P o C S What's The Story?

Principles of Complex Systems, CSYS/MATH 300
University of Vermont, Fall 2017
Assignment 8 • code name: Sure as heckfire!

Dispersed: Saturday, October 21, 2017.
Due: 11:59 pm, Friday, November 10, 2017.
Some useful reminders:
Deliverator: Peter Dodds
Office: Farrell Hall, second floor, Trinity Campus
E-mail: peter.dodds+pocs@uvm.edu
Office hours: 1:15 pm to 2:30 pm on Tuesday, 1:15 pm to 4:45 pm Thursday
Course website: http://www.uvm.edu/pdodds/teaching/courses/2017-08UVM-300
Bonus course notes: http://www.uvm.edu/pdodds/teaching/courses/2017-08UVM-300/docs/dewhurst-pocs-notes.pdf

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All parts are worth 3 points unless marked otherwise. Please show all your workingses clearly and list the names of others with whom you collaborated.

Please obey the basic life rule: Never use Excel. Or any Microsoft product except maybe Xbox (which sadly will likely not help you here.)

Graduate students are requested to use LaTeX (or related TeX variant).

Email submission: PDF only! Please name your file as follows (where the number is to be padded by a 0 if less than 10 and names are all lowercase):
`CSYS300assignment%02d$firstname-$lastname.pdf` as in
`CSYS300assignment06michael-palin.pdf`

Please submit your project's current draft in pdf format via email. Please use this file name format (all lowercase after CSYS):
`CSYS300project-$firstname-$lastname-YYYY-MM-DD.pdf` as in
`CSYS300project-lisa-simpson-1989-12-17.pdf` where the date is the date of submission (and not, say, your birthdate).

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1. (3 + 3 + 3 + 3) This question is all about pure finite and infinite random networks

   We'll define a finite random network as follows. Take N labelled nodes and add links between each pair of nodes with probability p.

   (a) i. For a random node i, determine the probability distribution for its number of friends k, $P_k(p, N)$.

ii. What kind of distribution is this?

iii. What does this distribution tend toward in the limit of large N, if p is fixed?
(No need to do calculations here; just invoke the right Rule of the Universe.)

(b) Using $P_k(p, N)$, determine the average degree. Does your answer seem right intuitively?

(c) Show that in the limit of $N \to \infty$ but with mean held constant, we obtain a Poisson degree distribution.

Hint: to keep the mean constant, you will need to change p.

(d) i. Compute the clustering coefficients C_1 and C_2 for standard finite random networks (N nodes).

ii. Explain how your answers make sense.

iii. What happens in the limit of an infinite random network with finite mean?

2. (3 + 3)

Determine the clustering coefficient for toy model small-world networks [1] as a function of the rewiring probability p. Find C_1, the average local clustering coefficient:

$$C_1(p) = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i = \frac{1}{N} \sum_{i=1}^{N} \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2}$$

where N is the number of nodes, $a_{ij} = 1$ if nodes i and j are connected, and $\mathcal{N}_i$ indicates the neighborhood of i.

As per the original model, assume a ring network with each node connected to a fixed, even number m local neighbors ($m/2$ on each side). Take the number of nodes to be $N \gg m$.

Start by finding $C_1(0)$ and argue for a $(1-p)^3$ correction factor to find an approximation of $C_1(p)$.

Hint 1: you can think of finding C_1 as averaging over the possibilities for a single node.

Hint 2: assume that the degree of individual nodes does not change with rewiring but rather stays fixed at m. In other words, take the average degree of individuals as the degree of a randomly selected individual.

For what value of p is $C_1 \simeq 1/2$?

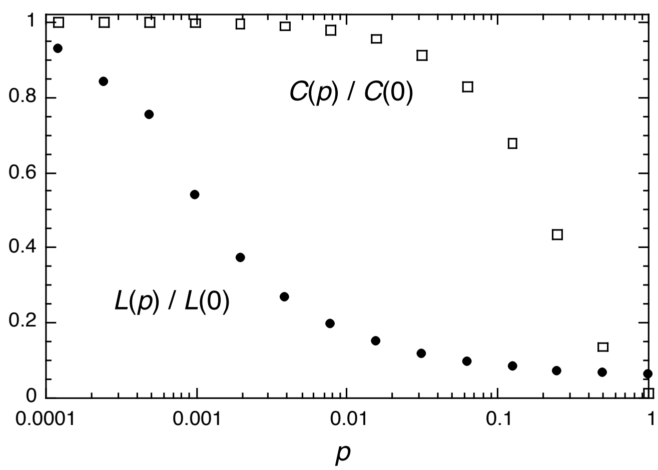
(3 points for set up, 3 for solving.)

3. Simulate the small-world model and reproduce Fig. 2 from the 1998 Watts-Strogatz paper showing how clustering and average shortest path behave with rewiring probability p [1].

   Please find and use any suitable code online, and feel free to Share with each other via Slack.

   Use $N = 1000$ nodes and $k = 10$ for average degree, and vary p from 0.0001 to 1, evenly spaced on a logarithmic scale (there are only 14 values used in the paper).

   Here's the figure you're aiming for:



4. (3 + 3)

   More on the peculiar nature of distributions of power law tails:

   Consider a set of N samples, randomly chosen according to the probability distribution $P_k = ck^{-\gamma}$ where $k \geq 1$ and $2 < \gamma < 3$. (Note that k is discrete rather than continuous.)

   (a) Estimate $\min k_{\max}$, the approximate minimum of the largest sample in the network, finding how it depends on N.

   (Hint: we expect on the order of 1 of the N samples to have a value of $\min k_{\max}$ or greater.)

Hint—Some visual help on setting this problem up:

Direct link: `http://www.youtube.com/watch?v=4tqlEuXA7QQ`

(b) Determine the average value of samples with value $k \geq \min k_{\max}$ to find how the expected value of $k_{\max}$ (i.e., $\langle k_{\max} \rangle$) scales with N.

For language, this scaling is known as Heap's law.

5. (3 + 3)

Let's see how well your answer for the previous question works.

For $\gamma = 5/2$, generate $n = 1000$ sets each of $N = 10$, 10^2, 10^3, 10^4, 10^5, and 10^6 samples, using $P_k = ck^{-5/2}$ with $k = 1, 2, 3, \ldots$

How do we computationally sample from a discrete probability distribution?

Hint: You can use a continuum approximation to speed things up. In fact, taking the exact continuum version from the first two assignments will work.

(a) For each value of sample size N, plot the maximum value of the $n = 1000$ samples as a function of sample number (which is not the sample size N). So you should have $k_{\max}$ for $i = 1, 2, \ldots, n$ where i is sample number. These plots should give a sense of the unevenness of the maximum value of k, a feature of power-law size distributions.

(b) For each set, find the maximum value. Then find the average maximum value for each N. Plot $\langle k_{\max} \rangle$ as a function of N and calculate the scaling using least squares.

Does your scaling match up with your theoretical estimate?

References

[1] D. J. Watts and S. J. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393:440–442, 1998. pdf