Lognormals and friends

Principles of Complex Systems | @pocsvox
CSYS/MATH 300, Fall, 2017

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What's the Story?

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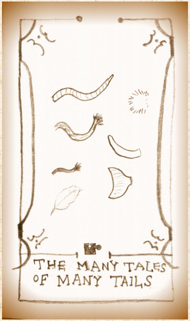
Outline

Lognormals
- Empirical Confusability
- Random Multiplicative Growth Model
- Random Growth with Variable Lifespan

References


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
THE WELL OF SCALES
THE LAWS OF DECAYING POWER
MINISTRY OF RANDOM WALKS
THE UNEXPECTED TRANSFIGURATION
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE KNOWLEDGE OF DEAD MONKS
THE DOORS THAT SHOULD NOT BE OPENED
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
It was the best of times,
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE ARK OF THE RANDOM NETWORK
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE NETWORKS FREE OF SCALE
CLASSIFIED
PHOENIX
BIO HAZARD
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE RATIO OF GAIN
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE MECHANICS OF FAME
THE STORIES CONTAGIOUS
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE JOHN DORY
THE STORYTELLERS
THE ADJACENT NARRATIVES
THE END


THE MANY TALES OF MANY TAILS

Alternative distributions

There are other 'heavy-tailed' distributions:

1. The Log-normal distribution

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions

$$P(x)\mathrm{d}x = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}}\mathrm{d}x$$

CCDF = stretched exponential.

3. Gamma distributions, and more.

lognormals

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- $\ln x$ is distributed according to a normal distribution with mean μ and variance σ.
- Appears in economics and biology where growth increments are distributed normally.

lognormals

- Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)$$

- For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu+\frac{1}{2}\sigma^2}, \qquad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2}-1)e^{2\mu+\sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu-\sigma^2}.$$

- All moments of lognormals are finite.

Derivation from a normal distribution

Take Y as distributed normally:

- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\mathrm{d}y$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

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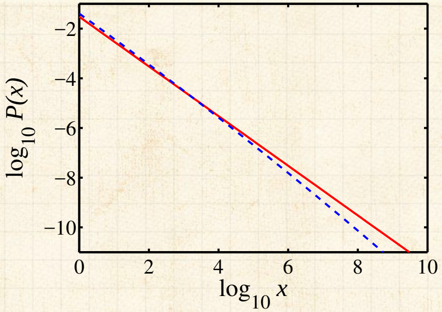
What's the Story?

Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude!

- For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- For power law (red), $\gamma = 1$ and $c = 0.03$.

Confusion

What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}$$

$$= -\ln x - \ln\sqrt{2\pi}\sigma - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\boxed{\ln P(x) \sim -\left(1 - \frac{\mu}{\sigma^2}\right)\ln x + \text{const.}} \Rightarrow \boxed{\gamma = 1 - \frac{\mu}{\sigma^2}}$$

Confusion

- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ,

$$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2} - 1\right)\ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu)\log_{10} e \simeq 0.05(\sigma^2 - \mu)$$

- $\Rightarrow$ If you find a -1 exponent,
  you may have a lognormal distribution...

Generating lognormals:

Random multiplicative growth:

- $$x_{n+1} = r x_n$$

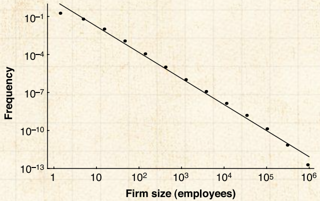
  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:

  $$\ln x_{n+1} = \ln r + \ln x_n$$

- $\Rightarrow \ln x_n$ is normally distributed
- $\Rightarrow x_n$ is lognormally distributed

Lognormals or power laws?

- Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).



Freq $\propto (\text{size})^{-\gamma}$

$\gamma \simeq 2$

- One piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size. [1].

An explanation

- Axtel cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 2$
- The set up: N entities with size $x_i(t)$
- Generally:

$$x_i(t+1) = r x_i(t)$$

  where r is drawn from some happy distribution
- Same as for lognormal but one extra piece.
- Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(r x_i(t), c \langle x_i \rangle)$$

Some math later...

Insert question from assignment 7

- Find $P(x) \sim x^{-\gamma}$
- where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$

N = total number of firms.

- Now, if $c/N \ll 1$ and $\gamma > 2$ $\quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$
- Which gives $\gamma \sim 1 + \frac{1}{1-c}$
- Groovy... c small $\Rightarrow \gamma \simeq 2$

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.

Averaging lognormals

$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln\frac{x}{m})^2}{2t}\right) \mathrm{d}t$$

- Insert fabulous calculation (team is spared).
- Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln\frac{x}{m})^2}}$$

The second tweak

- $$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln\frac{x}{m})^2}}$$
- Depends on sign of $\ln\frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
- Double-Pareto distribution
- First noticed by Montroll and Shlesinger [7, 8]
- Later: Huberman and Adamic [3, 4]: Number of pages per website

Summary of these exciting developments:

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take-home message: Be careful out there...

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