

Groovy... c small $\Rightarrow \gamma \simeq 2$

Some math later...

Insert question from assignment 7

- $$\text{Find} \quad P(x) \sim x^{-\gamma}$$
- where γ is implicitly given by

$$N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{(c/N)^{\gamma-1}-1}{(c/N)^{\gamma-1}-(c/N)}\right]$$

N = total number of firms.

- Now, if $c/N \ll 1$ and $\gamma > 2$ $\quad N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{-1}{-(c/N)}\right]$
- Which gives $\quad \gamma \sim 1 + \frac{1}{1-c}$
- Groovy... c small $\Rightarrow \gamma \simeq 2$

Some math later...

Insert question from assignment 7

- $$\text{Find} \quad P(x) \sim x^{-\gamma}$$
- where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$

N = total number of firms.

- Now, if $c/N \ll 1$ and $\gamma > 2$ $\quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$
- Which gives $\quad \gamma \sim 1 + \frac{1}{1-c}$
- Groovy... c small $\Rightarrow \gamma \simeq 2$

Some math later...

Insert question from assignment 7

- $$P(x) \sim x^{-\gamma}$$
  Find
- where γ is implicitly given by

$$N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{(c/N)^{\gamma-1}-1}{(c/N)^{\gamma-1}-(c/N)}\right]$$

N = total number of firms.

- Now, if $c/N \ll 1$ and $\gamma > 2$ $\quad N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{-1}{-(c/N)}\right]$

- Which gives $\quad \gamma \sim 1 + \frac{1}{1-c}$

- Groovy... c small $\Rightarrow \gamma \simeq 2$

PoCS

Principles of Complex Systems

@pocsvox

What's the Story?

Some math later...

Insert question from assignment 7

- Find $P(x) \sim x^{-\gamma}$
- where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$

N = total number of firms.

- Now, if $c/N \ll 1$ and $\gamma > 2$ $N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$

- Which gives $\gamma \sim 1 + \frac{1}{1 - c}$

- Groovy... c small $\Rightarrow \gamma \simeq 2$

PoCS
Principles of Complex Systems
@pocsvox
What's the Story?

Outline

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x-\mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.

Averaging lognormals

$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \mathrm{exp}\left(-\frac{(\ln \frac{x}{m})^2}{2t}\right) \mathrm{d}t$$

- Insert fabulous calculation (team is spared).
- Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln \frac{x}{m})^2}}$$

Averaging lognormals

$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(- \frac{(\ln \frac{x}{m})^2}{2t} \right) \mathrm{d}t$$

- Insert fabulous calculation (team is spared).
- Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda (\ln \frac{x}{m})^2}}$$

Averaging lognormals

$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln \frac{x}{m})^2}{2t}\right) \mathrm{d}t$$

- Insert fabulous calculation (team is spared).
- Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda (\ln \frac{x}{m})^2}}$$

The second tweak

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln \frac{x}{m})^2}}$$

- Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
- Double-Pareto distribution
- First noticed by Montroll and Shlesinger [7, 8]
- Later: Huberman and Adamic [3, 4]: Number of pages per website

The second tweak

- $$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln \frac{x}{m})^2}}$$
- Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
- Double-Pareto distribution
- First noticed by Montroll and Shlesinger [7, 8]
- Later: Huberman and Adamic [3, 4]: Number of pages per website

The second tweak

- $$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln\frac{x}{m})^2}}$$
- Depends on sign of $\ln\frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
- Double-Pareto distribution
- First noticed by Montroll and Shlesinger [7, 8]
- Later: Huberman and Adamic [3, 4]: Number of pages per website

PoCS
Principles of Complex Systems
@pocsvox
What's the Story?

The second tweak

- $$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln\frac{x}{m})^2}}$$
- Depends on sign of $\ln\frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
- Double-Pareto distribution
- First noticed by Montroll and Shlesinger [7, 8]
- Later: Huberman and Adamic [3, 4]: Number of pages per website

The second tweak

- $$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln\frac{x}{m})^2}}$$
- Depends on sign of $\ln\frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
- Double-Pareto distribution
- First noticed by Montroll and Shlesinger [7, 8]
- Later: Huberman and Adamic [3, 4]: Number of pages per website

The second tweak

- $$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln\frac{x}{m})^2}}$$
- Depends on sign of $\ln\frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
- Double-Pareto distribution
- First noticed by Montroll and Shlesinger [7, 8]
- Later: Huberman and Adamic [3, 4]: Number of pages per website

The second tweak

- $$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln \frac{x}{m})^2}}$$
- Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
- Double-Pareto distribution
- First noticed by Montroll and Shlesinger [7, 8]
- Later: Huberman and Adamic [3, 4]: Number of pages per website

Summary of these exciting developments:

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take-home message: Be careful out there...

Summary of these exciting developments:

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take-home message: Be careful out there...

Summary of these exciting developments:

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take-home message: Be careful out there...

Summary of these exciting developments:

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take-home message: Be careful out there...

Summary of these exciting developments:

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take-home message: Be careful out there...

References I

[1] R. Axtell.
Zipf distribution of U.S. firm sizes.
Science, 293(5536):1818–1820, 2001. pdf

[2] R. Gibrat.
Les inégalités économiques.
Librairie du Recueil Sirey, Paris, France, 1931.

[3] B. A. Huberman and L. A. Adamic.
Evolutionary dynamics of the World Wide Web.
Technical report, Xerox Palo Alto Research Center, 1999.

[4] B. A. Huberman and L. A. Adamic.
The nature of markets in the World Wide Web.
Quarterly Journal of Economic Commerce, 1:5–12, 2000.

References II

[5] O. Malcai, O. Biham, and S. Solomon.
Power-law distributions and lévy-stable intermittent fluctuations in stochastic systems of many autocatalytic elements.
Phys. Rev. E, 60(2):1299–1303, 1999. pdf

[6] M. Mitzenmacher.
A brief history of generative models for power law and lognormal distributions.
Internet Mathematics, 1:226–251, 2003. pdf

[7] E. W. Montroll and M. W. Shlesinger.
On $1/f$ noise and other distributions with long tails.
Proc. Natl. Acad. Sci., 79:3380–3383, 1982. pdf

References III

[8] E. W. Montroll and M. W. Shlesinger.
Maximum entropy formalism, fractals, scaling phenomena, and $1/f$ noise: a tale of tails.
J. Stat. Phys., 32:209–230, 1983.