From
The Book of Strang

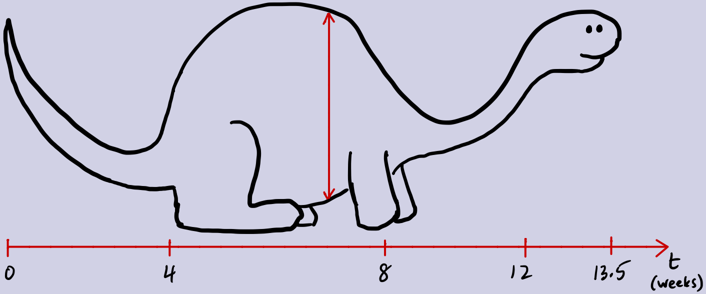
Matrixology

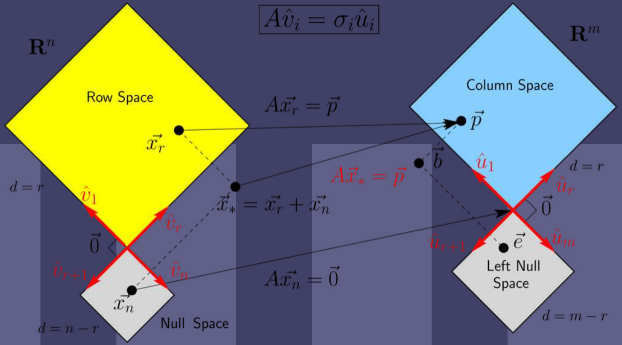
(linear algebra)

Prof Peter Sheridan Dodds

Recorded in 2016

Melvin the Course Difficulty Dinosaur:





The Central problem of Matrixology:

Given a matrix $\mathbb{A}_{m\times n}$ and a vector $\vec{b}_{m\times 1}$

find all $\vec{x}_{n\times 1}$ such that

$$\underset{m\times n}{\mathbb{A}}\ \underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$$

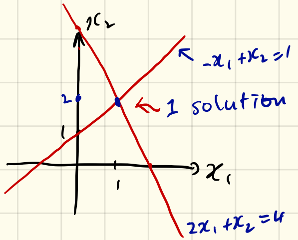
$$\left.\begin{aligned} -\overset{1.}{x_1} + \overset{1.}{x_2} &= 1 \quad ① \\ 2x_1 + \overset{1.}{x_2} &= 4 \quad ② \end{aligned}\right\}$$

system of linear equations

2 x 2 system

m rows = # equations

n columns = # variables



Row picture

Usual way:

$\boxed{-x_1} + x_2 = 1$ ... ①

$2x_1 + x_2 = 4$ ... ②

eq ③ = eq ② + 2 eq ①

$3x_2 = 6$ .. ③

$\Rightarrow x_2 = 2$

substitute into eq ①

$-x_1 + 2 = 1$

$\Rightarrow x_1 = 1$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

notes

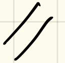
- Found intersection of two lines
- Both equations are true at this one point

yes!!

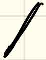
Algebra ⇄ Geometry

Three possibilities:

a) 1 soln ;



b) no soln ; c)



same line. ∞ly many soln.

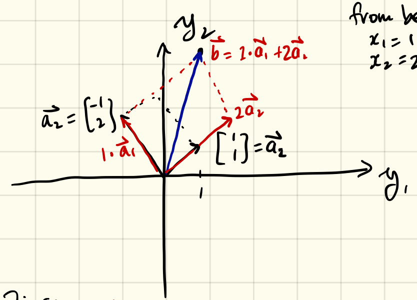
Column picture

Rewrite system as

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$\vec{a}_1$ $\vec{a}_2$ building blocks $\vec{b}$

from before:
$x_1 = 1$
$x_2 = 2$



Three possibilities:

a) $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$

⇒ ∞ly many solns

b) $\vec{a}_1 \nparallel \vec{a}_2$

→ 1 soln (can always make $\vec{b}$ in one way only)

c) $\vec{a}_1 \parallel \vec{a}_2 \nparallel \vec{b}$

→ 0 solns.

E1ap2

$$\underbrace{\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}}} = \underset{m\times 1}{\vec{b}}$$

Matrix Picture:

$$\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Two ways to multiply matrices.

① dot products of rows and columns

$$\underset{2\times 2}{\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}}\underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} -x_1+x_2 \\ 2x_1+x_2 \end{bmatrix}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Row picture: $-x_1+x_2=1$

$2x_1+x_2=4$

②

$$\begin{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

row vector 1×2

E16p1

3x3 example: $A\vec{x} = \vec{b}$

Matrix picture:

$$m\left\{\underset{\substack{3\times3\\ m\ \ \ n}}{\overbrace{\begin{bmatrix}2&1&0\\-1&1&2\\0&3&1\end{bmatrix}}^{n}}\right.\underset{\substack{n\times1\\3\times1}}{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}=\underset{\substack{m\times1\\3\times1}}{\begin{bmatrix}2\\2\\6\end{bmatrix}}$$

← find $\vec{x}$ such that A transforms $\vec{x}$ into $\vec{b}$

Multiply out:

Row picture:

$$\begin{aligned}2x_1 + x_2 \qquad &= 2\\ -x_1 + x_2 + 2x_3 &= 2\\ 3x_2 + x_3 &= 6\end{aligned}$$

← equations of planes in 3-d

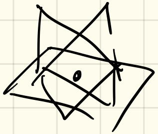
More sneakily

Column picture:

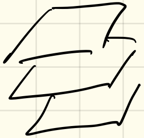
$$\underset{0}{x_1}\begin{bmatrix}2\\-1\\0\end{bmatrix}+\underset{2}{x_2}\begin{bmatrix}1\\1\\3\end{bmatrix}+\underset{0}{x_3}\begin{bmatrix}0\\2\\1\end{bmatrix}=\begin{bmatrix}2\\2\\6\end{bmatrix}$$

See: $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\2\\0\end{bmatrix}$ (Always takes more work than this !!!)

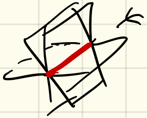
Row picture:



1 soln



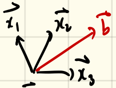
0 soln



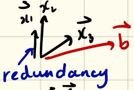
← ∞ly many solns

↖ way too hard in 4-d and above...

Column picture:



1 soln.



0 or ∞ly many soln, } depends on $\vec{b}$.

0 solns or ∞ly many }

← easy in many dimensions

Story: We (people + computers) solve systems of linear equations by "Elimination"

← Gaussian & Gauss-Jordan

Menu:

- Perform Elimination using Row Operations
- Anatomy of Row operations • Triangles !!
- Back Substitution
- key: Pivots D_i, multipliers l_{ij}, upper triangular matrix $\mathcal{U}$
  Augmented Matrix
- when things go "wrong"

$A\vec{x} = \vec{b}$

$$\begin{cases} \overset{-1}{-x_1} + 3x_2 = 1 \quad \dots ① \quad R_1 \\ ②x_1 + x_2 = 5 \quad \dots ② \quad R_2 \end{cases}$$

eliminate

2x2 system

$m=2$ = # eqs

$n=2$ = # variables

$D_1 = -1$

$-x_1 + 3x_2 = 1 \quad \dots \quad R_1$

$0x_1 + 7x_2 = 7 \quad \dots$

upper triangular

D_2

$$R_2' = R_2 - \underbrace{\left(\frac{2}{-1}\right)}_{+2} R_1$$

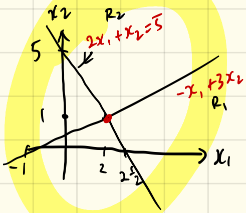
l_{21}

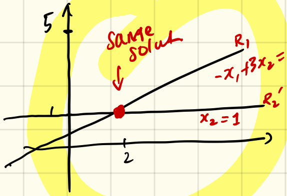
always use this form

D_1 = first Pivot without division

$D_2 = 7$

R_2 (drop primes): $x_2 = 1$





E2ap1

We have $x_2 = 1$, now solve for x_1 using back substitution:

$$-x_1 + 3x_2 = 1 \Rightarrow -x_1 + 3 = 1$$

↖ $x_2 = 1$

$$x_1 = 2$$

solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ✓

For later, we can go further and avoid back substitution.

Gauss-Jordan elimination:

$$\begin{cases} -x_1 + 3x_2 = 1 \quad \dots \quad R_1 \\ 0 + 1x_2 = 1 \quad \dots \quad R_2 \end{cases}$$

$$R_1' = R_1 - \left(\frac{3}{1}\right) R_2$$

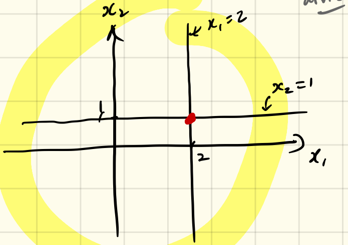
$D_2 = 7$ before division

$$\begin{cases} -x_1 + 0 = -2 \\ 0 + x_2 = 1 \end{cases}$$

$R_2' = R_2$

$\Rightarrow +x_1 = 2$

$x_2 = 1$



(E2ap2

Basic Elimination rules:

① Create upper trianglur system by systematic by row operations

② Swap rows if needed when pivots = 0

$$\begin{aligned} 0 + x_2 &= 3 \\ 3x_1 - 7x_2 &= 0 \end{aligned} \quad \underset{R_1 \leftrightarrow R_2}{\sim} \quad \begin{aligned} 3x_1 - 7x_2 &= 0 \\ x_2 &= 3 \end{aligned}$$

Augmented Matrix approach:

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_2 + x_2 &= 5 \end{aligned} \Rightarrow \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Row picture

$A\vec{x} = \vec{b}$, matrix

$$D_1 \rightarrow \left[\begin{array}{cc|c} -1 & 3 & 1 \\ 2 & 1 & 5 \end{array}\right] \sim R_2' = R_2 - \left(\frac{2}{-1}\right) \underset{D_1}{R_1} \quad \left[\begin{array}{cc|c} \overset{D_1}{-1} & 3 & 1 \\ 0 & \underset{D_2}{7} & 7 \end{array}\right]$$

$[A \mid \vec{b}]$

~ means systems have same solut.

Menu:

- 3x3 example of solving $A\vec{x}=\vec{b}$ with Elimination and Row Swaps
- Turn $A\vec{x}=\vec{b}$ into $U\vec{x}=\vec{c}$

upper triangul

Row picture:

$2x_1 - 3x_2 + 0.x_3 = 3$ eq1

$4x_1 - 5x_2 + 1.x_3 = 7$ eq2

$2x_1 - 1x_2 - 3x_3 = 5$ eq3

Three planes

Column Picture:

$$x_1\underset{\vec{a}_1}{\begin{bmatrix}2\\4\\2\end{bmatrix}} + x_2\underset{\vec{a}_2}{\begin{bmatrix}-3\\-5\\-1\end{bmatrix}} + x_3\underset{\vec{a}_3}{\begin{bmatrix}0\\1\\-3\end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix}3\\7\\5\end{bmatrix}}$$

Matrix Picture

$$\begin{bmatrix}2&-3&0\\4&-5&1\\2&-1&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3\\7\\5\end{bmatrix} \qquad A\vec{x}=\vec{b}$$

Augmented Matrix version of row picture:

$$\begin{bmatrix}2&-3&0&|&3\\4&-5&1&|&7\\2&-1&-3&|&5\end{bmatrix}\begin{matrix}R_1\\R_2\\R_3\end{matrix}$$

① ② ③ 0s

$$[\underbrace{A}_{m\times n} \mid \underbrace{\vec{b}}_{m\times 1}] \quad m\times(n+1)$$

ⓘ = order of eliminat.

multiplier

$$R_2' = R_2 - \left(\frac{4}{2}\right)R_1$$

l_{21} D_1

$$\begin{bmatrix}2&-3&0&|&3\\0&1&1&|&1\\2&-1&-3&|&5\end{bmatrix}$$

$$R_3' = R_3 - \left(\frac{2}{2}\right)R_1$$

l_{31} D_1

$$\begin{bmatrix}2&-3&0&|&3\\0&1&1&|&1\\0&2&-3&|&2\end{bmatrix}$$

D_1 D_2

$$R_3' = R_3 - \left(\frac{2}{1}\right)R_2$$

l_{32} D_2

$$\begin{bmatrix}2&-3&0&|&3\\0&1&1&|&1\\0&0&-5&|&0\end{bmatrix}$$

main diagonal

$$\underset{A}{U}\vec{x} = \underset{\vec{b}}{\vec{c}} \Rightarrow \begin{bmatrix}2&-3&0\\0&1&1\\0&0&-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3\\1\\0\end{bmatrix}$$

easy to solve with back substitut

Back substitution:

Step back out to equations and work upwards:

R_3: $-5x_3 = 0 \Rightarrow x_3 = 0$

R_2: $x_2 + \underset{0}{x_3} = 1 \Rightarrow x_2 = 1$

R_1: $2x_1 - 3\underset{=1}{x_2} = 3 \Rightarrow 2x_1 - 3 = 3$

$2x_1 = 6$

$x_1 = 3$

Solution:

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3\\1\\0\end{bmatrix}$$

Important:

$l_{21} = 2$ $\quad l_{32} = 2$

$l_{31} = 1$ $\quad$ multipliers

Pivots: find in U:

$D_1 = 2,\ D_2 = -1,\ D_3 = 5$

Menu:

- What can happen when a pivot is zilch....
- Singular system

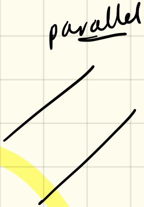
ex 1

$-x_1 + x_2 = 1$ .... R_1

$x_1 - x_2 = 5$ ... R_2

+1

$$\Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 1 \\ 1 & -1 & 5 \end{array}\right]$$



"has same solution as"

$$R_2' = R_2 - \left(\frac{1}{-1}\right) R_1$$

l_{21}

D_1

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 0 & 0 & 6 \end{array}\right]$$

0 pivot

pivot is "missing"

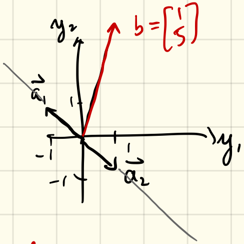
R_2: $0x_1 + 0x_2 = 6$

$0 = 6$

not true!

Column picture:

$$x_2 \underset{\vec{a}}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} + x_2 \underset{\vec{b}}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \overset{\vec{b}}{\begin{pmatrix} 1 \\ 5 \end{pmatrix}}$$



example of

Singular System

→ no unique solutn

may have 0 or ∞ly many

E2cp1

ex 2

$-x_1 + x_2 = 1$ ..... R_1

$2x_1 - 2x_2 = -2$ ..... R_2

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 2 & -2 & -2 \end{array}\right]$$

$A x = \vec{b}$

$$R_2' = R_2 - \left(\frac{2}{-1}\right) R_1$$

$l_{21} = -2$

D_1

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$U\vec{x} = \vec{c}$

singular matrix

eqs: $-x_1 + x_2 = 1$ ... R_1

~~$0 = 0$~~ ... R_2

later

pivot variable

free variable

Let $x_2 \in R^1$ ie real numbers → x_1 now depends on x_2

$-x_1 = 1 - x_2$

$x_1 = x_2 - 1$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 - 1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$1 \cdot x_2 - 1$

$1 \cdot x_2 + 0$

replace pivot variables with free variables

dialable piece

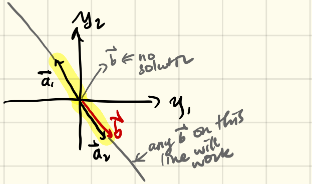
fixed

where $x_2 \in R^1$

Column pic

$$x_1 \underset{\vec{a}_1}{\begin{pmatrix} -1 \\ 2 \end{pmatrix}} + x_2 \underset{\vec{a}_2}{\begin{pmatrix} 1 \\ -2 \end{pmatrix}} = \underset{\vec{b}}{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

$\vec{a}_1 = -\vec{a}_2$



- Our task: Solve systems of linear equations
- Three pictures: row, column, & matrix.

where solving happens (row) · understanding (column) · deep understanding (matrix)

2x2 example from Episode 2

$$\underbrace{\begin{array}{ll} -x_1+3x_2=1 & \leftarrow \text{Row1} \\ 2x_1+x_2=5 & \nwarrow \text{Row2} \end{array}}_{\text{row pic}} \Leftrightarrow \underbrace{x_1\overset{\vec{a}_1}{\begin{bmatrix}-1\\2\end{bmatrix}}+x_2\overset{\vec{a}_2}{\begin{bmatrix}3\\1\end{bmatrix}}=\overset{\vec{b}}{\begin{bmatrix}1\\5\end{bmatrix}}}_{\text{column pic}} \Leftrightarrow \underbrace{\overset{A}{\begin{bmatrix}-1&3\\2&1\end{bmatrix}}\overset{\vec{x}}{\begin{bmatrix}x_1\\x_2\end{bmatrix}}=\overset{\vec{b}}{\begin{bmatrix}1\\5\end{bmatrix}}}_{\text{matrix pic}}$$

Solve by Gaussian Elimination

Equations (first) ⟷ same ⟷ Augmented Matrix (second) ↙ tidy

$-x_1+3x_2=1$ ... R1

$2x_1+x_2=5$ ... R2

multiplier ℓ_{21} ↓

$\Rightarrow$ $-x_1+3x_2=1$ ... R1

$0x_1+7x_2=7$ ... $R2'=R2-\left(\frac{2}{-1}\right)R_1$

↑ echelon form

$$\left[\begin{array}{cc|c}-1&3&1\\2&1&5\end{array}\right]$$

$$R_2'=R_2-\left(\frac{2}{\boxed{-1}}\right)R_1 \sim \left[\begin{array}{cc|c}-1&3&1\\0&7&7\end{array}\right]$$

$D_1=-1=$ first pivot

multiplier $\ell_{21}=-2$

↑ echelon form

Matrix picture:

$$A\vec{x}=\vec{b} \Rightarrow \underset{\begin{bmatrix}-1&3\\0&7\end{bmatrix}}{U}\vec{x}=\vec{c} \searrow \begin{bmatrix}1\\7\end{bmatrix}$$

The Gaussian Eliminator 9000:

$$\left[\begin{array}{ccc|c}\overset{A}{2}&-3&0&\overset{\vec{b}}{3}\\ ①\rightarrow\textcircled{4}&-5&1&7\\ \underset{②\nearrow}{2}&\underset{③\nearrow}{-1}&-3&5\end{array}\right] \leftarrow \text{Augmented Matrix for } A\vec{x}=\vec{b}$$

multiplier $\ell_{21}=2$

$$R_2'=R_2-\left(\frac{4}{2}\right)R_1 \quad \left[\begin{array}{ccc|c}2&-3&0&3\\0&1&1&1\\2&-1&-3&5\end{array}\right]$$

$\nwarrow D_1$

big tilde → $\sim$ ↑ equivalent to

rowoperatorify

↓

$\ell_{31}=1$

$$R_3'=R_3-\left(\frac{2}{2}\right)R_1 \quad \left[\begin{array}{ccc|c}2&-3&0&3\\0&1&1&1\\0&2&-3&2\end{array}\right]$$

$\nwarrow D_1$

$\sim$

↓

$\ell_{32}=2$

$$R_3'=R_3-\left(\frac{2}{1}\right)R_2 \quad \left[\begin{array}{ccc|c}\overset{D_1}{2}&-3&0&3\\0&\overset{D_2}{1}&1&1\\0&0&\overset{D_3}{-5}&0\end{array}\right]$$

$\nwarrow D_2$

$\sim$

#eliminated ↗ (0, 0, 0) ↖ echelon form

$$U\vec{x}=\vec{c}$$

⇓

easy to solve with back substitution

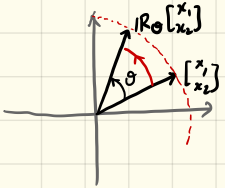
Menu:
- Using Elimination matrices to do the work for us
- Surprising help for our understanding will be possible
- Somehow, elimination makes two triangles.

Observation:

Matrices can do sneaky, gadgety things for us

ex Rotate a vector in 2-d through θ radians

$$\underbrace{\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}}_{R_\theta}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$



ex Permute entries in a vector:

$$\begin{bmatrix}0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix} = \begin{bmatrix}x_2 \\ x_3 \\ x_4 \\ x_1\end{bmatrix}$$

← cycle by 1.

Plan: encode row operations as
(1) elimination matrices ← normal elimination steps
& (2) permutation matrices ← row swaps

E3ap1

Augmented Matrix approach:

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_2 + x_2 &= 5 \end{aligned} \Rightarrow \begin{bmatrix}-1 & 3 \\ 2 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}1 \\ 5\end{bmatrix}$$

Row picture

$A\vec{x}=\vec{b}$, matrix

D_1

$$\left[\begin{array}{cc|c} \boxed{-1} & 3 & 1 \\ \boxed{2} & 1 & 5 \end{array}\right] \sim R_2' = R_2 - \underbrace{\left(\frac{2}{-1}\right)}_{\ell_{21} = -2} R_1 \quad \left[\begin{array}{cc|c} -1 & 3 & 1 \\ 0 & 7 & 7 \end{array}\right]$$

D_1 D_1 D_2

$[A \mid \vec{b}]$

~ means systems have same solut.

replace w. matrix multiplication

E_{21} = elimination matrix that removes the 2,1 entry in A or 1st entry in 2nd row.

row column

here

$$E_{21} = \begin{bmatrix}1 & 0 \\ \underbrace{-\frac{2}{-1}}_{-\ell_{21}} & 1\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 2 & 1\end{bmatrix}$$

Let's see how this works:

E3ap2

$$\underset{\mathbb{E}_{21}}{\begin{bmatrix}1&0\\2&1\end{bmatrix}}\underset{\mathbb{A}}{\begin{bmatrix}-1&3\\2&1\end{bmatrix}}\underset{\vec{x}}{\begin{bmatrix}x_1\\x_2\end{bmatrix}} = \underset{\mathbb{E}_{21}}{\begin{bmatrix}1&0\\2&1\end{bmatrix}}\underset{\vec{b}}{\begin{bmatrix}1\\5\end{bmatrix}}$$

↑ premultiply both sides

$$\underset{\mathbb{U}}{\begin{bmatrix}-1&3\\0&7\end{bmatrix}}\underset{\vec{x}}{\begin{bmatrix}x_1\\x_2\end{bmatrix}} = \underset{\vec{c}}{\begin{bmatrix}1\\7\end{bmatrix}}$$

Anatomy of $\mathbb{E}_{21}$:

$$\begin{bmatrix}1&0\\2&1\end{bmatrix}$$

keep copy of first row $R_1' = R_1$

add 2× first row to second row to make new second row

$$R_2' = R_2 - \underset{\nwarrow 2}{\ell_{21}} R_1$$

3×3 example:

We need $\mathbb{E}_{21}$ ✓, $\mathbb{E}_{31}$, & $\mathbb{E}_{32}$

(ℓ_{21}), (ℓ_{31}), (ℓ_{32})

ex

$$\begin{bmatrix}1&0&-1\\\boxed{2}&3&1\\3&6&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2\\2\\4\end{bmatrix}$$

Row op

$$R_2' = \underset{\uparrow}{1}R_2 - \overset{\ell_{21}}{\left(\frac{2}{\underset{D_1}{1}}\right)} R_1$$

Eliminat matrix

$$\mathbb{E}_{21} = \begin{bmatrix}1&0&0\\\underset{-\ell_{21}}{\boxed{-2}}&1&0\\0&0&1\end{bmatrix}$$

← mostly indentity matrix $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$$R_3' = 1\cdot R_3 - \overset{\ell_{31}=3}{\left(\frac{3}{\underset{D_2}{1}}\right)} R_1$$

$$\mathbb{E}_{31} = \begin{bmatrix}1&0&0\\0&1&0\\\underset{-\ell_{31}}{\boxed{-3}}&0&1\end{bmatrix}$$

Must use elimination matrices to get to $\mathbb{E}_{32}$

E3ap3

$$\underset{\mathbb{E}_2}{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}\underset{\mathbb{A}}{\begin{bmatrix}1&0&-1\\2&3&1\\3&6&2\end{bmatrix}}\underset{\vec{x}}{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}} = \underset{\mathbb{E}_{21}}{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}\underset{\vec{b}}{\begin{bmatrix}-2\\2\\4\end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix}1&0&-1\\0&3&3\\3&6&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2\\6\\4\end{bmatrix}$$

next: premultiply by $\mathbb{E}_{31} = \begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}1&0&-1\\0&\textcircled{3}&3\\0&\textcircled{6}&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2\\6\\10\end{bmatrix}$$

Important: can now see next row op

$$R_3' = R_3 - \overset{\ell_{21}=2}{\left(\frac{6}{\underset{D_2}{3}}\right)} R_2 \quad \Longleftrightarrow \quad \mathbb{E}_{32} = \begin{bmatrix}1&0&0\\0&1&0\\0&\underset{-\ell_{32}}{-2}&1\end{bmatrix}$$

As before
Premultiply by
elimination matrix $\mathbb{E}_{32}$

LHS:

$$\underbrace{\mathbb{E}_{32}}\,\underbrace{\mathbb{E}_{31}}\,\underbrace{\mathbb{E}_{21}}\,\mathbb{A} = \mathbb{U} =$$

$$= \underset{\mathbb{U}}{\begin{bmatrix}1&0&-1\\0&3&3\\0&0&-1\end{bmatrix}} \leftarrow \text{pivots} \quad \begin{cases}D_1=1\\D_2=3\\D_3=-1\end{cases}$$

RHS

$$\mathbb{E}_{32}\mathbb{E}_{31}E_{21}\vec{b}$$

$$= \begin{bmatrix}-2\\6\\-2\end{bmatrix}$$

To find solution, now use back substitute

E3ap4

Note: $\mathbb{E}_{ij}$ are always $m \times m$ lower triangular matrices (0's above main diagonal).

Sometimes row swaps are necessary.

ex// $\underset{\mathbb{P}_{12}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$

ex// 3×3 that swaps rows 2 & 3.

$$\mathbb{P}_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{keep: } R_1' = R_1 \\ \leftarrow R_2' = R_3 \\ \leftarrow R_3' = R_2 \end{array}$$

Usually, do row swaps first

3×3 example

$$\mathbb{U} = \mathbb{E}_{32}\mathbb{E}_{31}\mathbb{E}_{21}\underset{\substack{\uparrow \\ \text{row swaps}}}{\mathbb{P}}\mathbb{A}$$

$$\|$$

$$\vec{c} = \mathbb{E}_{32}\mathbb{E}_{31}\mathbb{E}_{21}\mathbb{P}\vec{b}$$

Menu:
- Matrix operations
- How to add, scale, and multiply
- The Sneakiness of Matrix multiplication

① Scalar multiplication:

$$3\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 & 3\cdot 1 \\ 3\cdot(-1) & 3\cdot 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -3 & 9 \end{bmatrix}$$

$\mathbb{B} = c\mathbb{A}$

$b_{ij} = ca_{ij}$

Notation: write ij th entry of $\overset{m\times n}{\mathbb{A}}$ as a_{ij} — row index $1 \le i \le m$, column index $1 \le j \le n$

$\mathbb{B}$ as b_{ij}

Sometimes: $\mathbb{A} = [a_{ij}]$

② Addition:

$\mathbb{A} + \mathbb{B}$ is only possible if $\mathbb{A}$ & $\mathbb{B}$ are the same shape

ex.

$$\underset{3\times 2}{\overset{\mathbb{A}}{\begin{bmatrix} 2 & 3 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}}} + \underset{3\times 2}{\overset{\mathbb{B}}{\begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}}} = \underset{3\times 2}{\overset{\mathbb{C}}{\begin{bmatrix} 3 & 2 \\ 3 & -1 \\ -1 & 2 \end{bmatrix}}}$$

entrywise addition

$$c_{ij} = a_{ij} + b_{ij}$$

③ Multiplication:

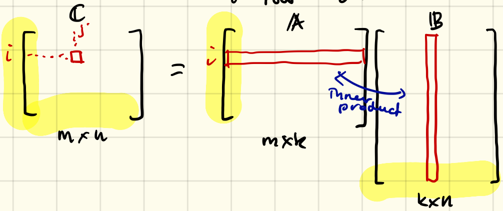
$\underset{m\times k}{\mathbb{A}}\ \underset{k\times n}{\mathbb{B}}$ is only possible if inner dimensions match

$$\underset{m\times n}{\mathbb{C}} = \underset{m\times k}{\mathbb{A}}\ \underset{k\times n}{\mathbb{B}}$$

Defn:

* c_{ij}, the entry for $\mathbb{C}$ in the i^{th} row and j^{th} column is the dot (inner) product of the i^{th} row of $\mathbb{A}$ and the j^{th} row of $\mathbb{B}$

* $$c_{ij} = \sum_{\ell=1}^{k} \underbrace{a_{i\ell}}_{i^{th}\text{ row}} \underbrace{b_{\ell j}}_{j^{th}\text{ column}}$$



Rules matrix operations are pretty normal...

$$\mathbb{A} + \mathbb{B} = \mathbb{B} + \mathbb{A}$$ commutative law for addition

$$\mathbb{A}\mathbb{B}\mathbb{C} = (\mathbb{A}\mathbb{B})\mathbb{C} = \mathbb{A}(\mathbb{B}\mathbb{C})$$

One bananapants exception:

$\mathbb{A}\,\mathbb{B}$ most often does not equal $\mathbb{B}\mathbb{A}$!!!!

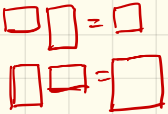
~~Two~~ Three problems

(1) $\underset{m\times k}{\mathbb{A}}\;\underset{k\times n}{\mathbb{B}}$ ✓ $\qquad \underset{k\times n}{\mathbb{B}}\;\underset{m\times k}{\mathbb{A}}$

if $n \neq m$, $\mathbb{B}\mathbb{A}$ does not make sense

(2) If $n = m$, products are both ok.

$\underbrace{\underset{m\times k}{\mathbb{A}}\;\underset{k\times m}{\mathbb{B}}}_{m\times m} \qquad \underbrace{\underset{k\times m}{\mathbb{B}}\;\underset{m\times k}{\mathbb{A}}}_{k\times k}$



if $k \neq m$, no good either

(3) So $m = n = k$ is required for us to even have a chance that $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$

Observe: Only possible for $n\times n$ square matrices

E4ap2

Even then, $\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}$ often

ex/

$$\begin{bmatrix}1&2\\-1&2\end{bmatrix}\begin{bmatrix}0&-1\\3&2\end{bmatrix}=\begin{bmatrix}6&3\\6&5\end{bmatrix} \neq$$

$$\begin{bmatrix}0&-1\\3&2\end{bmatrix}\begin{bmatrix}1&2\\-1&2\end{bmatrix}=\begin{bmatrix}1&-2\\1&10\end{bmatrix}$$

If $\underset{n\times n}{\mathbb{A}}\;\underset{n\times n}{\mathbb{B}} = \underset{n\times n}{\mathbb{B}}\;\underset{n\times n}{\mathbb{A}}$, we get very excited and say $\mathbb{A}$ & $\mathbb{B}$ commute

↑ special. spesh

Warning: Never slide matrices around in products and always be careful with order. //

Menu:
- Wizard-level matrix multiplication skills
- ~~inner~~ and outer products
- dot $A\vec{x}$, $\vec{y}^T B$, AB
- Block multiplication in general

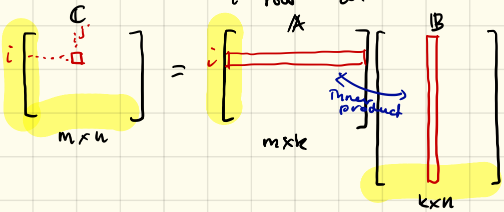
from before:

$$\underset{m \times n}{C} = \underset{m \times k}{A}\ \underset{k \times n}{B}$$

Defn:

* C_{ij}, the entry for C in the i^{th} row and j^{th} column is the dot (inner) product of the i^{th} row of A and the j^{th} row of B

* $$C_{ij} = \sum_{l=1}^{k} \underbrace{a_{il}}_{i^{th}\text{ row}} \underbrace{b_{lj}}_{j^{th}\text{ column}}$$



E4b p1

ex 1

$$\overset{A}{\begin{bmatrix}3&0&2\\1&-2&2\end{bmatrix}}\overset{B}{\begin{bmatrix}-1&0\\2&1\\0&2\end{bmatrix}} = \begin{bmatrix}-3&4\\-5&2\end{bmatrix}$$

2×3 3×2 2×2

C_{11} C_{12} C_{21} C_{22}

$$C_{11} = \begin{bmatrix}3&0&2\end{bmatrix}\begin{bmatrix}-1\\2\\0\end{bmatrix}$$

1st row of A

1st col of B

$$C_{12} = \begin{bmatrix}3&0&2\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$C_{21} = \begin{bmatrix}1&-2&2\end{bmatrix}\begin{bmatrix}-1\\2\\0\end{bmatrix} = -5$$

$$C_{22} = \begin{bmatrix}1&-2&2\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix} = 2$$

ex 2

$$\underset{1\times 3\ \checkmark}{\begin{bmatrix}1 & 2 & -1\end{bmatrix}}\underset{3\times 1\ \checkmark}{\begin{bmatrix}0\\1\\-2\end{bmatrix}} = \underset{1\times 1}{4}$$

inner product

ex 3

$$\underset{\underline{3}\times\textcircled{1}\ \checkmark}{\begin{bmatrix}0\\1\\-2\end{bmatrix}}\underset{\textcircled{1}\times \underline{3}\ \checkmark}{\begin{bmatrix}1 & 2 & -1\end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix}0 & 0 & 0\\1 & 2 & -1\\-2 & -4 & 2\end{bmatrix}}$$

amazingly important construction

outer product

later: see this is a rank r = 1 matrix.

See

$$= \begin{bmatrix}0\cdot\begin{bmatrix}1 & 2 & -1\end{bmatrix}\\ 1\cdot\begin{bmatrix}1 & 2 & -1\end{bmatrix}\\ -2\cdot\begin{bmatrix}1 & 2 & -1\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}1\cdot\begin{bmatrix}0\\1\\-2\end{bmatrix} & 2\begin{bmatrix}0\\1\\-2\end{bmatrix} & -1\begin{bmatrix}0\\1\\2\end{bmatrix}\end{bmatrix}$$

ex 4 block multiplication: E4bp2

$$\begin{bmatrix}3 & \vert & 0 & \vert & 2\\1 & \vert & -2 & \vert & 2\end{bmatrix}\begin{bmatrix}-1 & 0\\ \hline 2 & 1\\ \hline 0 & 2\end{bmatrix}$$

row of 2×1's

column of 1×2's

$$= \begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}-1 & 0\end{bmatrix} + \begin{bmatrix}0\\-2\end{bmatrix}\begin{bmatrix}2 & 1\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix}\begin{bmatrix}0 & 2\end{bmatrix}$$

$$= \begin{bmatrix}-3 & 0\\-1 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\-4 & -2\end{bmatrix} + \begin{bmatrix}0 & 4\\0 & 4\end{bmatrix}$$

$$= \begin{bmatrix}-3 & 4\\-5 & 2\end{bmatrix}$$

$=$

ex 5

$$\begin{bmatrix}\begin{bmatrix}3 & 0\\1 & -2\end{bmatrix} & \begin{bmatrix}2\\2\end{bmatrix}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix}\\ \begin{bmatrix}0 & 2\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}3 & 0\\1 & -2\end{bmatrix}\begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix} + \underbrace{\underset{2\times 1}{\begin{bmatrix}2\\2\end{bmatrix}}\underset{1\times 2}{\begin{bmatrix}0 & 2\end{bmatrix}}}_{2\times 2} \quad \checkmark$$

E4bp3

$\mathbb{A}\vec{x}$: $\underset{m\times n}{\mathbb{A}}\ \underset{n\times 1}{\vec{x}}$

$$\begin{array}{c} \mathbb{A} \\ m \text{ rows} \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \\ n \text{ columns} \end{array} \begin{array}{c} \vec{x} \\ \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} \Big\} n \text{ rows.} \end{array}$$

See this as the columns of $\mathbb{A}$ being combined with weights in vector $\vec{x}$;

$$\mathbb{A}\vec{x} = \begin{bmatrix} | & | & & | \\ \vec{a}_{*1} & \vec{a}_{*2} & \cdots & \vec{a}_{*n} \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

column vectors inside $\mathbb{A}$

$* =$ run over all indices

$$= x_1 \begin{bmatrix} | \\ \vec{a}_{*1} \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ \vec{a}_{*2} \\ | \end{bmatrix} + \dots + x_n \begin{bmatrix} | \\ \vec{a}_{*n} \\ | \end{bmatrix}$$

$\vec{a}_{*1}$ first column vector

ex/ $$\underset{2\times 3}{\overset{\mathbb{A}}{\begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}}} \underset{3\times 1}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} = (-1)\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (2)\begin{bmatrix} 0 \\ -2 \end{bmatrix} + (0)\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

scalings

$$= \begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

$$\parallel$$

$$\underset{2\times 1}{\begin{bmatrix} -3 \\ -5 \end{bmatrix}} =$$

"transpose"

$\vec{y}^T\mathbb{A}$: $\underset{1\times m}{\vec{y}^T}\ \underset{m\times n}{\mathbb{A}}$

$$\overset{\vec{y}^T\ 1\times m}{\underset{1\times m}{\begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}}} \overset{m\times n}{\overset{\mathbb{A}}{\underset{m\times n}{\begin{bmatrix} - & \vec{a}_{1*} & - \\ - & \vec{a}_{2*} & - \\ & \vdots & \\ - & \vec{a}_{m*} & - \end{bmatrix}}}}$$

n.b. $\vec{y}$ is $m\times 1$

column vector

row vectors inside $\mathbb{A}$

$\vec{a}_{1*}$ first row vector in $\vec{a}$

$$\vec{y}^T\mathbb{A} = y_1[- \ \vec{a}_{1*} \ -] + y_2[- \ \vec{a}_{2*} \ -] \ \vdots + y_m[- \ \vec{a}_{m*} \ -]$$

See this as the rows of $\mathbb{A}$ being combined with weights in vector $\vec{y}^T$.

ex $$\underset{1\times 3}{\begin{bmatrix} 3 & 0 & 2 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}} = \underset{1\times 2}{\begin{bmatrix} -3 & 4 \end{bmatrix}}$$

$$\parallel \qquad\qquad \parallel$$

$$(3)\begin{bmatrix} -1 & 0 \end{bmatrix} + (0)\begin{bmatrix} 2 & 1 \end{bmatrix} + (2)\begin{bmatrix} 0 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 4 \end{bmatrix}$$

E4bp4

$$\underset{m\times k}{\mathbb{C}} = \underset{m\times n}{\mathbb{A}}\,\underset{n\times k}{\mathbb{B}} \xrightarrow[\text{into columns}]{\text{break } \mathbb{B}} \mathbb{A}\begin{bmatrix} | & | & & | \\ \vec{b}_{*1} & \vec{b}_{*2} & \cdots & \vec{b}_{*k} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ (\underset{m\times n}{\mathbb{A}}\underset{n\times 1}{\vec{b}_{*1}}) & (\mathbb{A}\vec{b}_{*2}) & \cdots & (\mathbb{A}\vec{b}_{*k}) \\ | & | & & | \end{bmatrix}$$

column vector

← ℂ's columns are made up of A's columns

break A into rows ↓

2 views → ↓

$$\begin{bmatrix} -\ \vec{a}_{1*}\ - \\ -\ \vec{a}_{2*}\ - \\ \vdots \\ -\ \vec{a}_{m*}\ - \end{bmatrix}\mathbb{B} = \begin{bmatrix} -\ (\vec{a}_{1*}\mathbb{B})\ - \\ -\ (\vec{a}_{2*}\mathbb{B})\ - \\ \vdots \\ -\ (\underset{1\times n}{\vec{a}_{m*}}\,\underset{n\times k}{\mathbb{B}})\ - \end{bmatrix}$$

$1\times k$

ℂ's rows are made up of IB's rows

ex

$$\overset{\mathbb{A}}{\underset{2\times 3}{\begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}}}\ \overset{\mathbb{B}}{\underset{3\times 2}{\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}}} = \left[\ \overset{\mathbb{A}}{\underset{2\times 3}{\begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}}}\overset{\vec{b}_{*1}}{\underset{3\times 1}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}}\quad \overset{\mathbb{A}}{\begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}}\overset{\vec{b}_{*2}}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}\ \right]$$

$$= \left[\ \underset{2\times 1}{\begin{bmatrix} -3 \\ -5 \end{bmatrix}}\quad \begin{bmatrix} 4 \\ 2 \end{bmatrix}\ \right]$$

$$= \left[\begin{matrix} \overset{\vec{a}_{1*}}{\underset{1\times 3}{\begin{bmatrix} 3 & 0 & 2 \end{bmatrix}}}\ \overset{\mathbb{B}\ 3\times 2}{\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}} \\ \overset{\vec{a}_{2*}}{\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}}\ \underset{\mathbb{B}}{\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}} \end{matrix}\right] = \begin{bmatrix} \overset{1\times 2}{\begin{bmatrix} -3 & 4 \end{bmatrix}} \\ \underset{1\times 2}{\begin{bmatrix} -5 & 2 \end{bmatrix}} \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -5 & 2 \end{bmatrix}$$

✓

Matrix Inverses ← delicious

Square Matrices Only

Menu:

- General goodness of inverses
- Identity matrix
- Solving $\mathbb{A}\vec{x}=\vec{b}$ • Advanced goodness
- How to w. example

Ex

$$\frac{1}{8}\overbrace{\begin{bmatrix}4&2\\2&3\end{bmatrix}}^{\mathbb{A}^{-1}}_{2\times2}\overbrace{\begin{bmatrix}3&-2\\-2&4\end{bmatrix}}^{\mathbb{A}}_{2\times2}=\frac{1}{8}\begin{bmatrix}\overset{8}{4\times3+2\times-2} & \overset{0}{4\times-2+2\times4}\\ \overset{0}{2\times3+3\times-2} & \overset{8}{2\times-2+3\times4}\end{bmatrix}$$

$$=\begin{bmatrix}1&0\\0&1\end{bmatrix}=\mathbb{I}$$

"1" for 2×2 matrices

Identity matrix $\mathbb{I}_n$

$n\times n$

leaves matrices unchanged under multiplication

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}2&3&-1\\4&2&1\end{bmatrix}=\begin{bmatrix}2&3&-1\\4&2&1\end{bmatrix}$$

$R_1'=R_1$
$R_2'=R_2$

$$\mathbb{I}_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \qquad \mathbb{I}_4=\begin{bmatrix}1&&&\\0&1&&\\0&0&1&\\0&0&0&1\end{bmatrix}$$

lots of zeros

E5ap1

Given $\overset{n\times n}{\mathbb{A}}$, if $\mathbb{A}^{-1}$ exists then

$$\mathbb{A}^{-1}\mathbb{A}=\mathbb{A}\mathbb{A}^{-1}=\mathbb{I}$$

order doesn't matter

Selection of Big Deals for $\mathbb{A}^{-1}$:

- Even knowing if $\mathbb{A}^{-1}$ exists is valuable ← may not have actual form.
- If $\mathbb{A}^{-1}$ exists then: premultiply both sides

$$\underbrace{\mathbb{A}^{-1}\underset{n\times n}{\mathbb{A}}}_{m=n}\underset{n\times1}{\vec{x}}=\mathbb{A}^{-1}\underset{n}{\vec{b}}$$

$$\Rightarrow \underset{n\times n}{\mathbb{I}}\,\underset{n\times1}{\vec{x}}=\mathbb{A}^{-1}\vec{b}$$

$$\Rightarrow \vec{x}=\mathbb{A}^{-1}\vec{b}$$

Done!!

But wait:

- Many systems have rectangular $\mathbb{A}$'s $m\neq n$
- $\mathbb{A}^{-1}$ may not exist
- Even if $\mathbb{A}^{-1}$ exists, computing $\mathbb{A}^{-1}$ is very hard work computationally (grows badly with n)

- if $\mathbb{A}^{-1}$ exists, then $\mathbb{A}\vec{x}=\vec{b}$ has only one solution, always. (for all $\vec{b}$)

Simply: $\vec{x} = \underbrace{\mathbb{A}^{-1}\vec{b}}$

- if $\mathbb{A}^{-1}$ does not exist then we may have 0 or ∞ly many solutions

↖ more later

- If $\exists$ $\boxed{\vec{x} \neq \vec{0}}$ (there $\overset{\exists}{\text{exists}}$ an $\vec{x} \neq \vec{0}$)

such that $\mathbb{A}\vec{x} = \vec{0}$ $\text{"}= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

↖ "$\mathbb{A}$ maps $\vec{x}$ to $\vec{0}$"

"$\mathbb{A}$ crushes $\vec{x}$."

then $\mathbb{A}^{-1}$ does not exist

Proof $\underbrace{\mathbb{A}\vec{x}=\vec{0}}_{\text{true}} \Rightarrow \underbrace{\underset{n\times n}{\boxed{\mathbb{A}^{-1}}}\,\underset{n\times n}{\mathbb{A}}}\,\underset{n\times 1}{\vec{x}} = \underset{n\times n}{\mathbb{A}^{-1}}\,\underset{n\times 1}{\vec{0}}$

$\Rightarrow \mathbb{I}\,\vec{x} = \vec{0}$

$\vec{x} = \vec{0}$

contradiction!

$\Rightarrow \mathbb{A}^{-1}$ cannot exist

E5ap2

- Foreshadowing: if $\mathbb{A}\vec{x}=\vec{0}$ we say $\vec{x} \in N(\mathbb{A})$

↑ null space of $\mathbb{A}$

- $(\underset{n\times n}{\mathbb{A}}\,\underset{n\times n}{\mathbb{B}})^{-1} = \mathbb{B}^{-1}\mathbb{A}^{-1}$

See $\mathbb{B}^{-1}\underbrace{\mathbb{A}^{-1}(\mathbb{A}}\,\mathbb{B}) = \mathbb{B}^{-1}\overset{\text{"}\mathbb{B}}{\boxed{\mathbb{I}\,\mathbb{B}}} = \boxed{\mathbb{B}^{-1}\mathbb{B}}$

$(\mathbb{A}\underbrace{\mathbb{B})\,\mathbb{B}^{-1}}\mathbb{A}^{-1} = \underset{\mathbb{A}}{\boxed{\mathbb{A}\,\mathbb{I}}}\,\mathbb{A}^{-1} = \boxed{\mathbb{A}\,\mathbb{A}^{-1}} = \mathbb{I}$

$(\mathbb{A}\mathbb{B}\mathbb{C}\mathbb{D})^{-1}$

$= \mathbb{D}^{-1}\mathbb{C}^{-1}\mathbb{B}^{-1}\mathbb{A}^{-1}$

- If we have $\underset{n\times n}{\mathbb{A}}, \underset{n\times n}{\mathbb{Z}_L}, \underset{n\times n}{\mathbb{Z}_R}$ such that

↙ right inverse ↙ left inverse

$\mathbb{A}\,\mathbb{Z}_R = \mathbb{I}$ & $\mathbb{Z}_L\mathbb{A} = \mathbb{I}$

then $\mathbb{A}^{-1} = \mathbb{Z}_R = \mathbb{Z}_L$

Reason

$\overset{\mathbb{I}}{\boxed{\mathbb{Z}_L(\mathbb{A}}}\,\mathbb{Z}_R) = \underbrace{\mathbb{Z}_L(\mathbb{I})}$

premultiply

$\cancel{\mathbb{I}}\,\mathbb{Z}_R = \mathbb{Z}_L$

Using Gauss-Jordan Elimination to find $\mathbb{A}^{-1}$

- general story (it's $\mathbb{A}\vec{x}=\vec{b}$ again!)
- example

Game: given $\underset{n\times n}{\mathbb{A}}$, find $\overset{n\times n}{\mathbb{A}^{-1}}$

$$\downarrow$$

$$\mathbb{A}^{-1}\mathbb{A} = \mathbb{A}\mathbb{A}^{-1} = \mathbb{I}$$

$\underset{m\times n}{\mathbb{A}}\underset{n\times 1}{\vec{x}}=\underset{m\times 1}{\vec{b}}$ ish

Consider: $\overset{n\times n}{\underset{✓}{\mathbb{A}}}\ \overset{n\times n}{\underset{?}{\mathbb{Z}}} = \overset{n\times n}{\underset{✓}{\mathbb{I}}}$

2×2 general ex

$$\underset{2\times 2}{\mathbb{A}}\begin{bmatrix} | & | \\ \vec{z}_1 & \vec{z}_2 \\ | & | \end{bmatrix}_{2\times 1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

⇓ wrangling

$$\begin{bmatrix} | & | \\ \underset{2\times 2}{\mathbb{A}}\underset{2\times 1}{\vec{z}_1} & \underset{2\times 2}{\mathbb{A}}\underset{2\times 1}{\vec{z}_2} \\ | & | \\ 2\times 1 & 2\times 1 \end{bmatrix} = \begin{bmatrix} \overset{\hat{i}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & \overset{\hat{j}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \end{bmatrix}$$

$$\Rightarrow \text{Solve } \mathbb{A}\vec{z}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \& \ \mathbb{A}\vec{z}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

↖ $\mathbb{A}\vec{x}=\vec{b}$!!

E56p1

Note: we would make $\mathbb{A}$ become $\mathbb{U}$ w. row reduction for both equations.

Do all at once with a super augmented matrix:

$$\underbrace{\left[\ \underset{n\times n}{\mathbb{A}}\ \middle|\ \underset{n\times n}{\mathbb{I}}\ \right]}_{n\times 2n}$$

#awesome

Use row ops to turn $\mathbb{A}$ into $\mathbb{U}$ then $\mathbb{I}$

$\mathbb{I}$ will change into $\mathbb{A}^{-1}$

- actually: finding right inverse of $\mathbb{A}$; later we show it's the true inverse
- only works if $\mathbb{A}$ has n pivots

Example:

$$\mathbb{A} = \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$$

$$[\mathbb{A} \mid \mathbb{I}] = \left[\begin{array}{cc|cc} \overset{D_1}{\boxed{3}} & -2 & 1 & 0 \\ \boxed{-2} & 4 & 0 & 1 \end{array}\right]$$

$$R_2' = R_2 - \left(\frac{-2}{\underset{D_1}{3}}\right) R_1 \qquad \ell_{21} = -\frac{2}{3}$$

$$\left[\begin{array}{cc|cc} \overset{D_1}{3} & \overset{\mathbb{U}}{-2} & 1 & 0 \\ 0 & \underset{D_2}{8/3} & 2/3 & 1 \end{array}\right]$$

$4 - \frac{4}{3}$

$12/3$

$$R_1' = R_1 - \underbrace{\left(\frac{-2}{\underset{D_2}{8/3}}\right)}_{+\frac{3}{4}} R_2$$

$1 + \frac{2}{3} \cdot \frac{3}{4}$

$$\left[\begin{array}{cc|cc} 3 & 0 & 3/2 & 3/4 \\ 0 & 8/3 & 2/3 & 1 \end{array}\right]$$

divide by pivots

$$R_1' = \frac{1}{\underset{D_1}{3}} R_1$$

$$R_2' = \frac{1}{\underset{D_2}{8/3}} R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 1/4 & 3/8 \end{array}\right]$$

$$\underbrace{\hphantom{1\ 0}}_{\mathbb{I}} \qquad \underbrace{\hphantom{1/2\ 1/4}}_{\mathbb{A}^{-1}}$$

tidying up

see E5ap1.

$$\mathbb{A} = \frac{1}{8} \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

turns out this is a very special number for $\mathbb{A}$

notation

- Determinant of $\mathbb{A}$
- $\mathrm{Det}(\mathbb{A})$
- $|\mathbb{A}|$

More later!!

3x3 plan

LE5bp3

order of elimination ①

$$\left[\begin{array}{ccc|ccc} 2 & \overset{⑥}{3} & \overset{⑤}{4} & 1 & 0 & 0 \\ \text{①}\,1 & -1 & \overset{④}{7} & 0 & 1 & 0 \\ \underset{②}{13} & \underset{③}{2} & 17 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cc} \mathbb{A} & \mathbb{I} \end{array}$$

Hidden Secrets of Inverses:

- $\mathbb{A}^{-1}$ and elimination matrices
- Inverses of elimination matrices
- Missing pivots → $\mathbb{A}^{-1}$ does not exist

Pratchett upon learning more about inverses →

- Curious things about columns...

$$\mathbb{A} = \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix} \qquad \mathbb{A}^{-1} = \frac{1}{8}\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

aside $\mathbb{A} = \mathbb{A}^T$ transpose

$(\mathbb{A}^{-1})^T = \mathbb{A}^{-1}$

Row reduction

⇒ Elimination matrices

for Solving $[\mathbb{A} \mid \mathbb{I}] \iff \mathbb{A}\mathbb{Z} = \mathbb{I}$

row op 1

$$\mathbb{E}_{21} = \begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix}$$

row 2 row 1 $-\ell_{21}$

row op 2

multiplier

$$\mathbb{E}_{12} = \begin{bmatrix} 1 & 3/4 \\ 0 & 1 \end{bmatrix}$$

ES cp1

pivot matrix

$$\mathbb{D} = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 8/3 \end{bmatrix}$$

$$\mathbb{D}^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 3/8 \end{bmatrix}$$

undo each other

$$\left[\begin{array}{cc|cc} 3 & -2 & 1 & 0 \\ \cancel{-4} & 4 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 1/4 & 3/8 \end{array}\right]$$

-2

[note: transcribed incorrectly in video]

$$\underbrace{\mathbb{D}^{-1}\mathbb{E}_{12}\mathbb{E}_{21}}_{\mathbb{A}^{-1}}\mathbb{A}\mathbb{Z} = \underbrace{\mathbb{D}^{-1}\mathbb{E}_{12}\mathbb{E}_{21}}_{\mathbb{A}^{-1}}\mathbb{I}$$

$$\mathbb{I}\mathbb{Z}_1 = \mathbb{A}^{-1}$$

found by row operations

made by $\mathbb{E}_{ij}$ & $\mathbb{D}$ matrices.

Big Deal: See $\mathbb{A}^{-1}$ is a product of $\mathbb{E}_{ij}$'s, $\mathbb{D}^{-1}$, $\mathbb{P}$

pivots

permutation for row swaps

Huge: Demonstrates that $\mathbb{A}^{-1}$ is a left and right inverse

Next: Elimination matrices have simple inverses.

$$\mathbb{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \underset{-\ell_{31}}{\boxed{-2}} & 0 & 1 \end{bmatrix} \longleftrightarrow R_3' = R_3 - 2R_1$$

undo with $R_3' = R_3 + 2R_1$

$$\Rightarrow \mathbb{E}_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \underset{+\ell_{31}}{\boxed{+2}} & 0 & 1 \end{bmatrix}$$

swap sign

check $\mathbb{E}_{31}\mathbb{E}_{31}^{-1} = \mathbb{E}_{31}^{-1}\mathbb{E}_{31} = \mathbb{I}$.

In general flip sign of (one) off diagonal element to turn $\mathbb{E}_{ij}$ into $\mathbb{E}_{ij}^{-1}$

Monks made us do this... Sneaky plan.

Permutation matrices:

$$\mathbb{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \leftarrow \begin{matrix} R_1' = R_1 \\ R_2' = R_3 \\ R_3' = R_2 \end{matrix}$$

here: $\mathbb{P}^{-1} = \mathbb{P}$

But in general $\mathbb{P}^{-1} = \mathbb{P}^T$

Missing pivots

E5cp2

What if $[\mathbb{A} \mid \mathbb{I}] \rightarrow$ one or more rows of zeros on left? i.e., missing pivots?

from before: if $\vec{x} \neq \vec{0}$ solves $\mathbb{A}\vec{x} = \vec{0}$ then $\mathbb{A}^{-1}$ cannot exist

so

$$[\mathbb{A} \mid \underset{\vec{b}}{\vec{0}}] \underset{\text{Row ops}}{\sim} [\underset{\text{upper triangular}}{\mathbb{U}} \mid \underset{\vec{c}}{\vec{0}}]$$

Row of 0's in $\mathbb{U}$ → ∞ly many solns
→ $\mathbb{A}\vec{x} = \vec{0}$ is solved by $\vec{x} \neq \vec{0}$
→ $\mathbb{A}^{-1}$ does not exist

ex

$$\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 6 & 4 & 0 \end{array}\right] \underset{R_2' = R_2 - \left(\frac{6}{3}\right)R_1}{\sim} \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & \underset{\text{missing pivot}}{\boxed{0}} & 0 \end{array}\right]$$

Upshot:

$\underset{n\times n}{\mathbb{A}}^{-1}$ exists

$\iff \mathbb{A}$ has n pivots

$\iff \mathbb{A}\vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as a solution

$\iff \underset{\text{later}}{\det}(\mathbb{A}) \neq 0$

parallelograms will be involved

LE5cp3

If $\mathbb{A}_{3\times3}$ has column 1 + column 2 = column 3

show $\mathbb{A}^{-1}$ does not exist.. (weird)

(a) See $\mathbb{A}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ ⇐ non-zero solut to $\mathbb{A}\vec{x}=\vec{0}$

→ $\mathbb{A}^{-1}$ does not exist ⇒ pivot missing.

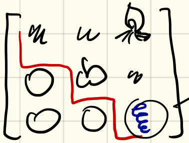
column picture

$$\underbrace{1\vec{a}_1}_{3\times1} + 1\cdot\vec{a}_2 - 1\cdot\vec{a}_3 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

(b) Another aspect:

Row operations destroy rows But
Column relationships are unchanged.

row reduct →



→ must be 0

→ 3rd pivot is missing

$C_1+C_2=C_3 \Rightarrow \xi = 0+0=0$

$$\begin{array}{ccc} \begin{bmatrix}1&2&3\\1&3&4\\1&4&5\end{bmatrix} \\ \vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 = \vec{a}_1+\vec{a}_2 \end{array} \cdots \sim \begin{bmatrix}1&2&3\\0&1&1\\0&2&2\end{bmatrix}$$

↑ col 1 + col 2 = col 3

$$\sim \begin{bmatrix}1&2&3\\0&1&1\\0&0&0\end{bmatrix}$$

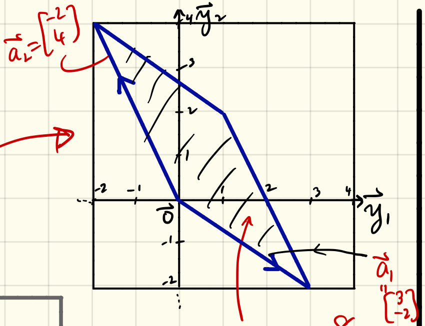
columns are "linearly dependent"

⇒ connects to $\mathbb{A}^{-1}$ not existing

Foreshadowing:

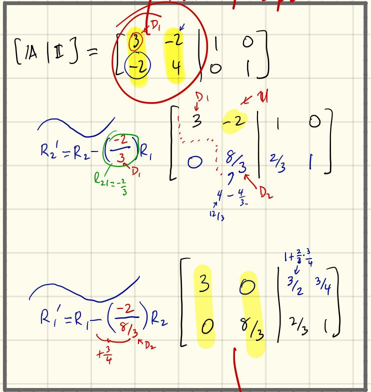
Determinant matrix.

E5dp1



area = 8

from p5bp2



$$[A \mid I] = \left[\begin{array}{cc|cc} \overset{D_1}{3} & -2 & 1 & 0 \\ -2 & 4 & 0 & 1 \end{array}\right]$$

$$R_2' = R_2 - \left(\frac{-2}{\underset{D_1}{3}}\right) R_1 \quad \left[\begin{array}{cc|cc} \overset{D_1}{3} & \overset{U}{-2} & 1 & 0 \\ 0 & \underset{D_2}{8/3} & 2/3 & 1 \end{array}\right]$$

$l_{21} = -\frac{2}{3}$

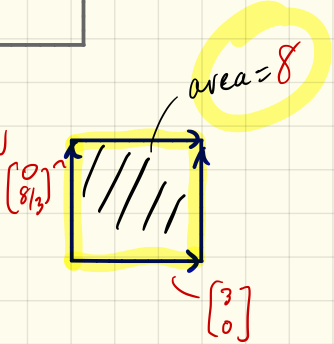
$4 - \frac{4}{3}$

$12/3$

$$R_1' = R_1 - \left(\frac{-2}{\underset{D_2}{8/3}}\right) R_2 \quad \left[\begin{array}{cc|cc} 3 & 0 & \overset{1+\frac{2}{3}\cdot\frac{3}{4}}{3/2} & 3/4 \\ 0 & 8/3 & 2/3 & 1 \end{array}\right]$$

$+\frac{3}{4}$

$$|A| = 3 \times 4 - (-2)(-2)$$
$$= 12 - 4 = 8$$



Triangle × Triangle = Rectangle

Menu:

- Our first factorization: $\mathbb{L}\mathbb{U}$
- Method first
- The l_{ij}'s serve us well
  (as promised by mysterious monks)

← t-shirt for each factorization

Ex

$$x_1 \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\vec{a}_1} + x_2 \underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{\vec{a}_2} + x_3 \underbrace{\begin{bmatrix}1\\3\\6\end{bmatrix}}_{\vec{a}_3} = \underbrace{\begin{bmatrix}5\\7\\11\end{bmatrix}}_{\vec{b}}$$

$$\underset{\underset{m\times n}{3\times 3}}{\mathbb{A}}\ \underset{3\times 1}{\vec{x}} = \underset{3\times 1}{\vec{b}}$$

Normal plan:

set up $[\mathbb{A} \mid \vec{b}] \xrightarrow[\text{ops}]{\text{row}} [\mathbb{U} \mid \vec{c}] \xrightarrow[\text{sub}]{\text{back}} \vec{x}$

Now: focus on reducing $\mathbb{A}$ by itself

↖ very good if $\vec{b}$ is changed.

$$\mathbb{A} = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 3 & 6\end{bmatrix}$$

$R_2' = R_2 - \left(\frac{1}{1}\right) R_1$ — $l_{21} = 1$

$R_3' = R_2 - \left(\frac{1}{1}\right) R_1$ — $l_{31} = 1$

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 2 & 5\end{bmatrix}$$

$R_3' = R_3 - \left(\frac{2}{1}\right) R_2$ — $l_{32} = 2$

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} \leftarrow \mathbb{U}$$

$D_1 = 1$
$D_2 = 1$
$D_3 = 1$

← not exciting

Elimination matrix story

E6ap1

$$\mathbb{A} \rightarrow \mathbb{U} = \mathbb{E}_{32}\mathbb{E}_{31}\mathbb{E}_{21}\mathbb{A}$$

↑ powerful encoding of our row operations

Monks whisper: "invert $\mathbb{E}_{ij}$'s"

RHS ↕

$$\underbrace{\mathbb{E}_{21}^{-1}\mathbb{E}_{31}^{-1}\underbrace{\mathbb{E}_{32}^{-1}\mathbb{E}_{32}}_{\mathbb{I}}\mathbb{E}_{31}\mathbb{E}_{21}}_{\mathbb{I}}\mathbb{A}$$

↕ LHS

$$= \mathbb{E}_{21}^{-1}\mathbb{E}_{31}^{-1}\mathbb{E}_{32}^{-1}\mathbb{U}$$

$$\Rightarrow \quad \mathbb{A} = \underbrace{\mathbb{E}_{21}^{-1}\mathbb{E}_{31}^{-1}\mathbb{E}_{32}^{-1}}_{\text{row operations in reverse}}\mathbb{U}$$

← upper triangular

Tells us how to combine $\underbrace{\text{rows of } \mathbb{U}}_{\text{simple-ish}}$ to make rows of $\mathbb{A}$

Amazingly: we'll see

$$\underset{m\times n}{\mathbb{A}} = \underset{\text{square } m\times m}{\mathbb{L}}\ \underset{m\times n}{\mathbb{U}}$$

lower triangular

$$\mathbb{L} = \begin{bmatrix}1 & 0 & 0\\+l_{21} & 1 & 0\\+l_{31} & +l_{32} & 1\end{bmatrix}_{m\times m}$$

upper triangular

$$\mathbb{U} = \begin{bmatrix}D_1 & * & *\\0 & D_2 & *\\0 & 0 & D_3\end{bmatrix}$$

Now

$$\mathbb{A} = \overbrace{\underbrace{\mathbb{E}_{21}^{-1}\,\mathbb{E}_{31}^{-1}\,\mathbb{E}_{32}^{-1}}_{\text{know these are simple}}}^{\mathbb{L}}\,\mathcal{U}$$

ex recall

$$\mathbb{E}_{21} = \begin{bmatrix}1&0&0\\-l_{21}&1&0\\0&0&1\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0\\+l_{21}&1&0\\0&0&1\end{bmatrix}$$

$R_2' = R_2 - l_{21}R_1$ $\qquad$ $R_2' = R_2 + l_{21}R_1$

Big Deals:

(1) $\mathbb{E}_{ij}^{-1}$ is $\mathbb{E}_{ij}$ with single off diagonal element flipped in sign

(2) $\mathbb{E}_{ij}$'s & $\mathbb{E}_{ij}^{-1}$'s are all lower triangular

(3) $\mathbb{E}_{ij}$ is $\mathbb{I}$ with $-l_{ij}$ replacing 0 in ij position

LE6ap2

(4) Remarkably:

$$\mathbb{E}_{21}^{-1}\,\mathbb{E}_{31}^{-1}\,\mathbb{E}_{32}^{-1} = \underbrace{\begin{bmatrix}1&0&0\\+l_{21}&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\+l_{31}&0&1\end{bmatrix}}\begin{bmatrix}1&0&0\\0&1&0\\0&+l_{32}&1\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&l_{32}&1\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix} = \mathbb{L}$$

$\mathbb{L}$ always has 1's on the diagonal.

Back to example:

$$\mathbb{L} = \begin{bmatrix}1&0&0\\1&1&0\\1&2&1\end{bmatrix}$$

So:

$$\mathbb{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = \mathbb{L}\,\mathbb{U}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \overset{\ell_{21}}{1} & 1 & 0 \\ \underset{\ell_{31}}{1} & \underset{\ell_{31}}{2} & 1 \end{bmatrix} \begin{bmatrix} \overset{D_1}{1} & 1 & 1 \\ 0 & \overset{D_2}{1} & 2 \\ 0 & 0 & \overset{D_3.}{1} \end{bmatrix}$$

//

Now Solve $\mathbb{A}\vec{x} = \vec{b}$ if $\vec{b} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$ by solving two (easy) triangular systems

$$\mathbb{L}\,\underbrace{\mathbb{U}\vec{x}}_{\vec{c}} = \underbrace{\begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}}_{\nwarrow \vec{b}.}$$

From before $\mathbb{U}\vec{x}=\vec{c}$

$$\rightarrow \quad \underset{\checkmark}{\mathbb{L}}\,\underset{?}{\vec{c}} = \underset{\checkmark}{\begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$$

← Solve by **forward** substitution

R1: $c_1 = 5$

R2: $\underset{5}{\cancel{c_1}} + c_2 = 7 \Rightarrow c_2 = 2$

R3: $\underset{5}{\cancel{c_1}} + 2\underset{2}{\cancel{c_2}} + c_3 = 11 \Rightarrow c_3 = 2$

$$\vec{c} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$$

E6ap3

Now solve $\underset{\checkmark}{\mathbb{U}}\,\underset{?}{\vec{x}} = \underset{\checkmark}{\vec{c}}$ with back substitution

$$\uparrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$$

R_3: $x_3 = 2$

R_2: $x_2 + 2\overset{2}{\cancel{x_3}} = 2 \rightarrow x_2 = -2$

$x_1 + \underset{-2}{\cancel{x_2}} + \underset{2}{\cancel{x_3}} = 5 \rightarrow x_1 = 5$

$$\vec{x} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$$

done

Big deal: Swap $\vec{b}$, easy to solve. Row reduction is done once and is encoded in $\mathbb{L}$ & $\mathbb{U}$.

// Extra pieces:

Our $\mathbb{A}$ was special b/c $\mathbb{A} = \mathbb{A}^T$

$\Rightarrow$ $\mathbb{L}$ and $\mathbb{U}$ are transposes of each other

But only b/c $D_1 = D_2 = D_3 = 1$

Also very useful:

Separate out pivots

$$\underset{\underset{\mathbb{A}}{m\times n}}{\begin{bmatrix}2&1&3\\2&0&0\\4&3&13\end{bmatrix}} = \underset{\substack{m\times m\\ \text{square}}}{\overset{\mathbb{L}}{\begin{bmatrix}1&0&0\\1&1&0\\2&-1&1\end{bmatrix}}} \overset{\mathbb{U}}{\begin{bmatrix}2&1&3\\0&-1&-3\\0&0&4\end{bmatrix}}$$

always 1s

$l_{21}=1$
$l_{31}=2$
$l_{32}=-1$

$D_1=2,\ D_2=-1,\ D_3=4$

Alternate factorization ($\mathbb{U}$ is different!)

$$\mathbb{A} = \underset{m\times m}{\begin{bmatrix}1&0&0\\1&1&0\\2&-1&1\end{bmatrix}} \underset{m\times m}{\begin{bmatrix}2&0&0\\0&-1&0\\0&0&4\end{bmatrix}} \begin{bmatrix}1&1/2&3/2\\0&1&3\\0&0&1\end{bmatrix}$$

$$= \mathbb{L}\,\mathbb{D}\,\mathbb{U}$$

* If $\underset{\underset{\text{next}}{\uparrow}}{\mathbb{A}=\mathbb{A}^T}$ then $\mathbb{A}=\mathbb{L}\mathbb{D}\underbrace{\mathbb{L}^T}_{\mathbb{U}}$

LE6ap4

Bad notation:

$\mathbb{U}$ in $\mathbb{L}\mathbb{U}$ $\neq$ $\mathbb{U}$ in $\mathbb{L}\mathbb{D}\mathbb{U}$

Must state which form we're using from the start.

=

Last thing: Row Swaps

^do at the start

$$\left[\begin{array}{l}\mathbb{P}\mathbb{A}=\mathbb{L}\mathbb{U}\\ \quad\text{or}\\ \mathbb{P}\mathbb{A}=\mathbb{L}\mathbb{D}\mathbb{U}\end{array}\right.$$

possible for every matrix $\mathbb{A}$

Amazing!!

Why $\mathbb{L}\mathbb{U}$ works:

mad!

Claim: $\mathbb{E}_{ij}$ matrices always combine to produce a lower triangular matrix with l_{ij}'s in the right spots & 1's along the main diagonal

$$\mathbb{L} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

why does $\mathbb{E}_{21}^{-1}\mathbb{E}_{31}^{-1}\mathbb{E}_{32}^{-1}$ work so simply?

Reason:

As we uncover $\mathbb{U}$ with row operations, we only use rows of $\mathbb{U}$ to modify lower rows of $\mathbb{A}$.

$\Rightarrow$ $\mathbb{L}$ will be lower triangular

↑

"combining" matrix tells us how to combine $\mathbb{U}$'s rows to produce $\mathbb{A}$.

3x3 example (ignoring row swaps):

Row 1 of $\mathbb{U}$ = Row 1 of $\mathbb{A}$

Row 2 of $\mathbb{U}$ = Row2 of $\mathbb{A}$ $- \, l_{21} \times \underbrace{\text{Row 1 of } \mathbb{U}}_{\text{Row 1 of } \mathbb{A}}$

Row 3 of $\mathbb{U}$ = Row 3 of $\mathbb{A}$
$- \, l_{31} \times$ Row 1 of $\mathbb{U}$
$- \, l_{32} \times$ Row 2 of $\mathbb{U}$

Invert

Row 1 of $\mathbb{A}$ = $\overset{1\times}{\text{Row 1 of } \mathbb{U}}$

Row 2 of $\mathbb{A}$ = $\overset{1\times}{\text{Row 2 of } \mathbb{U}}$
$+ \, l_{21} \times$ Row 1 of $\mathbb{U}$

Row 3 of $\mathbb{A}$ = $\overset{1\times}{\text{Row 3 of } \mathbb{U}}$
$+ \, l_{31} \times$ Row 1 of $\mathbb{U}$
$+ \, l_{32} \times$ Row 2 of $\mathbb{U}$

RHS is simple

Transposes and Symmetric Matrices ↖ ♡

Menu:
- Transposes
- Symmetric matrices
- properties of peculiar nature

//

Defn: $\underset{n\times m}{A^T}$ = the transpose of $\underset{m\times n}{A}$

= A flipped about the main diagonal

[A^T's columns are the rows of A

& " rows " " columns of A

ex $\;^{*}\begin{bmatrix}2 & 7 & 3\\ -1 & 2 & 4\end{bmatrix}^T = \begin{bmatrix}2 & -1\\ 7 & 2\\ 3 & 4\end{bmatrix}$

2×3 / $m\times n$ 3×2

ex. $\begin{bmatrix}3 & 9\\ 17 & 23\end{bmatrix}^T = \begin{bmatrix}3 & 17\\ 9 & 23\end{bmatrix}$

Defn again: $(A^T)_{ij} = (A)_{ji}$

$^{*}(A)_{21} = (A^T)_{12} = -1$

LE 7a p1

Big Deal:
The transpose of A will matter ridiculously everywhere and especially for solving $A\vec{x} = \vec{b}$

↖ our pal.

//

Defn: if $A = A^T$ (means A must be square) then we say A is symmetric and we are very happy.

↙ Super, super special matrices

//

Properties:

$$(A+B)^T = A^T + B^T \quad ✓$$

$$\underbrace{(\underbrace{\underbrace{A}_{m\times q}\underbrace{B}_{q\times n}}_{m\times n})^T}_{n\times m} \overset{?}{=} \cancel{\underbrace{A^T}_{q\times m}\underbrace{B^T}_{n\times q}}$$

can't be general...

$$= \underbrace{\underbrace{B^T}_{n\times q}\underbrace{A^T}_{q\times m}}_{n\times m} \leftarrow \text{always true}$$

← right dimensions.

$$\boxed{(\mathbb{A}\mathbb{B})^T = \mathbb{B}^T\mathbb{A}^T}$$ ← proof later

What about $(\mathbb{A}^{-1})^T$?

know
$$\mathbb{A}^{-1}\mathbb{A} = \mathbb{I} = \mathbb{A}\mathbb{A}^{-1}$$

take transposes:
$$(\mathbb{A}^{-1}\mathbb{A})^T = \mathbb{I}^T = (\mathbb{A}\mathbb{A}^{-1})^T$$

$$(\mathbb{A}^T)(\mathbb{A}^{-1})^T = \mathbb{I} = (\mathbb{A}^{-1})^T(\mathbb{A})^T$$

$$\Rightarrow (\mathbb{A}^T)^{-1} = (\mathbb{A}^{-1})^T$$

If $\mathbb{A}=\mathbb{A}^T$, $(\mathbb{A}^T)^{-1} = (\mathbb{A}^{-1})^T$

$$\begin{array}{c} \| \\ (\mathbb{A})^{-1} \end{array} \longrightarrow (\mathbb{A}^{-1})^T = \mathbb{A}^{-1}$$

So if $\mathbb{A}$ is symmetric, so its inverse

E7ap2

Crazily important objects:

Square matrices: $\underbrace{\mathbb{A}^T}_{n\times m}\underbrace{\mathbb{A}}_{m\times n}$ ($n\times n$) & $\underbrace{\mathbb{A}}_{m\times n}\underbrace{\mathbb{A}^T}_{n\times m}$ ($m\times m$)

$\mathbb{A}$ does not have to be square

$$(\mathbb{A}^T\mathbb{A})^T = (\mathbb{A}^T)\overset{\text{undo}}{(\mathbb{A}^T)^T} = \mathbb{A}^T\mathbb{A}$$

So $\mathbb{A}^T\mathbb{A}$ is always symmetric

Check: true for $\mathbb{A}\mathbb{A}^T$ as well.

ex

$$\mathbb{A} = \underset{3\times 2}{\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}}$$

$$\underset{2\times 2}{\mathbb{A}^T\mathbb{A}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

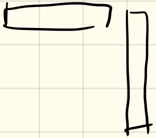
$$\underset{3\times 3}{\mathbb{A}\mathbb{A}^T} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

#awesome

E7ap3

Inner product:

row vector → $\vec{a}^T \vec{b}$

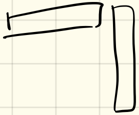


what happens with?:

$$(\vec{a}^T \vec{b})^T$$

$$= \vec{b}^T (\vec{a}^T)^T$$

$$= \underbrace{\underbrace{\vec{b}^T}_{1\times n} \underbrace{\vec{a}}_{n\times 1}}_{1\times 1}$$



More advanced inner producting:

$\underbrace{A\vec{x}}_{\text{transformation of } \vec{x}}$ & $\vec{y}$

ex:

$$(A\vec{x})^T \vec{y} = \vec{x}^T A^T \vec{y}$$

$$= \vec{x}^T (A\vec{y})$$

Inner product of $\vec{x}$ & $A^T \vec{y}$

Transform $\vec{y}$ with A^T first

More on the Transpose

E7bp1

menu:
- example of $(AB)^T = B^T A^T$
- three different proofs

ex

$$\left(\underset{A}{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}} \underset{B}{\begin{bmatrix} -1 & 4 \\ 3 & 1 \end{bmatrix}} \right)^T = \begin{bmatrix} 7 & 11 \\ -4 & 3 \end{bmatrix}^T = \begin{bmatrix} 7 & -4 \\ 11 & 3 \end{bmatrix}$$

$$\overset{B^T}{\begin{bmatrix} -1 & 4 \\ 3 & 1 \end{bmatrix}^T} \overset{A^T}{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}^T}$$

$$= \begin{bmatrix} -1 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 11 & 3 \end{bmatrix}$$

$$(AB)^T_{ij}$$

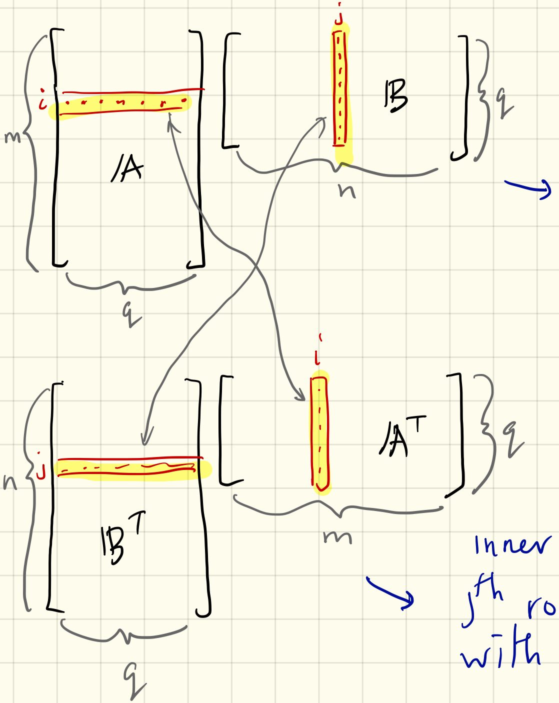
$$= (\underset{m\times q}{A}\,\underset{q\times n}{B})_{ji} = \sum_{k=1}^{q} a_{jk} b_{ki}$$

inner product of jth row of A & ith column of B.

$$= \sum_{k=1}^{q} (A^T)_{kj} (B^T)_{ik}$$

$$= \sum_{k=1}^{q} (B^T)_{ik} (A^T)_{kj}$$

$$= (B^T A^T)_{ij}$$



Inner product of i^{th} row of $\mathbb{A}$ with j^{th} column of $\mathbb{B}$

$$= (\mathbb{A}\mathbb{B})_{ij}$$

$$= ((\mathbb{A}\mathbb{B})^T)_{ji}$$

Inner product of j^{th} row of $\mathbb{B}^T$ with i^{th} column of $\mathbb{A}^T$

$$= (\mathbb{B}^T\mathbb{A}^T)_{ji}$$

$$(\mathbb{A}\mathbb{B})^T = \mathbb{B}^T\mathbb{A}^T$$

Yet another way:

column picture

$$(\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}})^T = \left(x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n\right)^T$$

$$= x_1\vec{a}_1^{\,T} + \dots + x_n\underbrace{\vec{a}_n^{\,T}}_{\text{row vector}}$$

$$= [x_1 \dots x_n]\begin{bmatrix} - & \vec{a}_1^{\,T} & - \\ - & \vec{a}_2^{\,T} & - \\ & \vdots & \\ - & \vec{a}_n^{\,T} & - \end{bmatrix}$$

$$= \boxed{\vec{x}^{\,T} A^T}$$

use here

$$(\underset{m\times q}{A}\,\underset{q\times n}{B})^T$$

$$= \left(A\begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_n \\ | & | & & | \end{bmatrix}\right)^T$$

$$= \left(\begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \dots & A\vec{b}_n \\ | & | & & | \end{bmatrix}\right)^T$$

E7bp3

$$= \begin{bmatrix} - & (A\vec{b}_1)^T & - \\ - & (A\vec{b}_2)^T & - \\ & \vdots & \\ - & (A\vec{b}_n)^T & - \end{bmatrix}$$

$$= \begin{bmatrix} - & \vec{b}_1^{\,T}A^T & - \\ - & \vec{b}_2^{\,T}A^T & - \\ & \vdots & \\ - & \vec{b}_n^{\,T}A^T & - \end{bmatrix}$$

$$= \begin{bmatrix} - & \vec{b}_1^{\,T} & - \\ & \vdots & \\ - & \vec{b}_n^{\,T} & - \end{bmatrix} A^T$$

$$= B^T A^T$$

Yes!!

"Paging Dr. Vector Spaceman"

Menu:

- Our new plan for $\mathbb{A}\vec{x}=\vec{b}$
- Vector spaces, introduction to

The Column picture for $\mathbb{A}\vec{x}=\vec{b}$

$$\mathbb{A} = \begin{array}{c}\begin{bmatrix}3 & 1 & -1\\ 0 & 1 & 0\\ 2 & 0 & 4\end{bmatrix}\\ \begin{array}{ccc}\uparrow & \uparrow & \uparrow\\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{array}\end{array} \qquad \vec{b} = \begin{bmatrix}2\\1\\3\end{bmatrix}$$

We solve

$$x_1\begin{bmatrix}3\\0\\2\end{bmatrix} + x_2\begin{bmatrix}1\\1\\0\end{bmatrix} + x_3\begin{bmatrix}-1\\0\\4\end{bmatrix} = \begin{bmatrix}2\\1\\3\end{bmatrix}$$

for $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

Game:

Find out how we may combine column vectors of A to create/generate/reach $\vec{b}$

New idea:

Understand the places ("spaces") where $\vec{a}$'s, $\vec{x}$'s, and $\vec{b}$'s live

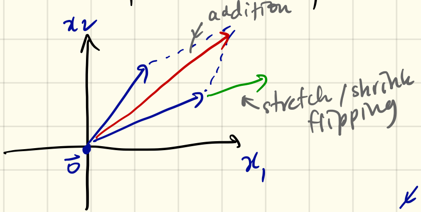
E8op1

Big things coming:

- Null space of $\mathbb{A}$
- Column Space of $\mathbb{A}$
- Row Space of $\mathbb{A}$
- Left Null Space of $\mathbb{A}$
- Beautiful connection to $\mathbb{A}^T$, $\mathbb{A}$'s transpose

---

Example Vector Space: Idealized plane



basic structure

restrictions

$$V = R^{\textcircled{2}} = \left\{ \begin{bmatrix}x_1\\x_2\end{bmatrix} ; x_1, x_2 \in R \right\}$$

curly braces

real numbers $R = R^1$

Two (pretty obvious) features of vector spaces:

They are closed under addition and scalar multiplication.

(1) If we add any two vectors in V, we get another vector that's still in V

(2) If we multiply a vector in V by a scalar (for us: a real number), the result is still in V.

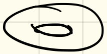
$$\underset{\in R^2}{\begin{bmatrix}3\\4\end{bmatrix}} + \underset{\in R^2}{\begin{bmatrix}773\\6\end{bmatrix}} = \underset{\in R^2}{\begin{bmatrix}776\\10\end{bmatrix}}$$

"is an element of"

$$\underset{\substack{\in R\\ \text{(real numbers)}}}{7}\underbrace{\begin{bmatrix}3\\4\end{bmatrix}}_{\in R^2} = \underbrace{\begin{bmatrix}21\\28\end{bmatrix}}_{\in R^2}$$

Examples of things that are and are not Vector Spaces: (E8ap2

(1) $\vec{v}_1 = 3$  $+ 4$ 

$\vec{v}_2 = 2$  $+ 1.3$ 

$V = \Big\{ \Big[$   $\Big] \Big\}$

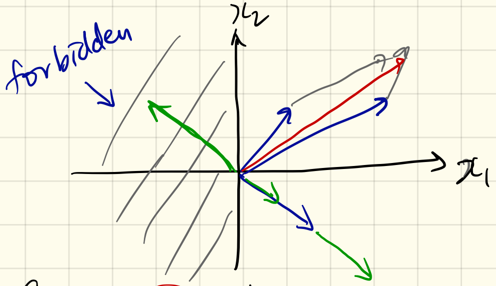
(2) $$V = \Big\{ f(x) = \overset{x}{c_2}x^2 + \overset{y}{c_1}x + \overset{z}{c_0} \overset{\text{`such that'}}{\Big|} c_2, c_1, c_0 \in R \Big\}.$$

$f_1(x) = 2x^2+3$

$f_2(x) = -7x^2+3x+4$

$f_1(x)+f_2(x) = -5x^2+3x+7$

(3)



$$V = \left\{ \vec{x} = \begin{bmatrix} \boxed{x_1} \\ x_2 \end{bmatrix} \;\middle|\; \underbrace{x_2 \in R}_{-\infty < x_2 < \infty},\ x_1 \geq 0 \right\}$$

Not a vector space

Observe:

Addition works

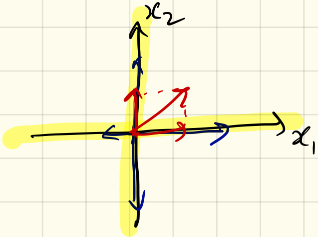
if $\vec{v}_1, \vec{v}_2 \in V$ then $\vec{v}_1 + \vec{v}_2 \in V$

Scalar multiplication fails!

$$\underbrace{-3\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\in V} = \underbrace{\begin{bmatrix} \boxed{-3} \\ -3 \end{bmatrix}}_{\notin V}$$

Note: we're starting to talk about subspaces

(4) All points on axes of R^2 ∠E8ap3



Now see

Scalar multiplication works but addition fails

$$\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\in V} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\in V} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

not in V

(5) All $m \times n$ matrices form a vector space

$\equiv R^{mn}$

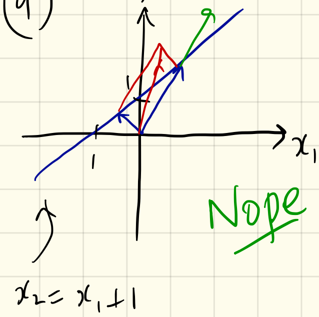
(6) What about the integers ✗

(7) " " rational numbers ✗

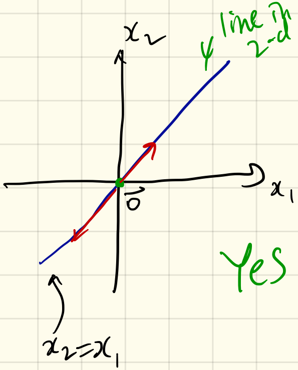
(8) " " real numbers ✓

LE8ap4

(a)



Nope



Yes

Vector Spaces Inside Vector Spaces

Menu:

- vector space requirements
- subspace requirements

General Requirements of a Vector Space:

VSP1 if $\vec{x}_1, \vec{x}_2 \in V$ then $\vec{x}_1 + \vec{x}_2 \in V$

VSP2 if $\vec{x} \in V$ then $c\vec{x} \in V$ for all $c \in R$

VSP3 $\vec{0} \in V$ and $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x}$ in V

vector space property

+ a series of increasingly boring conditions such as $c(\vec{x}_1 + \vec{x}_2) = cx_1 + cx_2$

E8bp1

Our focus: R^n, $n = 0, 1, 2, \ldots$

Super big deal:

Vector Spaces have vector spaces inside them and we call these subspaces

Need three properties for a subset S of V to be a subspace:

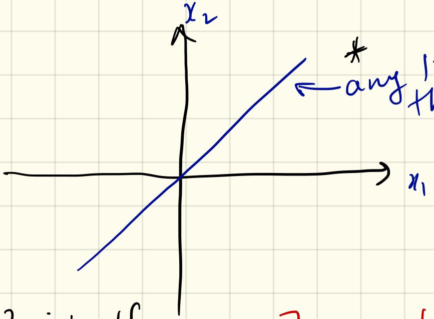
SSP1 if $\vec{x}_1 \& \vec{x}_2 \in S$ then $\vec{x}_1 + \vec{x}_2 \in S$

SSP2 if $\vec{x} \in S$ then $c\vec{x} \in S$ for all $c \in R$

SSP3 $\vec{0} \in S$

Examples of subspaces:

$\mathbb{R}^2$



* any line through $\vec{0}$

∞ly many

- * $\mathbb{R}^2$ itself
- * $\{\vec{0}\}$ works too

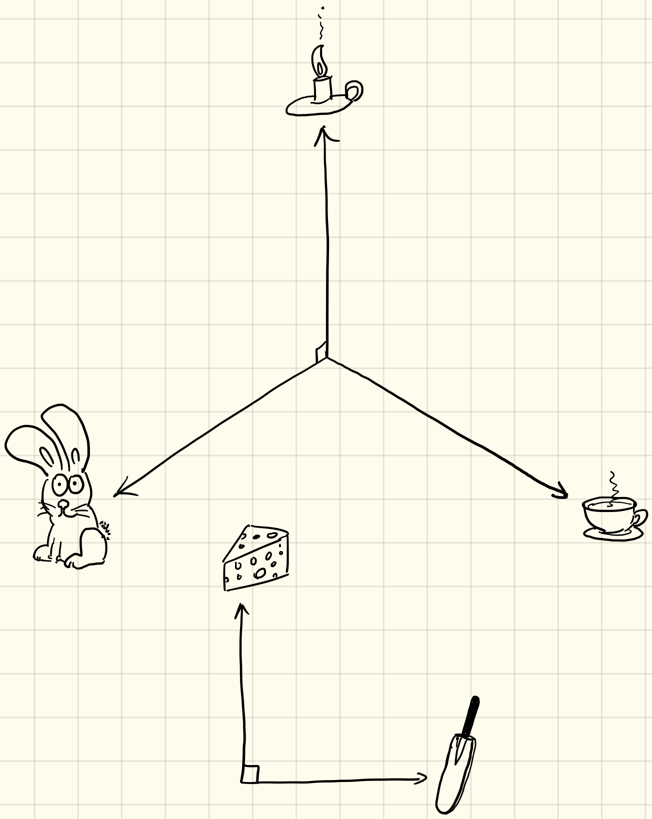
] important!

$\mathbb{R}^3$: Subspaces

- * $\mathbb{R}^3$ itself
- * $\{\vec{0}\}$
- * any line through $\vec{0}$
- * any plane " $\vec{0}$

E8bp2

Bonus very silly Spaces:



"Danger Will Robinson! We are entering column space!!"

Menu: • Column space for $A\vec{x}=\vec{b}$

→ the first of four awesome subspaces

Our [beloved / belated] problem $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$

delete as applicable

rows columns

The column picture:

$$\underset{\substack{\uparrow\\ \text{each column has m entries}}}{x_1\vec{a}_1} + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b}$$

Observation / Big deal:

Columns of A and $\vec{b}$ live in R^m

(not R^n)

↑

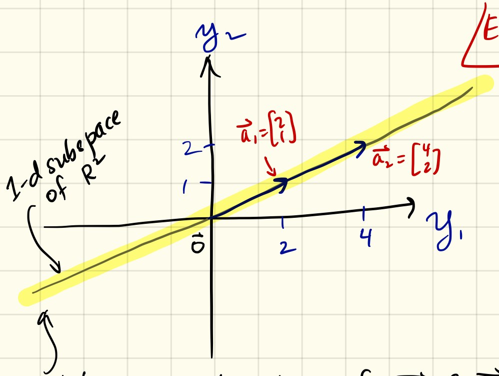
$\vec{x}$ lives here

ex $A = \begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}$

$\vec{a}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix}4\\2\end{bmatrix}$

↑ live in y-space

E8cp1



all linear combinations of $\vec{a}_1$ & $\vec{a}_2$, i.e. $x_1\vec{a}_1 + x_2\vec{a}_2$, live on this line which is a subspace of R^2

Huge: ← notation

$$C(\underset{m\times n}{A}) = \text{Column space of } A$$

$$= \text{Subspace of } R^m$$

Here:

$$C(A) \underset{\text{by eye}}{=} \left\{ \vec{y}\in R^2 \;\middle|\; \vec{y} = c\begin{bmatrix}2\\1\end{bmatrix},\ c\in R \right\}$$

"such that"

Big deal:

$A\vec{x} = \vec{b}$ has a solution (1 or ∞ly many) only if $\vec{b} \in C(A)$

" $\vec{b}$ lives in the column of A".

$\Rightarrow$ If $\vec{b} \notin A$, $A\vec{x} = \vec{b}$ has no solution.

For $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 38 \\ 19 \end{bmatrix} \in C(A)$ ← we'll see means there are ∞ly many solut.

$\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin C(A)$

↑ so no solution to $A\vec{x} = \vec{b}$

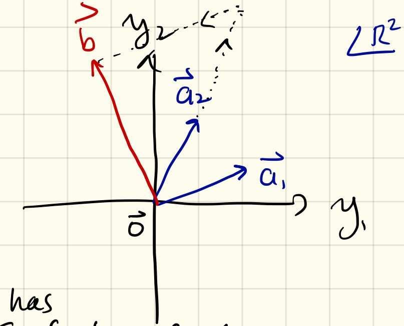
LE8cp2

LR^2

ex

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\vec{a}_1$ $\vec{a}_2$



See that $A\vec{x} = \vec{b}$ always has a solution. In fact, only 1.

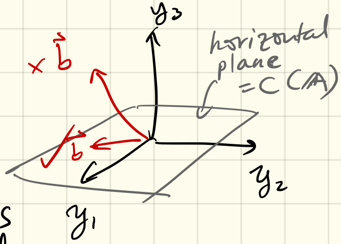
$\Rightarrow C(A) = R^2$

ex $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

3×2

tall

$\hat{i}$ $\hat{j}$ ↖ 2 vectors in 3-d.



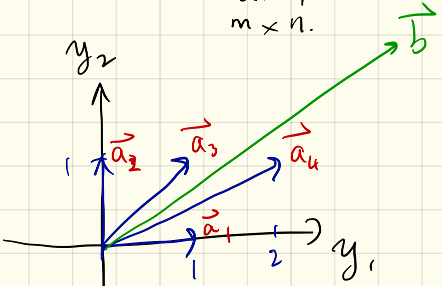
↓

$C(A) \neq R^3$ ← m=3

ex $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

wide A →

2×4
$m \times n.$



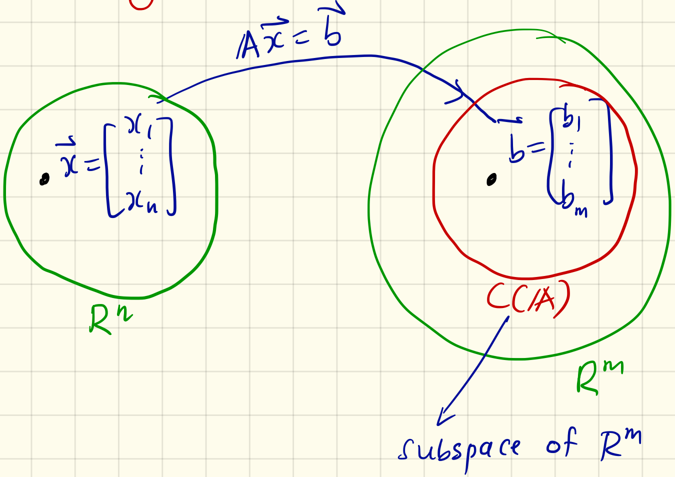
See: any two column vectors of A will work.

⇒ ∞ly many solutions

Again $C(A) = R^2 \leftarrow m=2$

E8cp3

Emerging Picture



A new realm opens up: Null Space

Menu:
- Definition of the Null space of A, N(A)
- what C(A) & N(A) mean for $A\vec{x}=\vec{b}$

Consider $A\vec{x}=\vec{0}$ ← very special $\vec{b}$

called Nullspace Equation
or Homogeneous Equation

how can we combine columns of A to produce nothing?

Immediate Observation: $A\vec{0}=\vec{0}$

So: Always a solution → $\vec{0}\in C(A)$ always

→ May be 1 or ∞ly many

Example: Solve $A\vec{x}=\vec{0}$ for:

$$A=\underset{m\,3\times 3\,n}{\begin{bmatrix}1&2&3\\1&-3&-2\\0&1&1\end{bmatrix}} \xrightarrow[\text{pain \& suffering}]{\text{row ops}} \vec{x}=c\begin{bmatrix}1\\1\\-1\end{bmatrix} \checkmark$$

where $c\in\mathbb{R}$

$$N(A)=\left\{\vec{x}\in\mathbb{R}^{3} \;\middle|\; \vec{x}=c\begin{bmatrix}1\\1\\-1\end{bmatrix},\ c\in\mathbb{R}\right\}$$

$3 \to n$

↑ Null space of A

↖ subspace of $\mathbb{R}^3$.

E9ap1

Notation:

$\vec{x}_n$ for a nullspace vector

Also $\vec{x}_h$ ← homogeneous

Now solve:

$$A\vec{x}=\begin{bmatrix}1&2&3\\1&-3&-2\\0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\overset{\vec{b}}{\begin{bmatrix}4\\-1\\1\end{bmatrix}}$$

non-zero interesting $\vec{b}$

more pain and suffering (row operations)

find $\vec{x}_r=\begin{bmatrix}1\\0\\1\end{bmatrix}$ ← could be called $\vec{x}_p$ where p is "particular"

we'll write $\vec{x}_r$ because some dying monk said we should

So $A\vec{x}_n=0$ & $A\vec{x}_r=\begin{bmatrix}4\\-1\\1\end{bmatrix}$ ← $\vec{b}$

⇒ See $\vec{x}_r$ is not a unique solution b/c $A(\vec{x}_r+\vec{x}_n)=\vec{b}$

E9ap2

More:

$$\underbrace{\begin{bmatrix}1&2&3\\1&-3&-2\\0&1&1\end{bmatrix}}_{A}\left(\underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\vec{x}_r}+c\underbrace{\begin{bmatrix}1\\1\\-1\end{bmatrix}}_{\vec{x}_n}\right)$$

$$=\underbrace{\begin{bmatrix}1&2&3\\1&-3&-2\\0&1&1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}}+c\underbrace{\begin{bmatrix}1&2&3\\1&-3&-2\\0&1&1\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix}}$$

$$=\begin{bmatrix}4\\-1\\1\end{bmatrix}+\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$=\begin{bmatrix}4\\-1\\1\end{bmatrix}=\vec{b}$$

$\Rightarrow$ There are infinitely many solutions b/c $N(A)\neq\{\vec{0}\}$

In general:

$$A(\vec{x}_r+\vec{x}_n)=\underbrace{A\vec{x}_r}_{\vec{b}}+\overset{\vec{0}}{\cancel{A\vec{x}_n}}$$

The Big Deals:

① All vectors $\vec{x}\in R^n$ for which $A\vec{x}=\vec{0}$ form a subspace of R^n

SSP1 $\quad A(\vec{x}_1+\vec{x}_2)=\overset{\vec{0}}{\cancel{A\vec{x}_1}}+\overset{\vec{0}}{\cancel{A\vec{x}_2}}=\vec{0}$

SSP2 $\quad A(c\vec{x})=c(A\vec{x})=c\vec{0}=\vec{0}$

SSP3 $\quad \vec{0}\in N(A)$ b/c $A\vec{0}=\vec{0}$

(2)
- If $N(A)=\{\vec{0}\}$ and $\vec{b}\in C(A)$ then $A\vec{x}=\vec{b}$ has one, unique solution
- If $N(A)\neq\{\vec{0}\}$ and $\vec{b}\in C(A)$ then $A\vec{x}=\vec{b}$ has infinitely many solutions
- if $\vec{b}\notin C(A)$ then $A\vec{x}=\vec{b}$ has no solutions

Row Reduction, as you wish.

menu:

- Turning $\mathbb{A}\vec{x}=\vec{b}$ into $\mathbb{R}_{\mathbb{A}}\vec{x}=\vec{d}$
- Reduced Row Echelon Forms (RREFs)
- Pivot and free variables
- The rank r of a matrix ← so much winning
- Fezzik

$$\mathbb{A}=\begin{bmatrix}2&4&3&4\\2&4&6&10\\6&12&12&18\end{bmatrix} \qquad \vec{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

Fezzik

3×4

$\vec{x}\in\mathbb{R}^4$

arbitrary

Monks tell us:

Solve $\mathbb{A}\vec{x}=\vec{b}$ for general $\vec{b}$

$$[\mathbb{A}\,|\,\vec{b}\,]=\left[\begin{array}{cccc|c}\overset{D_1}{2}&4&3&4&b_1\\2&4&6&10&b_2\\6&12&12&18&b_3\end{array}\right]$$

$$R_2'=1\cdot R_2-\left(\frac{2}{2_{D_1}}\right)R_1$$

$$R_3'=1\cdot R_3-\left(\frac{6}{2_{D_1}}\right)R_1$$

$$\left[\begin{array}{cccc|c}\overset{D_1}{2}&4&3&\overset{D_2}{4}&b_1\\0&0&3&6&b_2-b_1\\0&0&3&6&b_3-3b_1\end{array}\right]$$

$$R_3'=1\cdot R_3-\left(\frac{3}{3}\right)$$

E9bp1

$$\left[\begin{array}{cccc|c}\text{pivot column}&\text{free column}&\text{pivot column}&\text{free column}&\\2&4&3&4&b_1\\0&0&3&6&b_2-b_1\\0&0&0&0&b_3-b_2-2b_1\end{array}\right]$$

$U\vec{x}=\vec{c}$,

$(b_3-3b_1)-(b_2-b_1)$

Keep Going!!
(as with inverses).

$$R_1'=1\cdot R_1-\left(\frac{3}{3}\right)R_2$$

$$\left[\begin{array}{cccc|c}2&4&0&-2&2b_1-b_2\\0&0&3&6&b_2-b_1\\0&0&0&0&b_3-b_2-2b_1\end{array}\right]$$

last step:
divide through by pivots

$R_1'=\frac{1}{2}R_1$

$R_2'=\frac{1}{3}R_2$

1st 2 columns of Identity matrix

$$\left[\begin{array}{cccc|c}1&2&0&-1&\frac{1}{2}(2b_1-b_2)\\0&0&1&2&\frac{1}{3}(b_2-b_1)\\0&0&0&0&b_3-b_2-2b_1\end{array}\right]$$

$$=\left[\,\mathbb{R}_{\mathbb{A}}\,\middle|\,\vec{d}\,\right]$$

Reduced Row Echelon Form of $\mathbb{A}$

Big Deal Things:

- We can't reduce any further
- $\mathbb{R}_{\mathbb{A}}$ is unique for any $\mathbb{A}$
- Row swaps are still part of the game
- New — pivots may appear irregularly

- Pivot columns match Identity Matrix Columns
- We call x_j that match up with pivot columns, the pivot variables.
- Similarly: free columns $\leftrightarrow$ free variables

For Fezzik x_1 & x_3 are pivot variables

x_2 & x_4 are free variables

Very, very big deal:

Definition:

$$\boxed{\#\text{ pivot columns in } \mathbb{R}_{\mathbb{A}} = \text{rank of } \mathbb{A}}$$

Notation: rank of $\mathbb{A} = r$

For Fezzik: $r=2$

$\rightarrow m=3, n=4, r=2$

Huge idea: Inside every matrix $\mathbb{A}_{m\times n}$ there is an invertible $r\times r$ square matrix

The Search for Column Space … ← C(A)

Our Story:

C(A) = all $\vec{b}$'s for which $A\vec{x} = \vec{b}$ has a solution.

↓ subspace of R^m

Method 1 of 3 for finding C(A):

reduce $[A \mid \vec{b}]$ ← general $\vec{b}$ to $[R_A \mid \vec{d}]$ and for all rows of 0 in R_A, set matching entries in $\vec{d}$ to 0.

Our friend Fezzik:

$$[A \mid \vec{b}] = \left[\begin{array}{cccc|c} 2 & 4 & 3 & 4 & b_1 \\ 2 & 4 & 6 & 10 & b_2 \\ 6 & 12 & 12 & 18 & b_3 \end{array}\right]$$

previous row reduction suffering

↙ $\vec{d}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & \frac{1}{2}(2b_1 - b_2) \\ 0 & 0 & 1 & 2 & \frac{1}{3}(b_2 - b_1) \\ 0 & 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right] = [R_A \mid \vec{d}]$$

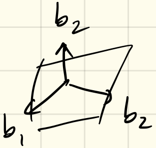
E9cp1

$$\underbrace{0x_1 + 0x_2 + 0x_3 + 0x_4}_{0} = \underbrace{b_3 - b_2 - 2b_1}_{\text{must be } 0}$$

→

$\Rightarrow b_3 - b_2 - 2b_1 = 0$

↖ eq of a plane in



true but not useful.

Better (but not only way):

Set $b_3 = b_2 + 2b_1$, where $b_1, b_2 \in R$

↖ b_3 depends on ↗

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_2 + 2b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

fixed vectors

always do this.

where $b_1, b_2 \in R$.

Formal result:

$$C(A) = \left\{ \vec{b} \in R^3 \;\middle|\; \vec{b} = b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ b_1, b_2 \in R \right\}$$

Big deal:

See $C(\mathbb{A})$ is a $\overset{\text{rank of } \mathbb{A},\ r}{\boxed{2}}$-d subspace (plane) of R^3 $(m=3)$

#awesome

Notes

* Because $C(\mathbb{A})$ does not fill up R^3 then $\mathbb{A}\vec{x} = \vec{b}$ may or may not have solutions

* Nothing super special about $\begin{bmatrix}1\\0\\2\end{bmatrix}$ & $\begin{bmatrix}0\\1\\1\end{bmatrix}$

×2

×−3

$\begin{bmatrix}2\\0\\4\end{bmatrix}$, $\begin{bmatrix}0\\-3\\-3\end{bmatrix}$ would work

*we had $b_3 - b_2 - 2b_1 = 0$

$$\underset{\uparrow \text{ dependent var}}{b_2} = b_3 - 2b_1$$

LE9cp2

very nutritious

$$* \quad b_3 - b_2 - 2b_1 = 0$$

$$\underbrace{\begin{bmatrix}-2 & -1 & 1\end{bmatrix}}_{1\times 3}\underbrace{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix}0\end{bmatrix}}_{1\times 1}$$

$$\mathbb{A} \quad \vec{x} = \vec{0}$$

Solve a nullspace problem to find $C(\mathbb{A})$

The Search for Null Space, $N(A)$:

Quest: find all $\vec{x}$ such that $\underbrace{A\vec{x}}=\vec{0}$

$\vec{x}$ is on right of A ⇒ Right null space

Again, with Fezzik:

$$[A\,|\,\cancel{\vec{b}}\,\vec{0}\,] = \left[\begin{array}{cccc|c} 2 & 4 & 3 & 4 & b_1 \\ 2 & 4 & 6 & 10 & b_2 \\ 6 & 12 & 12 & 18 & b_3 \end{array}\right]$$

previous row reduction suffering

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & \frac{1}{2}(2b_1-b_2) \\ 0 & 0 & 1 & 2 & \frac{1}{3}(b_2-b_1) \\ 0 & 0 & 0 & 0 & b_3-b_2-2b_1 \end{array}\right] = [\,R_A\,|\,\vec{d}\,]$$

$\vec{d}$ $\vec{0}$ $\vec{0}$

For $N(A)$, set $\vec{b}=\vec{0}$ (or start with $\vec{b}=\vec{0}$)

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

. hats not in video
. the shame

E9dp1

Usual story:

free variables

$$x_1 + 2x_2 \quad\quad -x_4 = 0$$
$$x_3 + 2x_4 = 0$$

pivot variables

Always do the following well defined procedure:
Express pivot variables (dependent) in terms of the free variables (independent)

$$\Rightarrow \begin{aligned} x_1 &= -2x_2 + x_4 \\ x_3 &= 0 \quad -2x_4 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2+x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

where $x_2, x_4 \in \mathbb{R}$

$A\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

$A\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \vec{0}$

always do this!

replace pivot vars with free vars

LE9dp2

Formally:

$$N(A) = \left\{ \vec{x} \in \mathbb{R}^{\overset{n}{\textcircled{4}}} \;\middle|\; \vec{x} = x_2 \begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + x_4 \begin{bmatrix}1\\0\\-2\\1\end{bmatrix} \right\}$$

$x_2, x_4 \in \mathbb{R}$

For Fezzik, this is a plane in 4 dimensions

Notes:

* b/c $N(A) \neq \{\vec{0}\}$, if $A\vec{x} = \vec{b}$ has a solution (i.e, $\vec{b} \in C(A)$) then there a ∞ly many solutions

* Soon we we'll see that the dimension of $N(A)$ is

$$\dim N(A) = \underset{\#\text{ columns}}{n} - \underset{\text{rank of } A}{r}$$

Fezzik: $4 - 2 = 2$ ✓

Solving $A\vec{x}=\vec{b}$ the Subspace Way:

- Fezzik with $\vec{b}$ in $C(A)$ & $\vec{b}\neq\vec{0}$

From before:

$$[A\,|\,\vec{b}\,] = \left[\begin{array}{cccc|c} 2 & 4 & 3 & 4 & b_1 \\ 2 & 4 & 6 & 10 & b_2 \\ 6 & 12 & 12 & 18 & b_3 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} P & F & P & F & \\ 1 & 2 & 0 & -1 & \frac{1}{2}(2b_1-b_2) \\ 0 & 0 & 1 & 2 & \frac{1}{3}(b_2-b_1) \\ 0 & 0 & 0 & 0 & b_3-b_2-2b_1 \end{array}\right] = [R_A\,|\,\vec{d}\,]$$

example: $\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$ b_1 b_2 b_3 ← see this is the first column!

$$[R_A\,|\,\vec{d}\,] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \leftarrow ✓$$

$\vec{b} \in C(A)$

E9ep1

Plan: Use same steps as for finding $N(A)$

$$\begin{array}{ccccc} P & F & P & F & \\ x_1 + 2x_2 & & & -x_4 & = 1 \\ & & x_3 & +2x_4 & = 0 \end{array}$$

each pivot variable appears only once in all equations

put free variables on RHS

$$x_1 = 1 - 2x_2 + x_4$$
$$x_3 = 0 + 0x_2 - 2x_4$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-2x_2+x_4 \\ 0+x_2+0x_4 \\ 0-2x_4+0x_2 \\ 0+x_4+0x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}_p} + \underbrace{x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\vec{x}_h}$$

p = particular

where $x_2, x_4 \in \mathbb{R}$

h = homogeneous

replace pivot vars w. free var expression.

General Story:

$$A(\overbrace{\vec{x}_p+\vec{x}_h}^{\vec{x}}) = \underbrace{A\vec{x}_p}_{\vec{b}} + \underbrace{\cancel{A\vec{x}_h}}_{\vec{0}} = \vec{b}$$

not unique

not necessarily the same

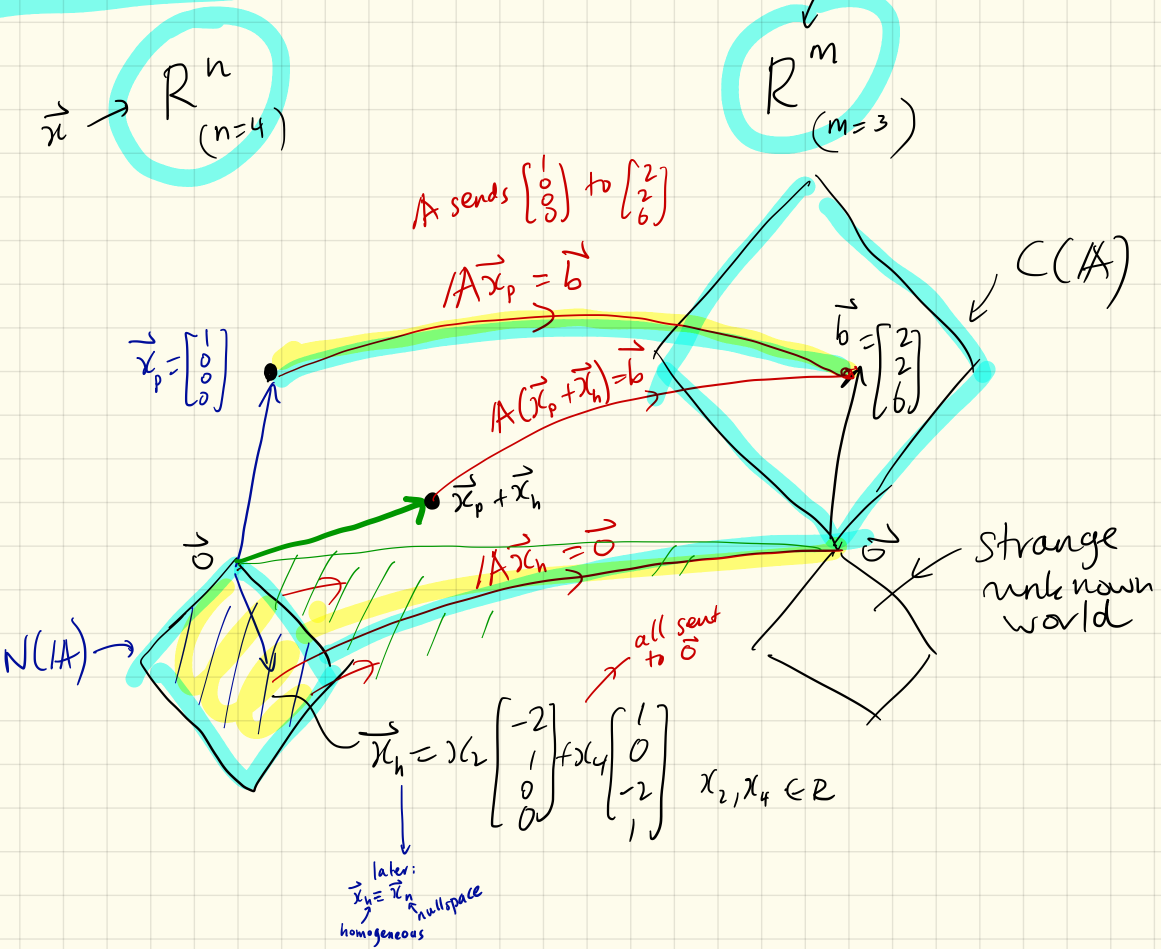
Later:

$$A(\underbrace{\vec{x}_r+\vec{x}_n}_{\vec{x}}) = \underbrace{A\vec{x}_r}_{\vec{b}} + \underbrace{\cancel{A\vec{x}_n}}_{\vec{0}}$$

think of as most excellent $\vec{x}_p$

unique ↑ row ↑ null

Big picture emerging for $A\vec{x}=\vec{b}$.



"I see null vectors"

- Jumping to the form of $N(\mathbb{A})$ from $\mathbb{R}_{\mathbb{A}}$

Fezzik's $\mathbb{R}_{\mathbb{A}}$:

$$\begin{array}{c}\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} p & F & p & F \end{array}\end{array} \longrightarrow \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \mathbb{F}$$

pivot columns: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{I}_2$ ↑ $r=2$

Our two null vectors from our earlier solut for $N(\mathbb{A})$:

make a matrix

$$\mathbb{N} = \begin{array}{c} p \\ F \\ p \\ F \end{array} \begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \qquad -\mathbb{F} = -\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \qquad \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

E9fp1

See $\underset{m\times n}{\mathbb{R}_{\mathbb{A}}}\ \underset{n\times(n-r)}{\mathbb{N}} = \underset{m\times(n-r)}{\mathbb{0}}$

↑ secret

↑ all zeros = kevin the destroyer of words

also $\mathbb{A}\,\mathbb{N} = \mathbb{0}$

↳ $\mathbb{A}$ & $\mathbb{R}_{\mathbb{A}}$ have the same Null space.

General Story

$$\underset{m\times n}{\mathbb{R}_{\mathbb{A}}} = \left[\begin{array}{c|c} \overset{r\times r}{\mathbb{I}} & \overset{r\times(n-r)}{\mathbb{F}} \\ \hline 0\ 0 \cdots 0 & \cdots \end{array}\right]$$

← permutation of x's

← $(m-r)\times n$ ← all zeros $\mathbb{0}$

$m-r$ rows of 0s

$$\underset{n\times(n-r)}{\mathbb{N}} = \begin{bmatrix} -\mathbb{F} \\ \hline \mathbb{I} \end{bmatrix}$$

$n-r = \dim N(\mathbb{A})$

$$\underset{m\times n}{\mathbb{R}_{\mathbb{A}}}\ \underset{n\times(n-r)}{\mathbb{N}} = \begin{bmatrix} -\overbrace{\mathbb{I}\,\mathbb{F}}^{r\times(n-r)} + \overbrace{\mathbb{F}\,\mathbb{I}}^{r\times r} \\ \hline \underset{(m-r)\times(n-r)}{\mathbb{0}} \end{bmatrix} = \underset{m\times(n-r)}{\mathbb{0}}$$

absent in videos due to cavalier attitude

E9gp1

Solving $A\vec{x}=\vec{b}$ the Subspace way: simpler examples

(1)

$$\underset{m\times n}{A} = \begin{bmatrix}1&0\\0&1\end{bmatrix} \leftarrow \text{the identity matrix}$$

$A\vec{x}=\vec{b}$ is always solvable!

$\uparrow I \rightarrow \vec{x}=\vec{b}$

see $R_A = A = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\qquad\qquad\quad\uparrow p \quad \uparrow p$

* **Column Space**

Solve $\left[\begin{array}{cc|c}1&0&b_1\\0&1&b_2\end{array}\right]$ for $C(A)$ $\leftarrow R_A$ already here

$\quad p \quad p$

$\Rightarrow b_1, b_2 \in R$ (no restrictions)

$$\vec{b} = \underbrace{\begin{bmatrix}b_1\\b_2\end{bmatrix}}_{\text{always}} = b_1\begin{bmatrix}1\\0\end{bmatrix} + b_2\begin{bmatrix}0\\1\end{bmatrix}$$

$b_1, b_2 \in R$

$\Rightarrow C(A) = R^2 \leftarrow m=2$

* **Null Space**

Solve $\left[\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right]$

$\Rightarrow \begin{array}{l}x_1=0\\x_2=0\end{array} \Rightarrow \vec{x} = \begin{bmatrix}0\\0\end{bmatrix} = \vec{0}$ is only solution

$N(A) = \{\vec{0}\}$.

Delectable Upshots
- Every $\vec{b} \in C(A)$ so $A\vec{x}=\vec{b}$ is always solvable
- Because $N(A)=\{\vec{0}\}$, every solution is unique.

$$\dim C(A) = 2 \;(= \overset{\text{later}}{r})$$

$$\dim N(A) = 0 \;(= \underset{2-2}{n-r})$$

(2)

*see E8cp1 for first examination of this $\mathbb{A}$

$$\mathbb{A} = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} \text{ with } \vec{b} = \begin{bmatrix}2\\4\end{bmatrix} \text{ \& then } \vec{b} = \begin{bmatrix}2\\2\end{bmatrix}$$

see columns are multiples of each other as are rows

Find $C(\mathbb{A})$:

$$[\mathbb{A} \mid \vec{b}\,] = \left[\begin{array}{cc|c} \overset{D_1}{1} & 2 & b_1\\ 2 & 4 & b_2\end{array}\right]$$

$$\sim \quad R_2' = R_2 - \left(\frac{2}{\underset{D_1}{1}}\right) R_1$$

rank $r=1$

$$\left[\begin{array}{cc|c} P & F & \\ 1 & 2 & b_1\\ 0 & 0 & b_2-2b_1\end{array}\right] \qquad \underbrace{0 \quad 0}_{\mathbb{R}_{\mathbb{A}}} \qquad \underbrace{b_2-2b_1}_{d} = 0$$

$\Rightarrow$ $b_2 - 2b_1 = 0$

$b_2 = 2b_1$ where $b_1 \in R$

$\uparrow$ dependent $\qquad$ $\uparrow$ independent.

$$\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix} = \begin{bmatrix}b_1\\2b_1\end{bmatrix} = b_1\begin{bmatrix}1\\2\end{bmatrix} \text{ where } b_1 \in R$$

E9gP2

Find $N(\mathbb{A})$:

$$[\mathbb{A} \mid \vec{0}\,] \sim \left[\begin{array}{cc|c} P & F & \\ 1 & 2 & 0\\ 0 & 0 & 0\end{array}\right]$$

$$\Rightarrow \quad \underset{P}{x_1} + \underset{F}{2x_2} = 0$$

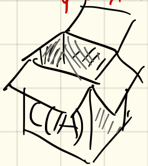
$$\Rightarrow \quad x_1 = -2x_2$$

$$\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \xrightarrow{\text{replace pivot var}} = \begin{bmatrix}-2x_2\\x_2\end{bmatrix} = x_2\begin{bmatrix}-2\\1\end{bmatrix} \text{ where } x_2 \in R$$

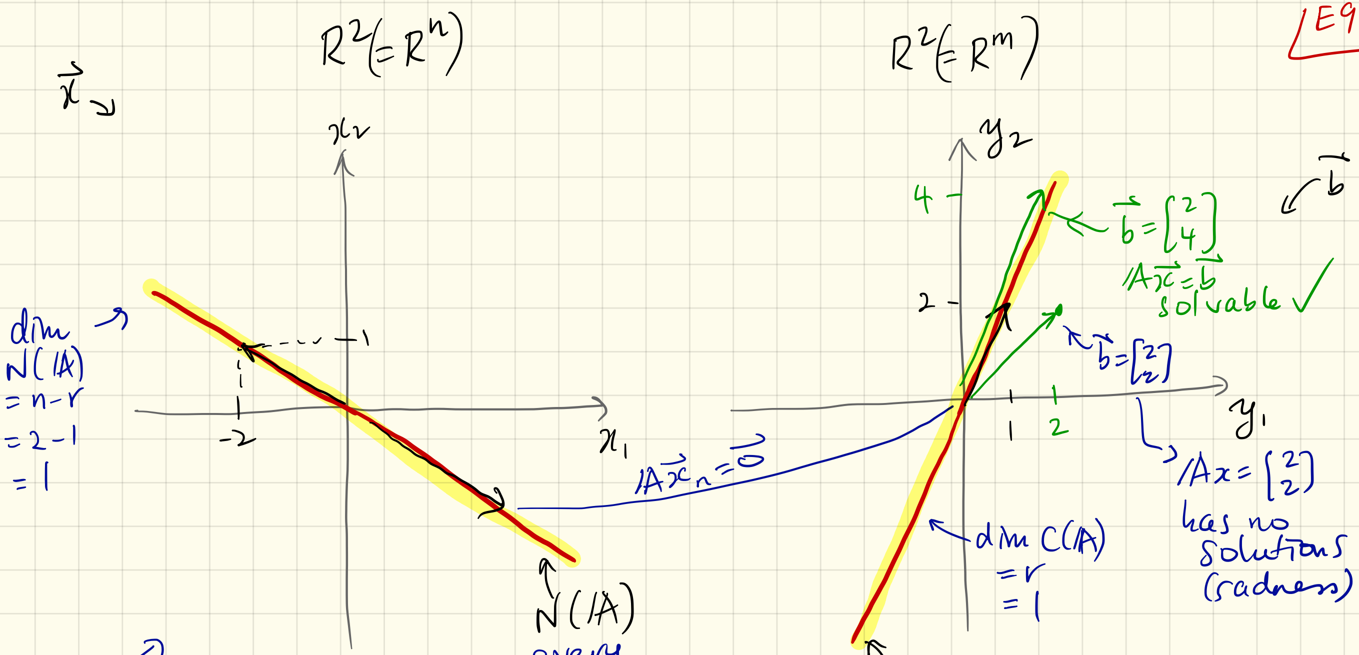
$$C(\mathbb{A}) = \left\{ \vec{b} \in R^2 \;\middle|\; \vec{b} = b_1\begin{bmatrix}1\\2\end{bmatrix},\ b_1 \in R \right\}$$

$$N(\mathbb{A}) = \left\{ \vec{x} \in R^2 \;\middle|\; \vec{x} = x_2\begin{bmatrix}-2\\1\end{bmatrix},\ x_2 \in R \right\}$$

box of vectors



E9gP3



every vector on this line is sent to zero by A.

b/c $N(A) \neq \{\vec{0}\}$ there are ∞ly many solution if $\vec{b} \in C(A)$

Any $\vec{b}$ in $C(A)$ can be made by $A\vec{x}$

$\iff$ $A\vec{x} = \vec{b}$ has a solution

What $\mathbb{R}_A$ tells us

Menu:

- The four basic kinds of $\mathbb{R}_A$
- How these forms for $\mathbb{R}_A$ dictate $A\vec{x}=\vec{b}$'s solution
- Later: $\mathbb{R}_{A^T}$ gives us the rest of what we need to know

---

The Story so far:

$\mathbb{R}_A$ provides us with

(1) The rank r of A (# pivot columns)

(2) Nullspace N(A) (solve $\mathbb{R}_A\vec{x}=\vec{0}$)

(3) The number of possible solutions to $A\vec{x}=\vec{b}$

(1),(2) → (3) because:

- If $r < m$, one or more rows of $\mathbb{R}_A$ are all 0's and therefore some $\vec{b}$'s will lead to no solution for $A\vec{x}=\vec{b}$
- If $N(A)\neq\{\vec{0}\} \Leftrightarrow \mathbb{R}_A$ has one or more free columns then $A\vec{x}=\vec{b}$ will have ∞ly many solutions if it is solvable (i.e., if $\vec{b}\in C(A)$)

Four examples:

E10ap1

(i)

$$\mathbb{R}_A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} \text{pivot} & \text{free} & f & p & f & p \\ 1 & \omega_{12} & \omega_{13} & 0 & \omega_{15} & 0 \\ 0 & 0 & 0 & 1 & \omega_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

from a wide A

no row of zeros

$m=3$, $n=6$, $r=3$

See: always a solution to $A\vec{x}=\vec{b}$

$\Rightarrow C(A) = R^3 = R^m = R^r$

Also N(A) is a 3-d subspace of R^6 (n=6)

$\vec{x}$ lives in R^6 — to be proven

Know $N(A)\neq\{\vec{0}\}$

So: $A\vec{x}=\vec{b}$ always has $\overbrace{\text{a solut}}^{C(A)}$ and there are always $\underbrace{\infty\text{ly many}}_{N(A)}$

Note: Wide A's always have at least 1 free variable

$\Rightarrow N(A)\neq\{\vec{0}\}$

(ii) $$\mathbb{R}_{/A} = \begin{array}{c} \begin{array}{ccc} p & p & p \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = \mathbb{I}$$

$m = n = r = 3$

⇒ No free variables

See: $A\vec{x} = \vec{b}$ is always solvable

and $N(A) = \{\vec{0}\}$

So: $A\vec{x} = \vec{b}$ always has 1, unique solut.

For square invertible matrices $(n \times n)$

$\mathbb{R}_{/A} = \mathbb{I}$ always.

↖ 1-1 mapping from $R^n \to R^n$

(iii) $$\mathbb{R}_{/A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

tall →

$m = 4$

$n = r = 3$

↖ tall matrices must always have a row of zeros

See:

+ $N(A) = \{\vec{0}\}$

Possible: no solutions

$A\vec{x} = \vec{b}$ has 0 or 1 solutn

$\vec{b} \notin C(A)$ ↗ (0); $b \in C(A)$ ↗ (1)

E10ap2

(iv)

m and $n > r$

$$\begin{array}{cccc} P & F & P & F \\ \hline 1 & \omega_{12} & 0 & \omega_{14} \\ 0 & 0 & 1 & \omega_{24} \\ 0 & 0 & 0 & 0 \end{array}$$

$m = 3$

$n = 4$

$r = 2$

↖ later: really a 2 by 2 matrix in a 3×4 matrix

A sends a plane in R^4 to a plane in R^3

See: $N(A) \neq \{\vec{0}\}$ (2 free variables)

$C(A)$ may or may not contain $\vec{b}$ (row of 0's in $\mathbb{R}_{/A}$)

⇒ either 0 or ∞ly many solutn.

E10ap3

Case:	example R_A	# Solutions
(i) $m=r$ $n=r$ square	$\begin{array}{c} \begin{matrix} p & p & p \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$	1 always $\left[\begin{array}{l} C(A)=R^m \\ N(A)=\{\vec{0}\} \end{array}\right.$
(ii) $m=r$ $n>r$ wide	$\begin{array}{c} \begin{matrix} P & F & P & F & P \end{matrix} \\ \begin{bmatrix} 1 & -2 & 0 & -4 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$	∞ always $\left[\begin{array}{l} C(A)=R^m \\ N(A)\neq\{\vec{0}\} \end{array}\right.$
(iii) $m>r$ $n=r$ tall	$\begin{array}{c} \begin{matrix} p & p & p \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$	0 or 1 $\left[\begin{array}{l} C(A)\neq R^m \\ N(A)=\{\vec{0}\} \end{array}\right.$
(iv) $m>r$ $n>r$ many possibil.	$\begin{bmatrix} 1 & -2 & 0 & 12 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	0 or ∞ $\left[\begin{array}{l} C(A)\neq R^m \\ N(A)\neq\{\vec{0}\} \end{array}\right.$

Next: · find bases for $C(A)$ & $N(A)$
· $\dim C(A)=r$, $\dim N(A)=n-r$

Getting to know your subspaces:

Menu:
- Care and feeding
- Spanning sets
- Bases
- Dimensions

New friend: Inigo $\quad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

Plan: explore $A\vec{x} = \vec{b}$ with Inigo

First ~ find C(A) and N(A)

Solve $$\left[\begin{array}{ccc|c} \overset{D_1}{1} & 2 & 1 & b_1 \\ 2 & 4 & 2 & b_2 \end{array}\right] = [A \mid \vec{b}]$$

$$R_2' = R_2 - \left(\frac{2}{\underset{D_1}{1}}\right)^{l_{21}} R_1 \quad \sim \quad \left[\begin{array}{ccc|c} P & F & F & \\ 1 & 2 & 1 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right] \approx [R_A \mid \vec{d}]$$

$m = 2$ rows
$n = 3$ columns
$r = 1$, rank

C(A): Must have $b_2 - 2b_1 = 0$ for solution to be possible

$\Rightarrow \overset{\text{dependent}}{b_2} = 2\overset{\text{independent}}{b_1}$

$$\Rightarrow \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 2b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$C(A) = \left\{ \vec{y} \in R^2 \;\middle|\; \vec{y} = c\begin{bmatrix} 1 \\ 2 \end{bmatrix}, c \in R \right\}$$

line through origin

r=1

1-d subspace of R^2 — m=2

N(A): Solve $A\vec{x} = \vec{0}$

$\Rightarrow$ Set $\vec{b} = 0$ in previous

$$\left[\begin{array}{ccc|c} P & F & F & \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = [R_A \mid \vec{0}]$$

$$\Rightarrow \underset{P}{x_1} + 2\underset{F}{x_2} + \underset{F}{x_3} = 0 \Rightarrow \underset{P}{x_1} = -2\underset{F}{x_2} - \underset{F}{x_3}$$

express pivot variables in terms of the free variables

replace pivot variables

$$\underbrace{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{always}} = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

where $x_2, x_3 \in R$

$$N(A) = \left\{ \vec{x} \in R^{\overset{n=3}{3}} \;\middle|\; \vec{x} = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; c_1, c_2 \in R \right\}$$

plane in R^3

Always true:

$$C(A) \underset{\text{“is a subspace of”}}{\subset} \underset{\vec{b}\text{'s}}{R^m} \;\&\; N(A) \subset \overset{\vec{x}\text{'s}}{R^n}$$

Boring but important:

How do we know C(A) & N(A) are really subspaces and not some wheedy subsets?

N(A) for Inigo comprises $\underbrace{\text{all linear combinations}}_{\text{automatic subspacification}}$ of $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\0\\1\end{bmatrix}$

Check supspace properties:

SSP1 if $\vec{x}_1, \vec{x}_2 \in N(A)$, $\vec{x}_1+\vec{x}_2 \in N(A)$

$$\vec{x}_1 = c_{11}\begin{bmatrix}-2\\1\\0\end{bmatrix} + c_{12}\begin{bmatrix}-1\\0\\1\end{bmatrix}, \quad \vec{x}_2 = c_{21}\begin{bmatrix}-2\\1\\0\end{bmatrix} + c_{22}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$\Rightarrow \vec{x}_1+\vec{x}_2 = (c_{11}+c_{21})\begin{bmatrix}-2\\1\\0\end{bmatrix} + (c_{12}+c_{22})\begin{bmatrix}-1\\0\\1\end{bmatrix} = \text{Vector in } N(A)$$

SSP2 if $\vec{x}_1 \in N(A)$, $c\vec{x}_1 \in N(A)$ for all $c \in \mathbb{R}$

Yes: $$\underbrace{c\times c_{11}}_{\text{still a real number}}\begin{bmatrix}-2\\1\\0\end{bmatrix} + \underbrace{c\times c_{21}}_{\text{same}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

E11ap2

SSP3 $\vec{0} \in N(A)$

Yes: set $c_{11}=c_{12}=0 \Rightarrow \vec{x}_1 = \begin{bmatrix}0\\0\\0\end{bmatrix}$

General Story:

Sets made up of all linear combinations of some set of vectors are automatically subspaces.

Terminology:

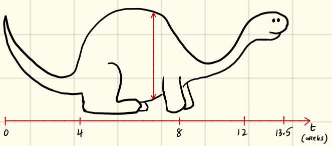
We say $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ "span" the nullspace of A

and that $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ are a "spanning set" for N(A)

"All your bases are belong to us"

Menu:
- Spanning sets for vector spaces & subspaces
- Bases for vector spaces & subspaces
- How bases are all about $\mathbb{A}\vec{x}=\vec{0}$ and the Nullspace of $\mathbb{A}$
- The dimensions of subspaces
- And this:

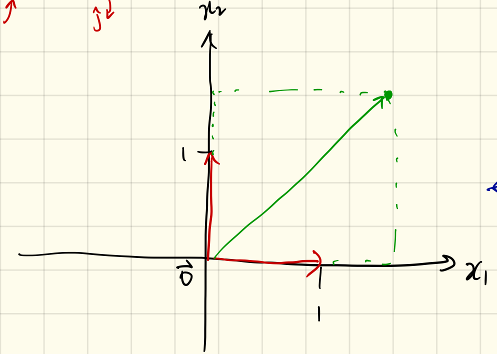
Melvin the Course Difficulty Dinosaur:



Three Examples of Spanning sets for R^2

(1) $\left\{ \overset{\vec{a}_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \overset{\vec{a}_2}{\begin{bmatrix}0\\1\end{bmatrix}} \right\}$ basis

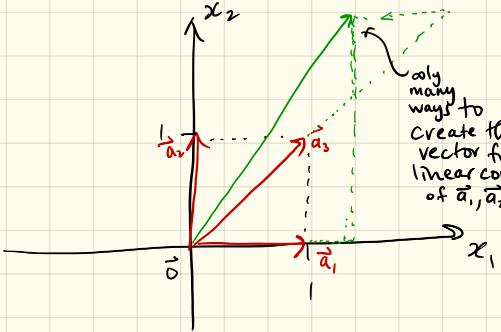
$\hat{i}$ $\hat{j}$



2 non-parallel vectors give unique representation of each vector

$E11bp1$

(2) $\left\{ \overset{\vec{a}_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \overset{\vec{a}_2}{\begin{bmatrix}0\\1\end{bmatrix}}, \overset{\vec{a}_3}{\begin{bmatrix}1\\1\end{bmatrix}} \right\}$ not a basis



∞ly many ways to create this vector from a linear combinat of $\vec{a}_1, \vec{a}_2$, and $\vec{a}_3$

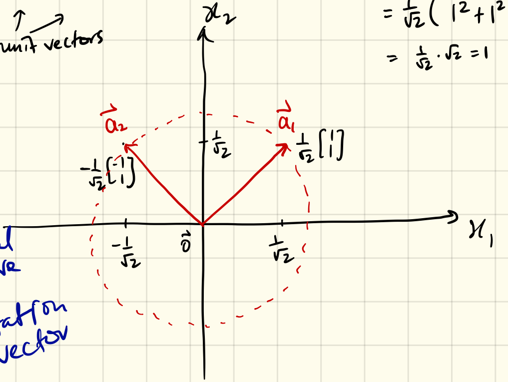
(3) $\left\{ \overset{\vec{a}_1}{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}}, \overset{\vec{a}_2}{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}} \right\}$ ← basis.

unit vectors

note: $\|\vec{a}_1\|$ ← length

$= \frac{1}{\sqrt{2}}(1^2+1^2)^{1/2}$

$= \frac{1}{\sqrt{2}}\cdot\sqrt{2} = 1$



Observe:

- Examples (1) & (3) are special because we need both vectors
- For (2), we could take any one vector away, and the remaining two would still span R^2

The right words for the above:

(1) & (3) have linearly independent sets of vectors

(2) has a linearly dependent set of vectors

Big Deal time:

Defn: A set of vectors

$$\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\} \text{ in } R^m$$

is linearly independent if

ER

$$\underline{x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{0}}$$

only has $x_1 = x_2 = \dots = x_n = 0$ as a solution

Nullspace Equation $A\vec{x} = \vec{0}$

(x_i is a scalar)

Why? If one or more $x_i \neq 0$, then we can express one vector in terms of the others

ex (2)

$$\underset{\vec{a}_3}{\begin{bmatrix}1\\1\end{bmatrix}} = \underset{\vec{a}_1}{\begin{bmatrix}1\\0\end{bmatrix}} + \underset{\vec{a}_2}{\begin{bmatrix}0\\1\end{bmatrix}}$$

$$\underset{\substack{\uparrow\\x_1}}{1}\cdot\vec{a}_1 + \underset{\substack{\uparrow\\x_2}}{1}\cdot\vec{a}_2 - \underset{\substack{\uparrow\\x_3}}{1}\cdot\vec{a}_3 = \vec{0}$$

Seeing things:

$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are linearly independent

$$\iff \underset{m \times n}{\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}} \underset{n \times 1}{\vec{x}} = \vec{0}$$

$\mathbb{A}$

has only $\vec{x} = \vec{0}$ as a solution

$$\iff N(\mathbb{A}) = \{\vec{0}\}$$

So exciting...

Defn: A spanning set that is linearly independent is called a **basis**

(plural: bases) ↖ "bay sieze"

Note: Bases are not unique (see ex(1) & (3) above) but some bases are better than others, and some are totally awesome

General goodness: Bases give us a unique representation of every point in the space they span.

Defn:

The dimension of a space is the number of vectors in any basis for that space

E11cp1

Why the dimension of $C(\mathbb{A})$ is the rank of $\mathbb{A}$, r

menu
- Including a second way to find $C(\mathbb{A})$
- Inigo & Fezzik

Claim: $\dim C(\mathbb{A}) = r = \#$ pivot columns in $\mathbb{R}_{\mathbb{A}}$

$\uparrow$ RREF

Two key points:

#1 When we perform row operations on a matrix, the relationships between the columns do not change.

Fezzik:

Identity matrix

$$\mathbb{A} = \begin{bmatrix} \text{P} & \text{F} & \text{P} & \text{F} \\ 2 & 4 & 3 & 4 \\ 2 & 4 & 6 & 10 \\ 6 & 12 & 12 & 18 \end{bmatrix} \sim \begin{bmatrix} \text{P} & \text{F} & \text{P} & \text{F} \\ 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbb{R}_{\mathbb{A}}$$

Observations: x_1 & x_3 are pivot variables
x_2 & x_4 are free variables

$c_2 = 2c_1$ in both $\mathbb{A}$ & $\mathbb{R}_{\mathbb{A}}$

$c_4 = -c_1 + 2c_3$ " " " " "

easiest to see relationships in $\mathbb{R}\mathbb{A}$

$\Downarrow$

Identity matrix in pivot columns is key

#2 Follows that in $\mathbb{A}$, the free columns can be made out of the pivot columns, and the pivot columns have to be linearly independent.

$\Rightarrow$ The pivot columns of $\mathbb{A}$ form a basis for $C(\mathbb{A})$

$\Rightarrow$ Because there a r pivot columns, then $\dim C(\mathbb{A}) = r$.

Second way of finding $C(\mathbb{A})$

Fezzik:

$$C(\mathbb{A}) = \left\{ \vec{y} \in \mathbb{R}^3 \;\middle|\; \vec{y} = c_1 \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix} \right\}$$

$c_1, c_2 \in \mathbb{R}$

basis for $C(\mathbb{A})$: $\left\{ \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix} \right\}$

E11cp2

Inigo:

$$\mathbb{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} \sim \mathbb{R}_{\mathbb{A}} = \begin{array}{c} \begin{array}{ccc} P & F & F \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

pivot column

Basis for $C(\mathbb{A}) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$\dim C(\mathbb{A}) = r = 1$

Notes: $C(\mathbb{A}) \neq C(\mathbb{R}_{\mathbb{A}})$

in general

$N(\mathbb{A}) = N(\mathbb{R}_{\mathbb{A}})$

about $\vec{x}$'s

($\mathbb{A}\vec{x} = \vec{b}$ has same solutions as $\mathbb{R}_{\mathbb{A}}\vec{x} = \vec{d}$)

Why the dimension of $N(A)$ is $n-r$

- see E9fp1 (p59ish)

Big Deal:

Our one true method of finding nullspace always produces a set of vectors that are linearly independent and are therefore a basis for $N(A)$

← vectors span $N(A)$

Inigo:

$$\mathbb{N} = \begin{array}{c} P \\ F \\ F \end{array}\begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ \multicolumn{2}{c}{I} \end{bmatrix}$$

I appears in free variable rows

Fezzik:

$$\mathbb{N} = \begin{array}{c} P \\ F \\ P \\ F \end{array}\begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$n-r$ = # free variables

monks ↙

b/c we express pivot variables in terms of the free variables when finding $N(A)$ (always)

E11dp1

$\mathbb{N}$ always has $n-r$ columns that are linearly independent

$\Rightarrow$ form a basis for $N(A)$

$\Rightarrow$ $\dim N(A) = n-r$.

Inigo: $\dim N(A) = 3-1 = 2$ ✓

Fezzik: $\dim N(A) = 4-2 = 2$ ✓

"It came from Row Space!"

- the row space of $\mathbb{A}$ is a thing
- what this means for $\mathbb{A}\vec{x} = \vec{b}$
- Many big deals
- The Big Picture

Story: Row Space of $\underset{m\times n}{\mathbb{A}}$ = all linear combinations of the rows of $\mathbb{A}$.

= subspace of $\mathbb{R}^n$ ← where $N(\mathbb{A})$ lives

Contrast: $C(\mathbb{A})$ = subspace of $\mathbb{R}^m$

Big Deal

BD #1: If $\vec{x} \in \mathbb{A}$'s Row Space, then $\mathbb{A}\vec{x} \neq \vec{0}$ unless $\vec{x} = \vec{0}$

← $\vec{x} \notin N(\mathbb{A})$

Example Inigo:

$$\mathbb{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} \Rightarrow \text{clearly, Row Space of } \mathbb{A} = \left\{ \vec{x} \in \mathbb{R}^m \;\middle|\; \vec{x} = c\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\ c \in \mathbb{R} \right\}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} \times c\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = c\begin{bmatrix} 6 \\ 12 \end{bmatrix} = 6c\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in C(\mathbb{A})$$

has to be true

Recall $\mathbb{A}\vec{x} = \vec{b}$

$$\vec{x} = \vec{x}_{\underset{\text{particular}}{p}} + \vec{x}_{\underset{}{h}} = \vec{x}_{\underset{\text{row}}{r}} + \vec{x}_{\underset{\text{null}}{n}}$$

homogeneous

note $\vec{x}_p \neq \vec{x}_r$ necessarily

E12ap1

$$\mathbb{A}(\vec{x}_p + \vec{x}_h) = \underbrace{\mathbb{A}\vec{x}_p}_{\vec{b}} + \underbrace{\mathbb{A}\vec{x}_h}_{\vec{0}}$$

← may be infinitely many $\in N(\mathbb{A}) \neq \{\vec{0}\}$

$\vec{x}_p$ must partly live in row space of $\mathbb{A}$.

BD #2

Any $\vec{x}$ in Row Space of $\mathbb{A}$ is $\perp$/orthogonal/at right angles to any $\vec{x}$ in Null space of $\mathbb{A}$.

$$\underset{\mathbb{A}}{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}} \left(\underbrace{c_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\in N(\mathbb{A}) \text{ from before}} \right) = \vec{0}$$

BD#3 The row space of $\mathbb{A}$ is the same as the row space of $\mathbb{R}_{\mathbb{A}}$

$\Rightarrow$ beautiful basis for row space of $\mathbb{A}$ = non-zero rows of $\mathbb{R}_{\mathbb{A}}$.

ex Inigo: $\mathbb{R}_{\mathbb{A}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Fezzik: $\mathbb{R}_{\mathbb{A}} = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$ basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

← rows are linearly independent b/c of $\mathbb{I}$ sitting in pivot columns

BD#4 Dim Row Space of $\mathbb{A}$ ⟸ because samesies

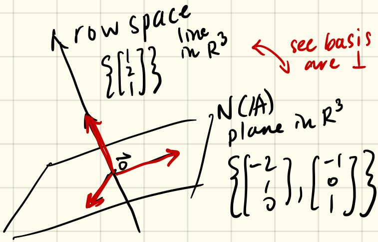
= Dim Row Space of $\mathbb{R}_\mathbb{A}$

= r = non-zero rows

(same as dim $C(\mathbb{A})$) amazing!!

BD#5 Dims of Row Space of $\mathbb{A}$ and Nullspace of $\mathbb{A}$ add up to n $(= r + (n-r))$.

Inigo:



see basis vectors are ⊥ to each other

dim RowSpace of $\mathbb{A} = r = 1$

dim $N(\mathbb{A})$ $= n - r = 3-1$ $= 2$

$1+2=3$

soon: "othogonal complements")

(Imagine loud organ music and lightning)

Row Space & $N(\mathbb{A})$ neatly divide up R^n

BD#6 E12ap2

Consider $\mathbb{A}^T$ for Inigo

$$\mathbb{A}^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \Rightarrow$$ now see Row Space of $\mathbb{A}$ is also the Column Space of $\mathbb{A}^T$

wow!

Notation:

$C(\mathbb{A}^T)$ = Row Space of $\mathbb{A}$

cat in a box:

repurpose b/c deep connection.

BD#7: We find a 3rd and final and totally bestest way for finding $C(\mathbb{A})$.

Find $\mathbb{R}_{\mathbb{A}^T}$ and then read off basis vectors for $C(\mathbb{A})$

row space of $\mathbb{A}^T$ = column space of $\mathbb{A}$

awesome!!

Note: $\mathbb{R}_{\mathbb{A}^T} \neq (\mathbb{R}_\mathbb{A})^T$ in general eek!!

E12ap3

ex: $\mathbb{A} = \begin{bmatrix}1&2&1\\2&4&2\end{bmatrix}$ $\qquad \mathbb{A}^T = \begin{bmatrix}1&2\\2&4\\1&2\end{bmatrix}$

Inigo:

$$\mathbb{A}^T \sim \begin{bmatrix}1&2\\0&0\\0&0\end{bmatrix} \rightarrow \left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$$

$R_2' = R_2 - \left(\frac{2}{1}\right)R_1$

$R_3' = R_3 - \left(\frac{1}{1}\right)R_1$

is a basis for $C(\mathbb{A})$

Fezzik:

$$\mathbb{A}^T = \begin{bmatrix}2&2&6\\4&4&12\\3&6&12\\4&10&18\end{bmatrix} \sim \begin{bmatrix}1&0&2\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}$$

row ops + tears

$$\Rightarrow \left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$ is a basis for $C(\mathbb{A})$

$\mathbb{I}$ matrix now always appears

The Known Unknowns of Left Nullspace

- What Left Nullspace is
- Connection to the other three subspaces

We have $C(A)$, $N(A)$, and $\underbrace{C(A^T)}_{\text{row space of } A}$

What about $N(A^T)$?

$\uparrow$ Left Nullspace of A

Reason: if $\vec{y} \in N(A^T) \subset R^m$

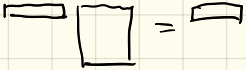
then $\underbrace{\underbrace{A}_{m\times n}{}^T}_{n\times m} \underbrace{\vec{y}}_{m\times 1} = \underbrace{\vec{0}}_{n\times 1}$ ← defn.

$\in R^m$ $\quad$ $\in R^n$ (where the $\vec{x}$'s are)

Take transpose of both sides:

$$(A^T \vec{y})^T = (\vec{0})^T \quad \leftarrow (BC)^T = C^T B^T$$

$$\underbrace{\vec{y}^T}_{\substack{1\times m \\ \text{row vector}}} \underbrace{A}_{m\times n} = \underbrace{\vec{0}^T}_{\substack{1\times n \\ \text{row vector}}}$$



$\vec{y}^T$ multiplies A from the left

So in fact $N(A)$ is the Right Nullspace of A

$$A\vec{x} = \vec{0}$$

↑ on the right.

Know immediately:

$$\dim N(A^T) = \#\text{ columns of } A^T - \text{rank of } A^T$$
$$= m - r$$

(for $N(A)$, we have $n - r$)

We find $N(A^T)$ just as we would find $N(A)$

Solve $A^T \vec{y} = \vec{0}$

$\vec{y}$ in R^m

Ex: Intro.

$$\left[\begin{array}{cc|c}1&2&0\\2&4&0\\1&2&0\end{array}\right] \sim \left[\begin{array}{cc|c}P&F&\\1&2&0\\0&0&0\\0&0&0\end{array}\right]$$

$\uparrow$ A^T $\qquad$ $\uparrow$ $\vec{0}$

Express pivot vars in terms of free

$$y_1 + 2y_2 = 0 \Rightarrow y_1 = -2y_2$$

$$\vec{y} = \begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}-2y_2\\y_2\end{bmatrix} = y_2\begin{bmatrix}-2\\1\end{bmatrix},\ y_2 \in R$$

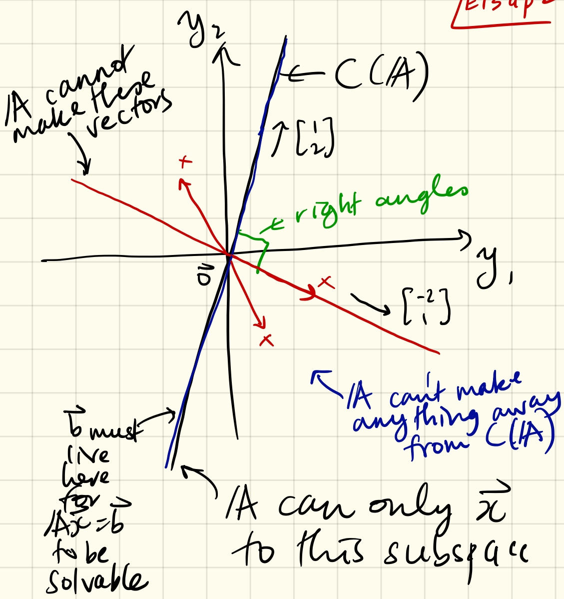
basis vector

Just as $N(A)$ & $C(A^T)$ divide up R^n so too do $N(A^T)$ & $C(A)$

basis $\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$ $\qquad$ basis $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

see these are 1-d orthogonal.

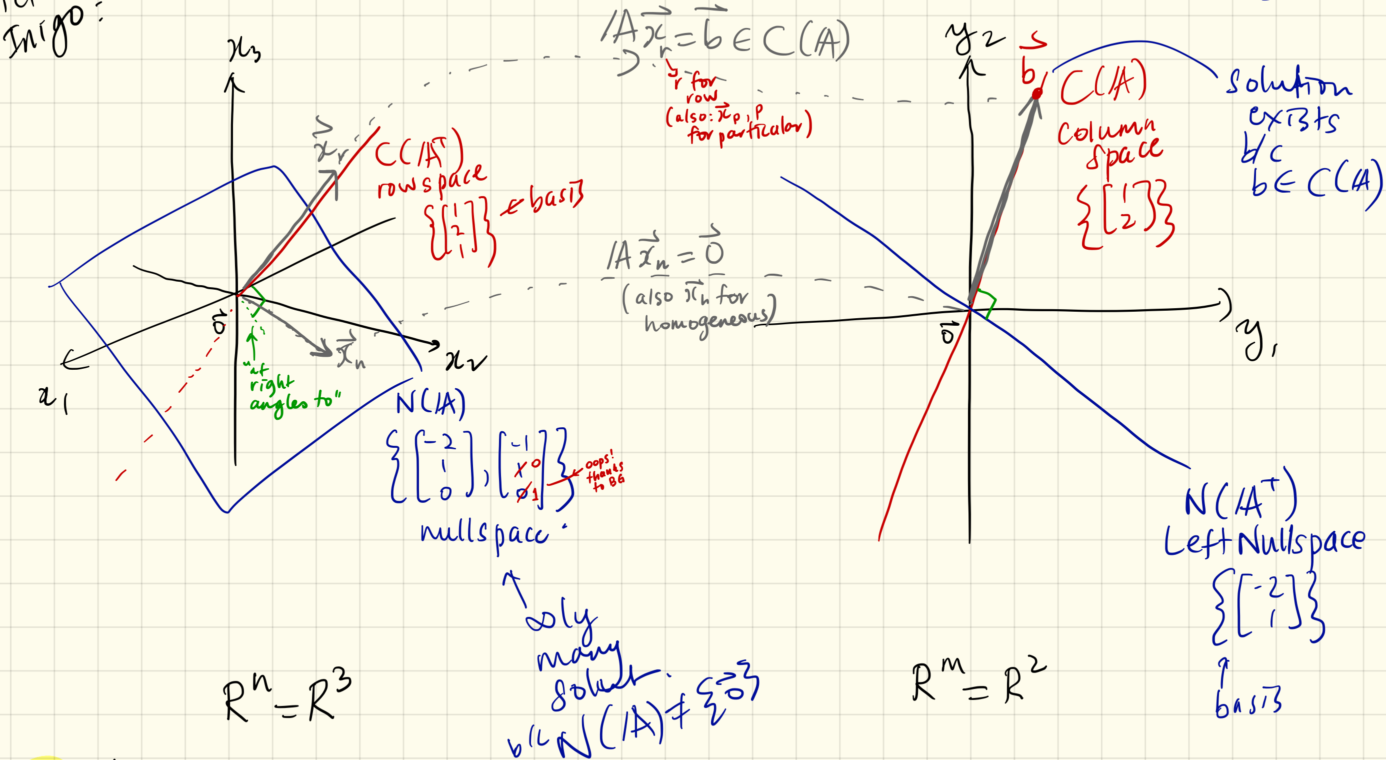
E13ap2



The Fundamental Theorem of Matrixology (almost)

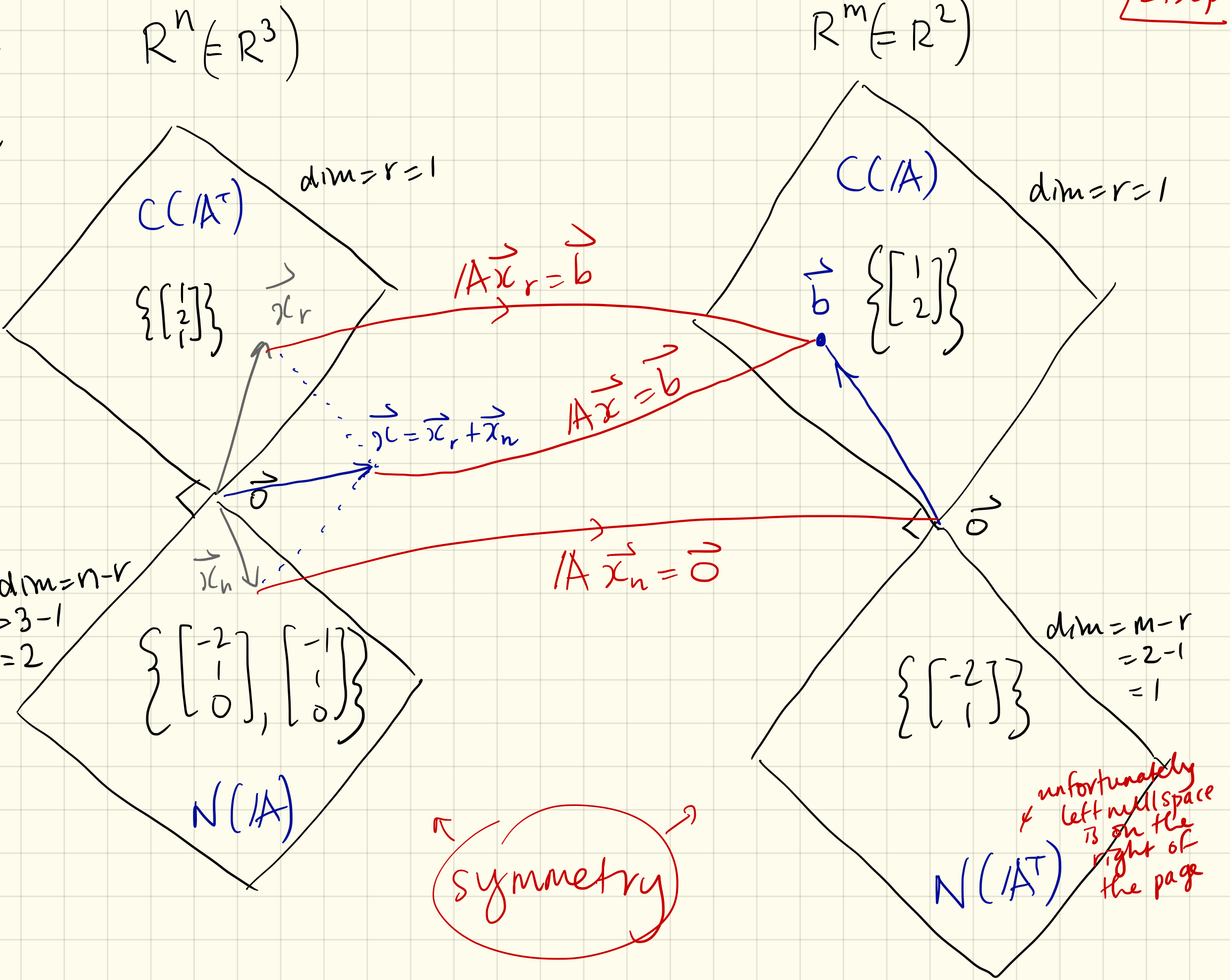
$\mathbb{A}\vec{x} = \mathbb{A}(\vec{x}_r + \vec{x}_n)$
$= \vec{b}$

Big picture for Inigo:



key:
Inigo sends a line to a line
Later: see Inigo $\sim \sqrt{30}$

Abstract big picture with Inigo's structure



Definitions we need:

(1) (old) if $\vec{x}^T\vec{y} = 0$ we say $\vec{x}$ & $\vec{y}$ are orthogonal

$//$

$\vec{y}^T\vec{x}$

(2) We say two supspaces S_1 & S_2 are orthogonal if all vectors in S_1 are orthogonal to all vectors in S_2

(3) If two subspaces S_1 & S_2 in a vector space V (dim n) are orthogonal and their dimensions add to n, we say S_1 & S_2 are orthogonal complements of each other

Notation: S and $S^\perp$

and $S \oplus S^\perp = V$

$\Uparrow$

any vector in V can be written as a sum of a vector in S and a vector in $S^\perp$

E13bp3

Fundamental Theorem of Matrixology (mostly):

$\dim C(A) = r$	column space
$\dim N(A^T) = m - r$	left null space
$\dim C(A^T) = r$	row space
$\dim N(A) = n - r$	nullspace

- $C(A)$ and $N(A^T)$ are orthogonal complements in R^m
- $C(A^T)$ and $N(A)$ are orthogonal complements in R^n
- The bases of $C(A)$ & $N(A^T)$ combine to give a basis of R^m
- The bases of $C(A^T)$ & $N(A)$ combine to give a basis of R^n

More near the end of course

E14ap1

Matrix-fu

The Man in Black, Westley:

$$\mathbb{A} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \qquad \begin{matrix} m=2 \text{ rows} \\ n=2 \text{ cols} \end{matrix}$$

$\underset{m \quad n}{2 \times 2}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$R_2' = R_2 - \left(\frac{3}{1}\right) R_1$

$R_1' = R_1$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} = \mathbb{R}_{\mathbb{A}}$$

$\uparrow$ P $\quad \uparrow$ F

see rank $\underline{r=1}$

x_1 is a pivot variable

x_2 is a free variable

$m=2,\ n=2,\ r=1$

$$\mathbb{A}^T = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}$$

$R_2' = R_2 - \left(\frac{-2}{1}\right) R_1$

$$\begin{array}{c} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \\ y_1 \quad y_2 \end{array} = \mathbb{R}_{\mathbb{A}^T}$$

Dimensions:

$$\dim \underset{\text{column space}}{C(\mathbb{A})} = \underset{\underset{1}{\parallel}}{r} = \dim \underset{\text{row space}}{C(\mathbb{A}^T)}$$

$$\dim \underset{\text{(Right) Nullspace}}{N(\mathbb{A})} = n - r = 2 - 1 = 1.$$

$$\dim \underset{\text{Left Nullspace}}{N(\mathbb{A}^T)} = m - r = 2 - 1 = 1.$$

Bases:

Nullspaces:

$$\mathbb{A}\vec{x} = \vec{0} \iff \mathbb{R}_{\mathbb{A}}\vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\underset{P}{x_1} - 2\underset{F}{x_2} = 0$$

$$\Rightarrow x_1 = 2x_2$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \underset{\substack{\uparrow \\ \text{replace pivot variables}}}{=} \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad x_2 \in R$$

$N(\mathbb{A}) =$

$$\left\{ \vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ x_2 \in R \right\}$$

∞ly many points

$\underline{\underline{A}}$ basis for $N(\mathbb{A})$ is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Also good

$$\begin{bmatrix} 2\infty \\ \infty \end{bmatrix},\ \begin{bmatrix} -2 \\ -1 \end{bmatrix},\ \underset{\text{unit vector}}{\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

Left Nullspace:

Solve $\mathbb{A}^T\vec{y}=\vec{0}$

$\Leftrightarrow \mathbb{R}_{\mathbb{A}^T}\vec{y}=\vec{0}$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \underset{\substack{\uparrow \\ P}}{y_1} + 3\underset{\substack{\uparrow \\ F}}{y_2} = 0.$

$y_1 = -3y_2$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -3y_2 \\ y_2 \end{bmatrix} = \underset{\substack{\uparrow \\ \text{free}}}{y_2}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$N(\mathbb{A}^T) = \left\{ \vec{y} = y_2\begin{bmatrix} -3 \\ 1 \end{bmatrix}, y_2 \in \mathbb{R} \right\}$$

A basis for $N(\mathbb{A}^T)$ is

$$\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}.$$

Column space $C(\mathbb{A})$.

① of 3 Solve $\mathbb{A}\vec{x}=\vec{b}$ for general $\vec{b}$.

$$\left[\begin{array}{cc|c} 1 & -2 & b_1 \\ 3 & -6 & b_2 \end{array}\right]$$

$$\xrightarrow{R_2' = R_2 - (\frac{3}{1})R_1} \left[\begin{array}{cc|c} 1 & -2 & b_1 \\ 0 & 0 & b_2 - 3b_1 \end{array}\right]$$

$0 = b_2 - 3b_1$ must hold if $\vec{b} \in C(\mathbb{A})$.

$\Rightarrow b_2 = 3b_1$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 3b_1 \end{bmatrix} = b_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$C(\mathbb{A}) = \left\{ \vec{y} = b_1 \underset{\text{1-d}}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}, b_1 \in \mathbb{R} \right\}.$$

basis: $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

② of 3 find pivot columns through $\mathbb{R}_{\mathbb{A}}$

$\Rightarrow$ Same columns in $\mathbb{A}$ form a basis for $C(\mathbb{A})$

here 1st column: $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

③ of 3. Take non-zero rows of $\mathbb{R}_{\mathbb{A}^T} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

Again: $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

{best way}

E14ap2

Row space: take non-zero rows of $\mathbb{R}_{\mathbb{A}} = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

E14ap3

$$\mathbb{A} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}$$

Bases $\overset{\mathbb{R}^m}{\rightarrow}$ $C(\mathbb{A}): \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $N(\mathbb{A}^T): \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\mathbb{R}^n \rightarrow$ $C(\mathbb{A}^T): \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $N(\mathbb{A}): \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\mathbb{A}\vec{x}$: ① $\mathbb{A}$ sends any multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ to some multiple of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

② $\mathbb{A}$ sends any multiple of $\underset{N(\mathbb{A})}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

③ $\mathbb{A}$ cannot make any vector which has some non-zero part of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Complementary Orthogonality of Subspaces

$$C(\mathbb{A}) \oplus N(\mathbb{A}^T) = \mathbb{R}^2 \leftarrow m=2$$

$$C(\mathbb{A}^T) \oplus N(\mathbb{A}) = \mathbb{R}^2 \leftarrow n=2,$$

$$\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0.$$

how $\mathbb{A}$ functions:

$$\mathbb{A}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

in C(A)

length $= \sqrt{1^2 + (-2)^2}$
$= \sqrt{5}$.

length
$5\sqrt{1+3^2}$
$= 5\sqrt{10}$

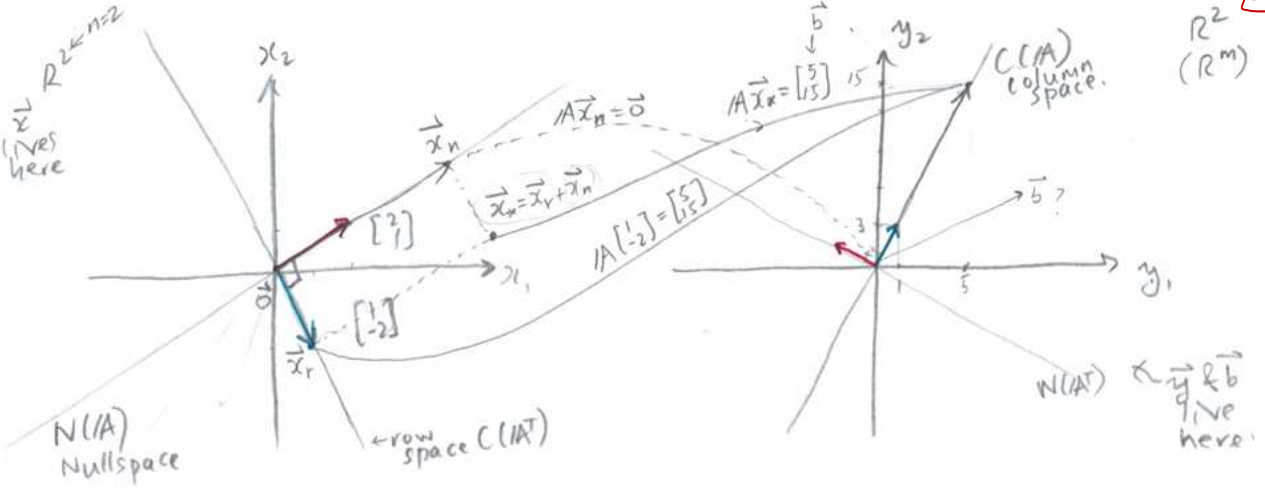
Stretch factor: $\dfrac{5\sqrt{10}}{\sqrt{5}} = \sqrt{5}\sqrt{10} = 5\sqrt{2}$.

So Westley is like $y = \underset{1\times 1 \leftarrow r=1}{5\sqrt{2}}\, x$

but only for vectors in row & column space $\leftarrow$ 1-d subspace.

$\mathbb{A}$ is $\underset{r\times r}{\vee}$ invertible in those subspaces

E14ap4

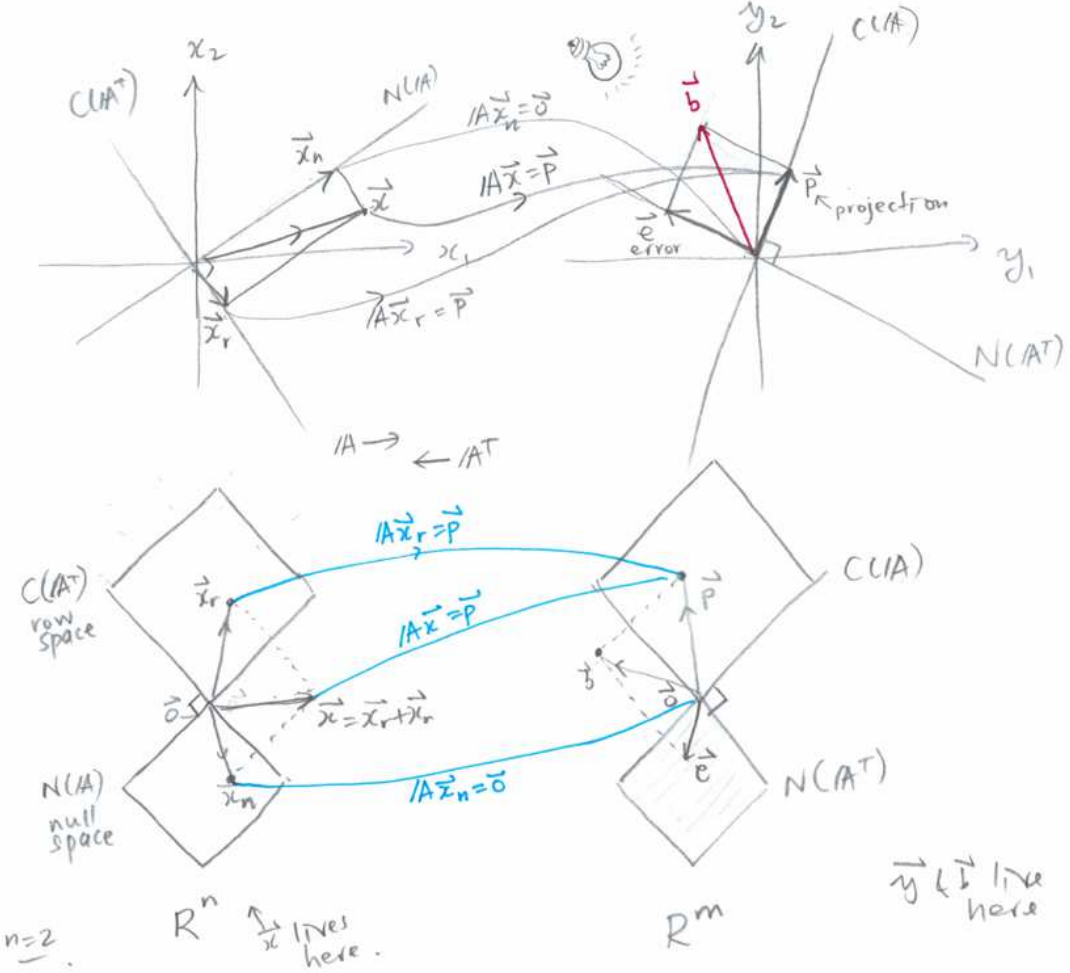


- $A\vec{x}=\vec{b}$ is solvable if $\vec{b}\in C(A)$.
- if $\vec{b}\in C(A)$, then there is one solutn if $N(A)=\{\vec{0}\}$ and ∞ly many otherwise.
  $\dim N(A)\geq 1$.

Ex// if $\vec{b}=5\begin{bmatrix}1\\3\end{bmatrix}$ then $\vec{x}=\underbrace{\overset{\vec{x}_r}{\begin{bmatrix}1\\-2\end{bmatrix}}}_{C(A^T)}+c\underbrace{\overset{\vec{x}_n}{\begin{bmatrix}2\\1\end{bmatrix}}}_{N(A)}$, $c\in R$.

Ex if $\vec{b}=5\begin{bmatrix}1\\3\end{bmatrix}+4\underbrace{\begin{bmatrix}-3\\1\end{bmatrix}}_{\in N(A^T)}=\begin{bmatrix}-7\\19\end{bmatrix}$, no solutn
#inconceivable

E14ap5


x_2
$C(A^T)$
$N(A)$
$\vec{x}_n$
$\vec{x}$
$A\vec{x}_n=\vec{0}$
$A\vec{x}=\vec{p}$
x_1
$\vec{x}_r$
$A\vec{x}_r=\vec{p}$
y_2
$C(A)$
$\vec{b}$
$\vec{p}$ ← projection
$\vec{e}$ error
y_1
$N(A^T)$
$A \rightarrow$
$\leftarrow A^T$
$C(A^T)$ row space
$\vec{x}_r$
$A\vec{x}_r=\vec{p}$
$A\vec{x}=\vec{p}$
$\vec{x}=\vec{x}_r+\vec{x}_n$
$\vec{0}$
$N(A)$ null space
x_n
$A\vec{x}_n=\vec{0}$
$C(A)$
$\vec{p}$
$\vec{y}$
$\vec{0}$
$\vec{e}$
$N(A^T)$
$n=2$
R^n
$\vec{x}$ lives here.
R^m
$\vec{y}$ & $\vec{b}$ live here

Inigo:

$\underset{2\times 3}{\mathbb{A}} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$ $m=2$ $n=3$

$\underset{3\times 2}{\mathbb{A}^T} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}$

$\mathbb{R}_{\mathbb{A}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ P & F & F \end{bmatrix}$ $\underline{r=1}$

$\mathbb{R}_{\mathbb{A}^T} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ P & F \end{bmatrix}$

Bases

$C(\mathbb{A})$: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

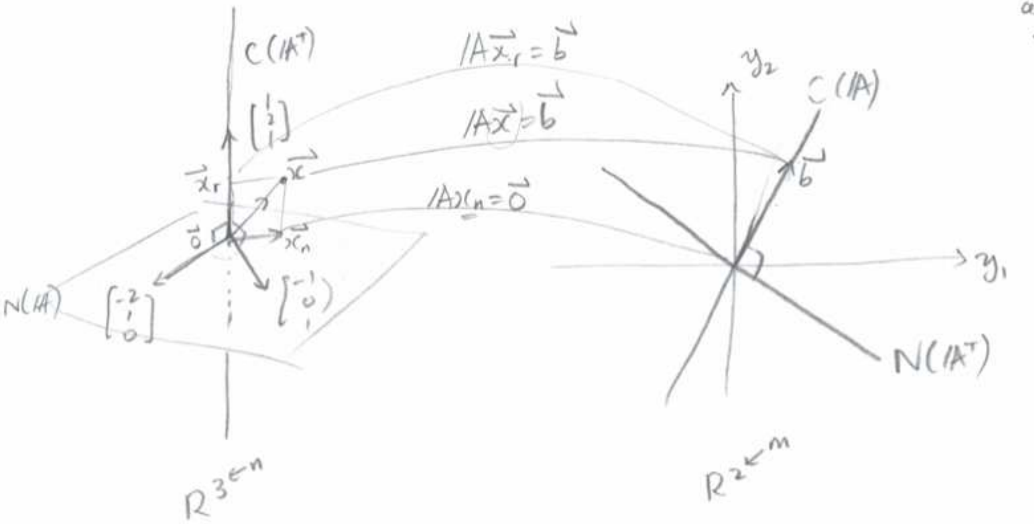
$N(\mathbb{A}^T)$: $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ $m-r=2-1=1$

$C(\mathbb{A}^T)$: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

$N(\mathbb{A})$: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

↗ dim: $n-r$
$= 3-1=2$

not
a beautiful basis
#morelater.



row space ⟷ colspace.

Inigo is "$\underset{r\times r}{1\times 1}$" matrix, equivalent to $\sqrt{30}$.

E14cp1

Fezzik:

$$\underset{3\times4}{\mathbb{A}} = \begin{bmatrix} 2 & 4 & 3 & 4 \\ 2 & 4 & 6 & 10 \\ 6 & 12 & 12 & 18 \end{bmatrix} \quad \begin{matrix} m=3 \\ n=4 \end{matrix}$$

$$\underset{4\times3}{\mathbb{A}^T} = \begin{bmatrix} 2 & 2 & 6 \\ 4 & 4 & 12 \\ 3 & 6 & 12 \\ 4 & 10 & 18 \end{bmatrix}$$

$$\mathbb{R}_{\mathbb{A}} = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} P & F & P & F \end{matrix} \end{array} \quad r=2$$

$$\mathbb{R}_{\mathbb{A}^T} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} P & P & F \end{matrix} \end{array}$$

bases

$C(\mathbb{A})$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

dim = r=2

$N(\mathbb{A}^T)$: $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$

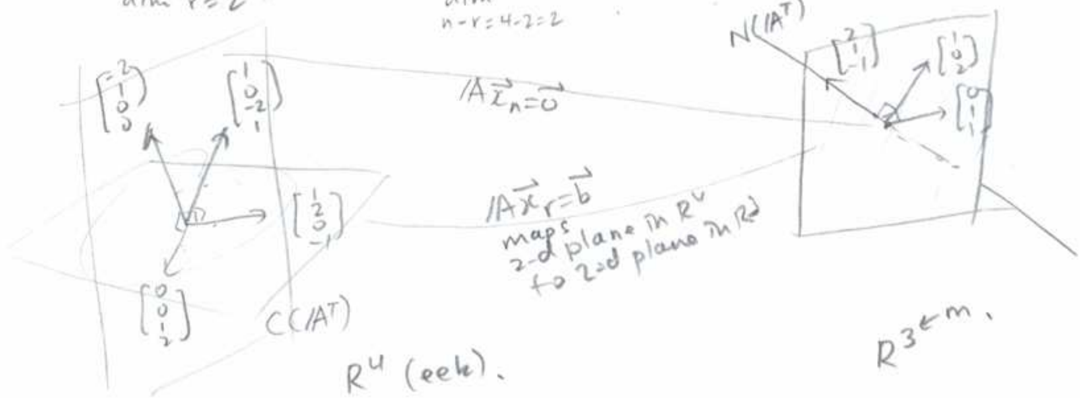
dim
m−r
=3−2=1

$C(\mathbb{A}^T)$: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

dim r=2

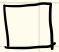
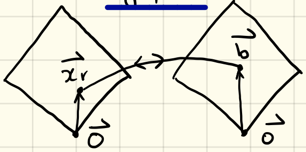
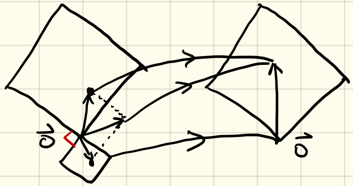
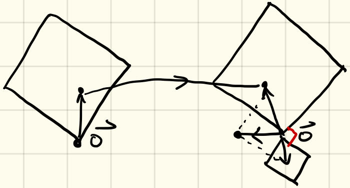
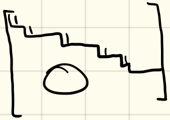
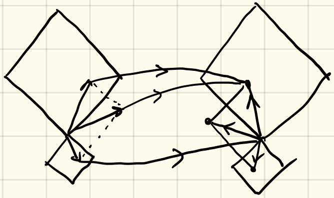
$N(\mathbb{A})$: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

dim
n−r=4−2=2



E14dp1

Everything hinges on $\mathbb{R}_{A}$ & $\mathbb{R}_{A^T}$ // Four main kinds of A.

Shape/rank:	big picture: m, n, r	# solutions to $A\vec{x}=\vec{b}$:	dim $N(A)$: how many solutions	dim $N(A^T)$: whether solution is possible
full rank $m=n=r$ square  invertible		1	0	0
$m=r$ $n>r$  wide		∞	$\geqslant 1$	0
$m>r$ $n=r$  tall		1 or 0 $\Updownarrow$ $b\in C(A)$	0	$\geqslant 1$
 $m, n > r$		∞ or 0	$\geqslant 1$	$\geqslant 1$

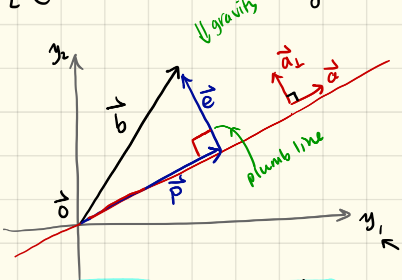
Projections:

menu:
- Project a vector onto a line
- Notion of an error vector $\vec{e}$
- Goal: Handle $A\vec{x}=\vec{b}$ when no solutions are possible. Big idea: Best approximation

Idea: Given a vector $\vec{b}$ and a direction described by a vector $\vec{a}$, break $\vec{b}$ into two components $\vec{b}=\vec{p}+\vec{e}$:

$\vec{p}$ = piece of $\vec{b}$ in direction of $\vec{a}$
$\vec{e}$ = " " " orthogonal to $\vec{a}$, in direction of $\vec{a}_\perp$

Picture: (for $\mathbb{R}^2$ but works in $\mathbb{R}^m$)



Big Deal: really projecting onto a subspace

↖ suspicious use of y's

$\vec{p}$ = projected component
$\vec{e}$ = error

One reason for doing this:

In solving $A\vec{x}=\vec{b}$, if $\vec{b}\notin C(A)$, we can still solve $A\vec{x}_*=\vec{p}$ where $\vec{p}$ is $\vec{b}$'s projection onto Column Space.

- Best Approximation
- Left Nullspace will matter!

How to find $\vec{p}$ & $\vec{e}$ given $\vec{b}$ & $\vec{a}$:

/E15ap1

We want $\vec{p}\parallel\vec{a}$ and $\vec{e}\perp\vec{a}$

Mathematically: $\vec{p}=x_*\vec{a}$ ↑ some number $\in\mathbb{R}$

$\vec{e}^T\vec{a}=\vec{a}^T\vec{e}=0$ ↖ inner (dot) product

Monks whisper: "Use the orthogonality..."

$$\vec{b}=\vec{p}+\vec{e}$$

Sneakiness:

$$\underset{\checkmark}{\vec{a}^T}(\underset{\checkmark}{\vec{b}}) = \underset{\checkmark}{\vec{a}^T}(\underset{?}{\vec{p}}+\underset{?}{\vec{e}})$$

$$\underbrace{\vec{a}^T\vec{b}}_{\text{number}} = \vec{a}^T\vec{p}+\cancelto{0}{\vec{a}^T\vec{e}}$$

$$= \vec{a}^T(x_*\vec{a})$$

$$= x_*\underbrace{\vec{a}^T\vec{a}}_{\text{number}}$$

$$\Rightarrow \quad x_* = \frac{(\vec{a}^T\vec{b})}{(\vec{a}^T\vec{a})}$$

$$\vec{p}=x_*\vec{a}=\underbrace{\frac{(\vec{a}^T\vec{b})}{(\vec{a}^T\vec{a})}}_{\text{some scaling of }\vec{a}}\vec{a}$$

$$\vec{e}=\vec{b}-\vec{p}$$

done.

Example:

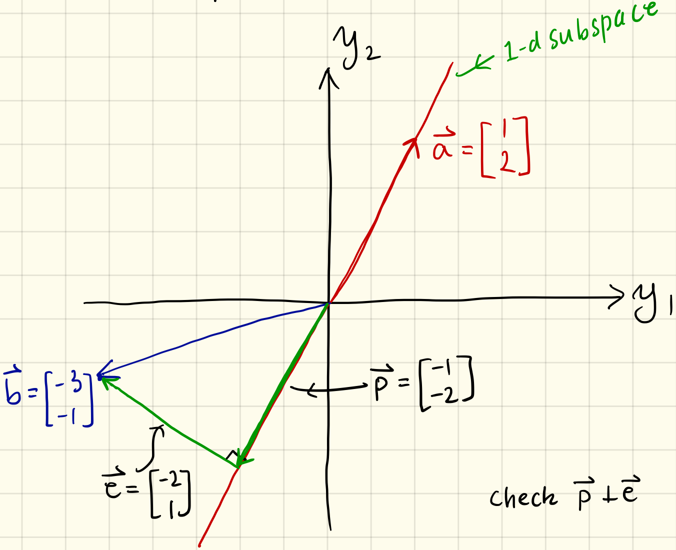
project $\vec{b} = \begin{bmatrix}-3\\-1\end{bmatrix}$ onto $\vec{a} = \underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{\text{direction is all that matters}}$

$$x_* = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} = \frac{[1\ 2]\begin{bmatrix}-3\\-1\end{bmatrix}}{[1\ 2]\begin{bmatrix}1\\2\end{bmatrix}} = \frac{-5}{5} = -1$$

$$\Rightarrow \vec{p} = x_*\vec{a} = (-1)\vec{a} = \begin{bmatrix}-1\\-2\end{bmatrix}$$

$$\Rightarrow \vec{e} = \vec{b} - \vec{p} = \begin{bmatrix}-3\\-1\end{bmatrix} - \begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$$

note: $\vec{p} \perp \vec{e}$ as required



check $\vec{p} \perp \vec{e}$

More Sneakiness:

We have $\vec{p} = \underbrace{\left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)}_{\substack{1\times1\\ \text{a number}}} \underset{m\times1}{\vec{a}}$

not great to have to recalculate for each $\vec{b}$

really want a gadget matrix that projects $\vec{b}$ as an operator

$$= \frac{\overbrace{(\vec{a}^T\vec{b})}^{1\times1}}{\underbrace{(\vec{a}^T\vec{a})}_{1\times1}} \underset{m\times1}{\vec{a}} = \frac{1}{\underbrace{(\vec{a}^T\vec{a})}_{1\times1}} \cdot \underbrace{(\vec{a}^T\vec{b})}_{1\times1} \underset{m\times1}{\vec{a}}$$

$$= \frac{1}{\underbrace{(\vec{a}^T\vec{a})}_{1\times1}} \underset{m\times1}{\vec{a}} \underbrace{(\overset{1\times m}{\vec{a}^T} \overset{m\times1}{\vec{b}})}_{1\times1} = \frac{1}{\underbrace{(\vec{a}^T\vec{a})}_{1\times1}} \underbrace{(\overset{m\times1}{\vec{a}}\, \overset{1\times m}{\vec{a}^T})}_{\substack{m\times m\\ \text{square}}} \underset{m\times1}{\vec{b}}$$

outer product

$$= \frac{1}{\underbrace{\|\vec{a}\|^2}_{(\text{length of } \vec{a})^2}} (\vec{a}\,\vec{a}^T)\vec{b} = \underbrace{\left(\frac{\overset{m\times1}{\vec{a}}}{\|\vec{a}\|}\ \frac{\overset{1\times m}{\vec{a}^T}}{\|\vec{a}\|}\right)}_{m\times m} \underset{m\times1}{\vec{b}}$$

good: length of $\vec{a}$ cannot matter

$$= \underbrace{\overset{m\times1}{\hat{a}}\ \overset{1\times m}{\hat{a}^T}}_{m\times m} \underbrace{\vec{b}}_{m\times1} = \mathbb{P}_{\hat{a}}\vec{b}$$

unit vector

outer product

Projection Operator

Example again:

project $\vec{b} = \begin{bmatrix}-3\\-1\end{bmatrix}$ onto $\vec{a} = \begin{bmatrix}1\\2\end{bmatrix}$

makes unit vector

$$\hat{a} = \frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{\sqrt{1^2+2^2}}\begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$$

$$\mathbb{P}_{\hat{a}} = \hat{a}\,\hat{a}^T = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$$

symmetry is guaranteed

So: $$\vec{p} = \mathbb{P}_{\hat{a}}\vec{b} = \frac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}-3\\-1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}-5\\-10\end{bmatrix} = \begin{bmatrix}-1\\-2\end{bmatrix}$$ ✓

$\vec{e} = \vec{b} - \vec{p}$ as before

Bonus:

$\cancel{(\overset{1\times1}{1} - \overset{m\times m}{\mathbb{P}_{\hat{a}}})}$ wrong.

$$\vec{e} = \vec{b} - \vec{p} = \underset{m\times1}{\vec{b}} - \underset{m\times m}{\mathbb{P}_{\hat{a}}}\,\underset{m\times1}{\vec{b}}$$

$$= \underset{m\times m}{\mathbb{I}}\,\underset{m\times1}{\vec{b}} - \underset{m\times m}{\mathbb{P}_{\hat{a}}}\,\underset{m\times1}{\vec{b}} = (\underbrace{\underset{m\times m}{\mathbb{I}} - \underset{m\times m}{\mathbb{P}_{\hat{a}}}}_{\text{Extracts } \vec{e} \text{ part of } \vec{a}.})\,\underset{m\times1}{\vec{b}}$$

much happiness over $\mathbb{P}_{\hat{a}}$

The Amazing Normal Equation:

Menu:

- Find the best approximation to $A\vec{x} = \vec{b}$ when $\vec{b} \notin C(A)$

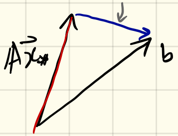
Before: We just gave up when $A\vec{x} = \vec{b}$ had no solution ⇐ betrayed a lack of ticker

New plan: find $\vec{x}_*$ so that $A\vec{x}_*$ is as close to $\vec{b}$ as possible.

denotes approximation

Mathematically: we want $\vec{x}^*$ that minimizes $\| \vec{b} - A\vec{x}_* \|$

distance between $\vec{b}$ and $A\vec{x}_*$



Big idea: See $\vec{b} = \vec{p} + \vec{e}$ where

$\vec{p} \in C(A)$

& $\vec{e} \in N(A^T)$ } $\vec{p} \perp \vec{e}$ guarantees

We project $\vec{b}$ onto $C(A)$ and solve $\boxed{A\vec{x}_* = \vec{p}}$ instead

How?

Same approach as for simple projections:

We want

$$\underbrace{\vec{b} = \vec{p} + \vec{e}}_{①} \text{ where } \underbrace{A\vec{x}_* = \vec{p}}_{②} \text{ \& } \underbrace{A^T\vec{e} = \vec{0}}_{③}$$

Monks

Start with $\underbrace{A^T\vec{e} = \vec{0}}_{③}$

$$\underbrace{\vec{0} =}_{③} A^T\vec{e} \underbrace{=}_{①} A^T(\vec{b} - \vec{p}) = A^T\vec{b} - A^T\vec{p}$$

$$\Rightarrow A^T\vec{b} = A^T\vec{p} \underbrace{=}_{②} \overset{\text{group.}}{A^T(A}\underbrace{\vec{x}_*}_{}) $$

means $\vec{p}$ is some linear combination of A's columns

Switch sides:

square, symmetric = awesome

$$\underbrace{(\underbrace{A^T}_{n\times m}\underbrace{A}_{m\times n})}_{n\times n}\underbrace{\vec{x}_*}_{n\times 1} = \underbrace{\underbrace{A^T}_{n\times m}\underbrace{\vec{b}}_{m\times 1}}_{n\times 1}$$

$\in \mathbb{R}^n$ $\in \mathbb{R}^n$

incredible! always solvable!

of the form:

$$\underbrace{A'}_{n\times n}\underbrace{\vec{x}_*}_{n\times 1} = \underbrace{\vec{b}'}_{n\times 1}$$

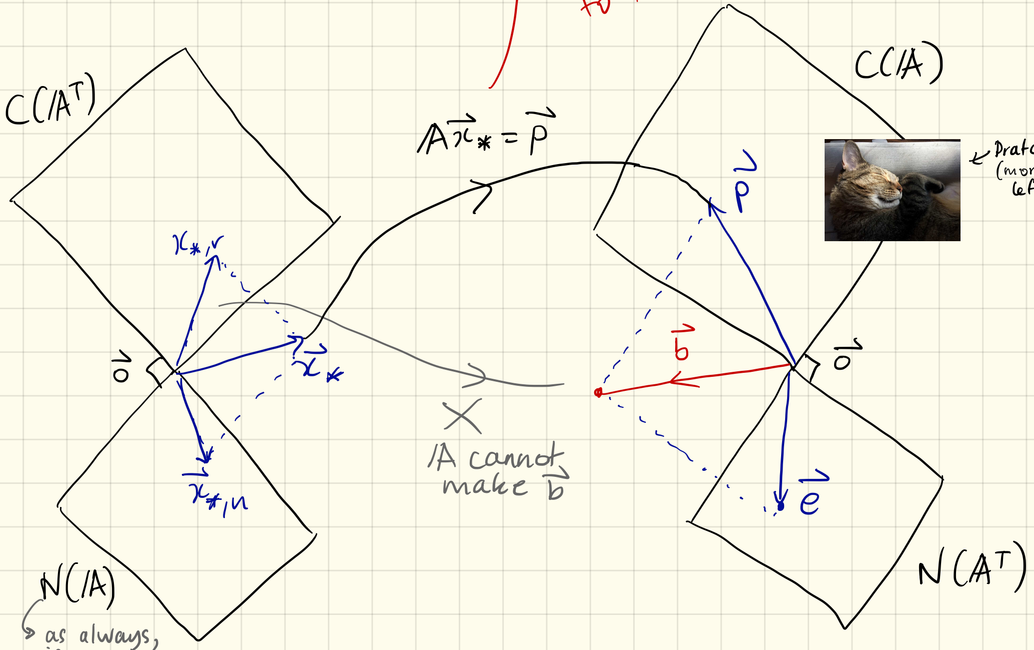
← prime

Abstract Big Picture

$$A^TA\vec{x}_* = A^T\vec{b}$$

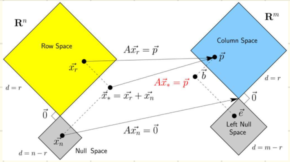
E15bp2

solve normal equations to find $\vec{x}_*$



as always, if non-zero, infinitely many solutions exist

Zoomable version



Example of using the Normal Equation

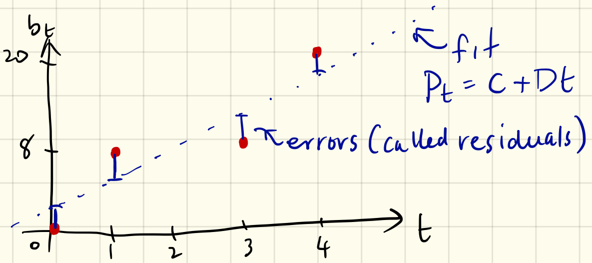
- fitting a straight line to a set of data points

← fundamental scientific activity!

Ex from Strang:

$b_t = 0, 8, 8, 20$

at times $t = 0, 1, 3, 4$



Want fit to be true:

$t=0$	$b_0 = 0 = 1C + D \times 0$
$t=1$	$b_1 = 8 = 1C + D \times 1$
$t=3$	$b_2 = 8 = 1C + D \times 3$
$t=4$	$b_3 = 20 = 1C + D \times 4$

E15cp1

Matrixify:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}_{\substack{\checkmark \mathbb{A} \\ 4\times 2}} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{\substack{\vec{x} \\ 2\times 1 \\ ?}} = \underbrace{\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}}_{\substack{\checkmark \vec{b} \\ 4\times 1}}$$

$\begin{bmatrix} 0 \\ 1 \\ 9 \\ 16 \end{bmatrix}$ E for $p_t = C + Dt + Et^2$

Clear $\vec{b}$ is not in $\mathbb{A}$'s Column Space

Solve $\mathbb{A}^T \mathbb{A} \vec{x}_* = \mathbb{A}^T \vec{b}$:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}}_{\mathbb{A}^T} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}_{\mathbb{A}} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{\vec{x}_*} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}}_{\mathbb{A}^T} \underbrace{\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}}_{\vec{b}}$$

always 2×2

$$\Rightarrow \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

← line fit for 10^6 (ex) data pts ⇒ still 2×2 problem

$[\mathbb{A}^T\mathbb{A} \mid \mathbb{A}^T\vec{b}]$

$$R_2' = R_2 - \left(\frac{8}{4}\right) R_1 \quad \begin{bmatrix} 4 & 8 & | & 36 \\ 0 & 10 & | & 40 \end{bmatrix} \leftarrow U$$

Back substitute: $4C_* + 8D_* = 36 \leftarrow C_* = 1$

$10D_* = 40 \leftarrow D_* = 4$

$\vec{x}_* = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ Best fit line $= P_t = 1 + 4t$

$$\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}; \quad \|\vec{e}\|^2 = 1^2 + 3^2 + 5^2 + 3^2 = 44$$

The Normal Equation and the Man in Black

Westley, our hero: $\mathbb{A} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}$

Solve $\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \leftarrow \vec{b} \notin C(\mathbb{A})$

$$\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 3 & -6 & 5 \end{array}\right] \xrightarrow[R_2' = R_2 - \left(\frac{3}{1}\right)R_1]{\sim} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 0 & -10 \end{array}\right] \leftarrow \text{eek: } 0 = -10$$

inconceivable! (or: no solution)

Time for the Normal Equation: $\mathbb{A}^T\mathbb{A}\vec{x}_* = \mathbb{A}^T\vec{b}$

$\mathbb{A}^T\mathbb{A}$ is always symmetric

$$\underbrace{\underset{n\times m}{\mathbb{A}^T}\ \underset{m\times n}{\mathbb{A}}}_{n\times n} = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} = \begin{bmatrix} 10 & -20 \\ -20 & 40 \end{bmatrix}$$

$$\underbrace{\underset{n\times m}{\mathbb{A}^T}\ \underset{m\times 1}{\vec{b}}}_{n\times 1} = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ -40 \end{bmatrix}$$

Now Solve

$$\left[\begin{array}{cc|c} 10 & -20 & +20 \\ -20 & 40 & -40 \end{array}\right] \xrightarrow[R_2' = R_2 - \left(\frac{-20}{10}\right)R_1]{\sim} \left[\begin{array}{cc|c} 10 & -20 & +20 \\ 0 & 0 & 0 \end{array}\right]$$

all good as promised

$$\xrightarrow[R_1' = \frac{1}{10}R_1]{\sim} \left[\begin{array}{cc|c} 1 & -2 & +2 \\ 0 & 0 & 0 \\ P & F & \end{array}\right]$$

$$\Rightarrow \underset{P}{x_{*,1}} - 2\underset{F}{x_{*,2}} = +2$$

$$\Rightarrow \underset{P}{x_{*,1}} = +2 + 2\underset{F}{x_{*,2}}$$

replace pivot vars in terms of free vars

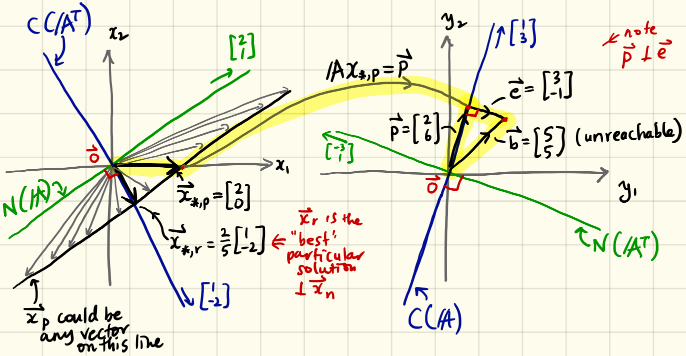
$$\vec{x}_* = \begin{bmatrix} \overset{P}{x_{*,1}} \\ \underset{F}{x_{*,2}} \end{bmatrix} = \begin{bmatrix} +2 + 2x_{*,2} \\ 0 + x_{*,2} \end{bmatrix} = \overset{\vec{x}_{*,p}}{\begin{bmatrix} +2 \\ 0 \end{bmatrix}} + \overset{\vec{x}_{*,h}}{x_{*,2}\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

particular homogeneous

where $x_{*,2} \in R$

$$\vec{p} = \mathbb{A}\vec{x}_* = \underbrace{\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} +2 \\ 0 \end{bmatrix}}_{\text{particular solution does the work}} + x_{*,2}\cancel{\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$



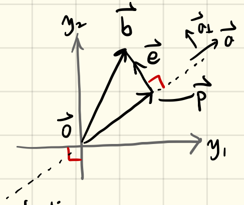
- Note $\vec{x}_p \neq \vec{x}_r$ in general; $\vec{x}_r$ is the special $\vec{x}_p$ that lives in A's row space
- How $\vec{b}$ was built:

$$\vec{b} = \underbrace{2\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\vec{p}} \underset{+}{-} \underbrace{1\begin{bmatrix} -3 \\ 1 \end{bmatrix}}_{\vec{e}} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Projecting a vector $\vec{b} \in R^m$ onto a subspace of R^m

#excitement

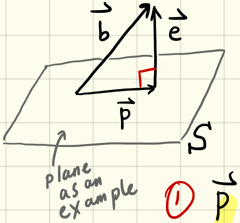
We know how to project a vector $\vec{b}$ onto a line defined by a vector $\vec{a}$:



$$\vec{p} = \underbrace{\overset{\text{outer product of unit vectors}}{\underset{m\times 1}{\hat{a}}\,\underset{1\times m}{\hat{a}^T}}}_{m\times m}\vec{b} = \mathbb{P}_{\hat{a}}\vec{b}$$

Projection matrix/ operator

Now:
Generalize to an r-dim subspace of R^m



We have ~~a~~ some basis for subspace S:

$$\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_r\}$$

$\vec{p} \subset C(\mathbb{A})$

① $$\underset{m\times 1}{\vec{p}} = \underset{m\times r}{\mathbb{A}}\,\underset{r\times 1}{\vec{x}_*} = \underbrace{\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_r \\ | & | & & | \end{bmatrix}}_{\text{linearly independent because } \vec{a}_i \text{ form a basis}}\vec{x}_*$$

② $$\mathbb{A}^T\vec{e} = \vec{0}$$

$\vec{e} \in N(\mathbb{A}^T)$

③ $$\vec{b} = \vec{p} + \vec{e}$$

Monks:

$$\vec{0} \overset{②}{=} \mathbb{A}^T\vec{e} \overset{③}{=} \mathbb{A}^T(\vec{b} - \vec{p}) = \mathbb{A}^T\vec{b} - \mathbb{A}^T\underbrace{\vec{p}}_{\overset{①}{=}\ \mathbb{A}\vec{x}_*}$$

E15ep1

$$\Rightarrow \vec{0} = \mathbb{A}^T\vec{b} - \mathbb{A}^T\mathbb{A}\vec{x}_*$$

$$\Rightarrow \mathbb{A}^T\mathbb{A}\vec{x}_* = \mathbb{A}^T\vec{b}$$

Solve for $\vec{x}_*$ then find $\vec{p}$ as $\vec{p} = \mathbb{A}\vec{x}_*$

Special deal: extra tofu knives

b/c $\mathbb{A}$'s columns are linearly independent, $\mathbb{A}^T\mathbb{A}$ is invertible

reason to follow

$$\Rightarrow \underbrace{(\mathbb{A}^T\mathbb{A})^{-1}(\mathbb{A}^T\mathbb{A})}_{\mathbb{I}}\vec{x}_* = (\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T\vec{b}$$

premultiply both sides by inverse

$$\Rightarrow \vec{x}_* = (\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T\vec{b}$$

$$\Rightarrow \vec{p} = \mathbb{A}\vec{x}_* = \underbrace{\underset{m\times r}{\mathbb{A}}\underbrace{(\underset{r\times m}{\mathbb{A}^T}\underset{m\times r}{\mathbb{A}})^{-1}}_{r\times r}\underset{r\times m}{\mathbb{A}^T}}_{}\underset{m\times 1}{\vec{b}}$$

$$\equiv \underset{m\times m}{\mathbb{P}}\,\underset{m\times 1}{\vec{b}}$$

Projection Matrix (good for low dimensions)

More goodness: expect $\mathbb{P}^2\vec{b} = \mathbb{P}^3\vec{b} = \dots = \vec{p}$

or

$\mathbb{P} = \mathbb{P}^n$ for all $n \geq 1$

Check: $\mathbb{P}^2 =$

$$\mathbb{A}\underbrace{\cancel{(\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T\mathbb{A}}}_{\mathbb{I}}(\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T$$

$$= \mathbb{A}(\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T = \mathbb{P}$$

cool! (right?)

Our Projection Matrix when we have a basis for S:

$$P = A(A^TA)^{-1}A^T; \quad P\vec{b} = \vec{p} \in S$$

Warning! $(A^TA)^{-1} \neq A^{-1}(A^T)^{-1}$ generally

may be equal sometimes

A may be rectangular!!!

$\underbrace{\overset{r\times m}{A^T}\,\overset{m\times r}{A}}_{r\times r}$ is always square and symmetric

Important Truth:
A^TA is invertible iff
A's columns are linearly independent

"if and only if"

$N(A^TA)=\{\vec{0}\}$, $\overset{r\times r}{A^TA}$ is full rank r.

$\iff A\vec{x}=\vec{0}$ only has $\vec{x}=\vec{0}$ as a solution

$\iff N(A)=\{\vec{0}\}$

Plan: Show A^TA & A have the same Nullspace always

Need to show that if $\vec{x}\in N(A)$ then $\vec{x}\in N(A^TA)$ and vice versa

LE15ep2

Assume $\vec{x}\in N(A)$: $A\vec{x}=\vec{0}$

$\Rightarrow A^T(A\vec{x}) = A^T(\vec{0})$

$\Rightarrow (A^TA)\vec{x} = \vec{0}$

So $\vec{x}\in N(A^TA)$

Second,
if $\vec{x}\in N(A^TA)$ then

$A^TA\vec{x}=\vec{0}$ ← by definition

$\Rightarrow \overset{1\times r}{\vec{x}^T}(A^TA\vec{x}) = \vec{x}^T\overset{r\times 1}{(\vec{0})}$

$\Rightarrow \vec{x}^TA^TA\vec{x} = \underset{1\times 1}{0}$

$\Rightarrow (A\vec{x})^T(A\vec{x}) = 0$

we $(BC)^T = C^TB^T$

$\Rightarrow \|\underbrace{\overset{m\times r}{A}\,\overset{r\times 1}{\vec{x}}}_{m\times 1 \text{ vector}}\|^2 = 0$

if length $=0$

$\Rightarrow A\vec{x}=\vec{0}$

So $\vec{x}\in N(A)$ // done

Note: we can do the same sort of thing for AA^T

Upshot: if $N(A)=\{\vec{0}\}$ then A^TA is invertible

A need not be square

square $r\times r$ matrix with rank r

Orthogonal and Orthonormal bases help us win friends and influence people

Menu:
- Motivation for orthogonality
- orthogonal Matrices

Next:
- the Gram-Schmidt Process
- What this all means for $A\vec{x}=\vec{b}$

Observation: We've been finding bases for our four fundamental subspaces, and we've so far taken whatever popped out.

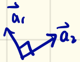
ex: $\left\{ \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$ ← Basis for Fezzik's $C(A)$ ↓ Describes 2-d subspace in R^3

Does the job BUT we really like orthogonality in our bases and $\begin{bmatrix}① & ⓪ & ②\end{bmatrix}\begin{bmatrix}⓪\\①\\①\end{bmatrix} = 2 \neq 0$

not orthogonal

ex1 $\left\{ \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix} \right\}$ = basis for a plane in R^3

$\vec{a}_1 \quad \vec{a}_2$



$\vec{a}_1^T\vec{a}_2 = \begin{bmatrix}① & 1 & 2\end{bmatrix}\begin{bmatrix}-2\\0\\1\end{bmatrix} = -2+0+2=0$ ✓

We call such a basis Orthogonal

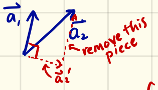
E16ap1

ex2. (from Monks) $\left\{ \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\3\\7\end{bmatrix} \right\}$ $\quad \vec{a}_1^T\vec{a}_2 = 1+3+14=18\neq 0$

$\vec{a}_1 \quad \vec{a}_2$

sneakiness

$\vec{a}_2 = \begin{bmatrix}-2\\0\\1\end{bmatrix} + 3\begin{bmatrix}1\\1\\2\end{bmatrix}$



← $\vec{a}_2$ contains some of $\vec{a}_1$

[Big idea: Systematically turn a basis into an orthogonal basis by removing non-orthogonal pieces

[Everything is connected: Projections will do the work for us.

Claim: Orthogonality makes a basis a happy basis ← math

- Main reason: Representation of vectors is very clean.

Information contained in each basis vector is distinct.

- Later: We will see we get orthogonal bases for free when working with a certain kind of Totally Awesome Matrices

- **Bonus:** A set of orthogonal vectors is automatically linearly independent and therefore must form a basis for the subspace they span

"Obvious" but proof is nutritious
↖ dangerous word

Given $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ with $\vec{a}_i^{\,T}\vec{a}_j = 0$ for all $1 \le i, j \le n$, $i \ne j$
& $\vec{a}_i \in \mathbb{R}^m$

$$\underset{m\times n}{\mathbb{A}} = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$$

Monks: Presume linear dependence
$\iff$ $\mathbb{A}\vec{x} = \vec{0}$ has a non-zero solution $\vec{x}$
$\iff$ $N(\mathbb{A}) \ne \{\vec{0}\}$

Must have

use $(\mathbb{B}\mathbb{C})^T = \mathbb{C}^T\mathbb{B}^T$

$$0 = \underset{1\times m}{\vec{0}^{\,T}}\underset{m\times 1}{\vec{0}} = (\mathbb{A}\vec{x})^T(\mathbb{A}\vec{x})$$

$$= \vec{x}^{\,T}\mathbb{A}^T\mathbb{A}\vec{x}$$

$$= \underset{\vec{x}^{\,T}}{[x_1 \cdots x_n]}\begin{bmatrix} - \vec{a}_1^{\,T} - \\ - \vec{a}_2^{\,T} - \\ \vdots \\ - \vec{a}_n^{\,T} - \end{bmatrix}\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}\underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$$

$$= [x_1 \ \dots \ x_n]\begin{bmatrix} \|\vec{a}_1\|^2 & & 0 \\ & \|\vec{a}_2\|^2 & \\ 0 & \ddots & \|\vec{a}_n\| \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

↙ all zeros

diagonal matrix ↗

$$= x_1^2\underbrace{\|\vec{a}_1\|^2}_{\ne 0} + x_2^2\underbrace{\|\vec{a}_2\|^2}_{\ne 0} + \dots + x_n^2\underbrace{\|\vec{a}_n\|^2}_{\ne 0}$$

$$= 0 \text{ only if } x_1 = x_2 = \dots = x_n = 0$$

Contradiction
$\Rightarrow N(\mathbb{A}) = \{\vec{0}\}$

E16ap2

Extra happy kind of basis:

An Orthonormal Basis

$\equiv$ Orthogonal Basis made up of unit vectors

Observation: Easy to do!

$$\underset{\text{orthogonal basis}}{\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}\right\}} \underset{\text{divide by lengths (easy)}}{\Rightarrow} \underset{\text{orthonormal basis}}{\left\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\2\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\0\\1\end{bmatrix}\right\}}$$

hard part ↙

Next: How to create an orthogonal basis in the first place

Transmuting a basis into an orthogonal one

Menu
- The Gram-Schmidt Process

Idea: Turn $\underset{\mathbb{R}^m}{\overset{\text{basis}}{\{\vec a_1, \vec a_2, \dots, \vec a_n\}}}$ into $\overset{\text{orthogonal basis}}{\{\vec q_1, \vec q_2, \dots, \vec q_n\}}$

and then $\underset{\text{orthonormal basis}}{\{\hat q_1, \hat q_2, \dots, \hat q_n\}}$ $\rightarrow \hat q_i := \frac{1}{\|\vec q_i\|}\vec q_i$

by incrementally removing parts of vectors.

n=3 general formula: ↙ 3d subspace in $\mathbb{R}^m$

① Set $\vec q_1 = \vec a_1$

② $\vec q_2 = \vec a_2 - \frac{\vec q_1^T \vec a_2}{\vec q_1^T \vec q_1}\vec q_1$ ← projection of $\vec a_2$ onto direction described by $\vec q_1$

③ $\vec q_3 = \vec a_3 - \frac{(\vec q_1^T \vec a_3)}{(\vec q_1^T \vec q_1)}\vec q_1 - \frac{(\vec q_2^T a_3)}{(\vec q_2^T \vec q_2)}\vec q_2$

↖ projections ↗

ⓝ $\vec q_n = \vec a_n - (\dots \dots)$

n-1 projections of $\vec a_n$ onto $\vec q_1, \vec q_2, \dots, \vec q_{n-1}$

E16 p1

We know $\frac{\overset{\leftarrow \text{number}}{(\vec q_1^T \vec a_2)}}{\underset{\leftarrow \text{number}}{(\vec q_1^T \vec q_1)}}\vec q_1 = \underbrace{\underset{m\times 1}{\hat q_1}\,\underset{1\times m}{\hat q_1^T}}_{m\times m}\,\underset{m\times 1}{\vec a_2}$ (outer product ↓, $\swarrow P_{\hat q_1}$)

So above 3 steps can be rewritten as:

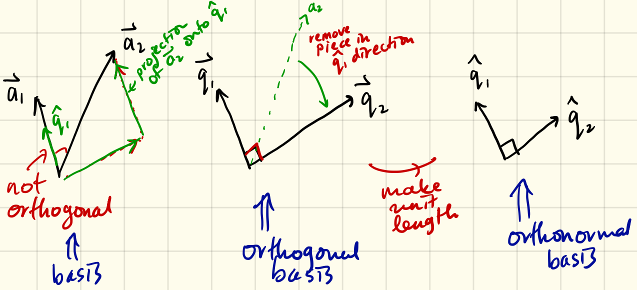
① $\vec q_1 = \vec a_1 \Rightarrow \hat q_1 = \frac{1}{\|\vec q_1\|}\vec q_1$

② $\vec q_2 = \vec a_2 - \hat q_1 \hat q_1^T \vec a_2 \Rightarrow \hat q_2 = \frac{1}{\|\vec q_2\|}\vec q_2$

③ $\vec q_3 = \vec a_3 - \hat q_1 \hat q_1^T \vec a_3 - \hat q_2 \hat q_2^T a_3 \Rightarrow \hat q_3 = \frac{1}{\|\hat q_3\|}\hat q_3$

↗ good for theory

What's happening:



Example calculation:

$$\left\{ \underset{\vec a_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \underset{\vec a_2}{\begin{bmatrix}4\\2\\0\end{bmatrix}}, \underset{\vec a_3}{\begin{bmatrix}5\\-4\\-7\end{bmatrix}} \right\}$$

① $\vec q_1 = \vec a_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

orthogonal

② $$\vec q_2 = \vec a_2 - \frac{\vec q_1^{\,T}\vec a_2}{\vec q_1^{\,T}\vec q_1}\vec q_1$$

$$= \begin{bmatrix}4\\2\\0\end{bmatrix} - \frac{\overset{6}{[1\ 1\ 1]\begin{bmatrix}4\\2\\0\end{bmatrix}}}{\underset{3}{[1\ 1\ 1]\begin{bmatrix}1\\1\\1\end{bmatrix}}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}4\\2\\0\end{bmatrix} - \frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\0\\-2\end{bmatrix}$$

③ $$\vec q_3 = \vec a_3 - \frac{\vec q_1^{\,T}\vec a_3}{\vec q_1^{\,T}\vec q_1}\vec q_1 - \frac{\vec q_2^{\,T}\vec a_3}{\vec q_2^{\,T}\vec q_2}\vec q_2$$

$$= \begin{bmatrix}5\\-4\\-7\end{bmatrix} - \frac{\overset{-6}{\cancel{[1\ 1\ 1]\begin{bmatrix}5\\-4\\-7\end{bmatrix}}}}{\underset{+3}{\cancel{[1\ 1\ 1]\begin{bmatrix}1\\1\\1\end{bmatrix}}}}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{\overset{24}{\cancel{[2\ 0\ -2]\begin{bmatrix}5\\-4\\-7\end{bmatrix}}}}{\underset{8}{\cancel{[2\ 0\ -2]\begin{bmatrix}2\\0\\-2\end{bmatrix}}}}\begin{bmatrix}2\\0\\-2\end{bmatrix}$$

$$= \begin{bmatrix}5\\-4\\-7\end{bmatrix} \overbrace{-\frac{-\cancel{6}}{\cancel{3}}}^{+2}\begin{bmatrix}1\\1\\1\end{bmatrix} \overbrace{-\frac{\cancel{24}}{\cancel{8}}}^{-3}\begin{bmatrix}2\\0\\-2\end{bmatrix} = \begin{bmatrix}1\\-2\\1\end{bmatrix}$$

Normalize:

$$\hat q_1 = \frac{1}{\sqrt3}\begin{bmatrix}1\\1\\1\end{bmatrix},\quad \hat q_2 = \frac{1}{\underset{2\sqrt2}{\boxed{\sqrt8}}}\begin{bmatrix}2\\0\\-2\end{bmatrix},\quad \hat q_3 = \frac{1}{\sqrt6}\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

$$\parallel$$

$$\frac{1}{\sqrt2}\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

Note: Gram-Schmidt method tends to produce many square roots

Check $\hat q_1^{\,T}\hat q_2 = \hat q_2^{\,T}\hat q_3 = \hat q_3^{\,T}\hat q_1 = 0$

- test $\overset{\nearrow}{\text{every}}$ pair of basis vectors
- must be orthogonal

Next: See $\vec a_i$'s can be rebuilt from $\hat q_i$'s $\Rightarrow \underset{m\times n}{A} = \underset{m\times n}{Q}\,\underset{n\times n}{R}$

A new factorization: $\underset{m\times n}{\mathbb{A}} = \underset{m\times n}{\mathbb{Q}}\,\underset{n\times n}{\mathbb{R}}$

Q's shape matches A's

unfortunate namespace overlap
↓
NOT $\mathbb{R}/\mathbb{A}$

Idea: We love to use matrices to encode our methods

empowering

ex $\mathbb{P}\mathbb{A} = \mathbb{L}\mathbb{U} \equiv$ Gaussian Elimination

So: let's turn the Gram-Schmidt Process into a factorization of $\mathbb{A}$

From a few pages back:

① $\vec q_1 = \vec a_1 \Rightarrow \hat q_1 = \frac{1}{\|\vec q_1\|}\vec q_1$

② $\vec q_2 = \vec a_2 - \overbrace{\hat q_1 \hat q_1^T}^{m\times m\ \mathbb{P}_{\hat q_1}}\vec a_2 \Rightarrow \hat q_2 = \frac{1}{\|\vec q_2\|}\vec q_2$

③ $\vec q_3 = \vec a_3 - \overbrace{\hat q_1 \hat q_1^T}^{m\times m}\vec a_3 - \overbrace{\hat q_2 \hat q_2^T}^{m\times m} a_3 \Rightarrow \hat q_3 = \frac{1}{\|\vec q_3\|}\hat q_3$

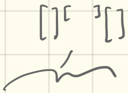
Monks say: Express the $\vec a_i$ in terms of the $\hat q_i$ using a column picture approach

$\Rightarrow$ Connect $\mathbb{A}$ to $\mathbb{Q} = \begin{bmatrix} | & | & & | \\ \hat q_1 & \hat q_2 & \cdots & \hat q_n \\ | & | & & | \end{bmatrix}$

Rearrange above so $\vec a_i = \cdots$:

put $\vec a_i$'s on left by themselves

E16cp1



① $\vec a_1 = \vec q_1$

② $\vec a_2 = \vec q_2 + \hat q_1 \hat q_1^T \vec a_2$

③ $\vec a_3 = \vec q_3 + \hat q_1 \hat q_1^T \vec a_3 + \hat q_2 \hat q_2^T \vec a_3$

need these to look the same

Sneakiness: See $\vec q_i$ as projection of $\vec a_i$ onto $\hat q_i$ direction

ex ③ above:

$$\underbrace{(\hat q_3 \hat q_3^T)}_{\mathbb{P}_{\hat q_3}}\vec a_3 = (\hat q_3 \hat q_3^T)\big(\vec q_3 + \overset{\perp}{\hat q_1 \hat q_1^T \vec a_3} + \overset{\perp}{\hat q_2 \hat q_2^T \vec a_3}\big)$$

$$= \hat q_3\underbrace{(\hat q_3^T \vec q_3)}_{\|\vec q_3\|} + 0 + 0$$

$$= \vec q_3$$

① $\vec a_1 = \hat q_1 \hat q_1^T \vec a_1$

② $\vec a_2 = \hat q_2 \hat q_2^T \vec a_2 + \hat q_1 \hat q_1^T \vec a_2$

③ $\vec a_3 = \hat q_3 \hat q_3^T \vec a_3 + \hat q_1 \hat q_1^T \vec a_3 + \hat q_2 \hat q_2^T \vec a_3$

Makes sense $\vec a_3$ has components in $\hat q_1, \hat q_2, \& \hat q_3$ directions

Reorder:

① $$\vec a_1 = \overset{m\times1}{\hat q_1}\,\overset{1\times m}{\hat q_1^T}\,\overset{m\times1}{\vec a_1}$$

numbers (inner products)

② $$\vec a_2 = \hat q_1 \hat q_1^T \vec a_2 + \hat q_2 \hat q_2^T \vec a_2$$

③ $$\vec a_3 = \hat q_1 \hat q_1^T \vec a_3 + \hat q_2 \hat q_2^T \vec a_3 + \hat q_3 \hat q_3^T \vec a_3$$

Column picture:

① $$\vec a_1 = \begin{bmatrix} | & | & | \\ \hat q_1 & \hat q_2 & \hat q_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} \hat q_1^T \vec a_1 \\ 0 \\ 0 \end{bmatrix}$$

② $$\vec a_2 = \begin{bmatrix} | & | & | \\ \hat q_1 & \hat q_2 & \hat q_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} \hat q_1^T \vec a_2 \\ \hat q_2^T \vec a_2 \\ 0 \end{bmatrix}$$

③ $$\vec a_3 = \begin{bmatrix} | & | & | \\ \hat q_1 & \hat q_2 & \hat q_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} \hat q_1 \vec a_3 \\ \hat q_2 \vec a_3 \\ \hat q_3 \vec a_3 \end{bmatrix}$$

TRIUMPHANCY:

$$\mathbb{A} = \underset{m\times n}{\begin{bmatrix} | & | & | \\ \vec a_1 & \vec a_2 & \vec a_3 \\ | & | & | \end{bmatrix}} = \underset{m\times n}{\overset{\mathbb{Q}}{\begin{bmatrix} | & | & | \\ \hat q_1 & \hat q_2 & \hat q_3 \\ | & | & | \end{bmatrix}}}\ \underset{n\times n}{\overset{\mathbb{R}}{\begin{bmatrix} \hat q_1^T \vec a_1 & \hat q_1^T \vec a_2 & \hat q_1^T \vec a_3 \\ 0 & \hat q_2^T \vec a_2 & \hat q_2^T \vec a_3 \\ 0 & 0 & \hat q_3^T \vec a_3 \end{bmatrix}}}$$

$\mathbb{Q}$ → $\mathbb{A}$ cleaned up

$\mathbb{R}$ ← upper triangular "combining" matrix

- $\mathbb{A} = \mathbb{Q}\mathbb{R}$ will help with $\mathbb{A}\vec x = \vec b$ (next)
- Delicious way to find $\mathbb{R}$:

$$\mathbb{Q}^T\mathbb{A} = \underbrace{\mathbb{Q}^T\mathbb{Q}}_{\mathbb{I}}\mathbb{R} \Rightarrow \mathbb{R} = \mathbb{Q}^T\mathbb{A}$$

↑ premultiply by $\mathbb{Q}^T$

$\mathbb{I}$ ← because $\mathbb{Q}$'s columns are unit vectors

Return to example:

$$\left\{\underset{\vec a_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \underset{\vec a_2}{\begin{bmatrix}4\\2\\0\end{bmatrix}}, \underset{\vec a_3}{\begin{bmatrix}5\\-4\\-7\end{bmatrix}}\right\} \Rightarrow \left\{\underset{\hat q_1}{\frac{1}{\sqrt3}\begin{bmatrix}1\\1\\1\end{bmatrix}}, \underset{\hat q_2}{\frac{1}{\sqrt2}\begin{bmatrix}1\\0\\-1\end{bmatrix}}, \underset{\hat q_3}{\frac{1}{\sqrt6}\begin{bmatrix}1\\-2\\1\end{bmatrix}}\right\}$$

$$\underset{\mathbb{A}}{\begin{bmatrix} 1 & 4 & 5 \\ 1 & 2 & -4 \\ 1 & 0 & -7 \end{bmatrix}} = \underset{\mathbb{Q}}{\begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt2} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & 0 & -\frac{2}{\sqrt6} \\ \frac{1}{\sqrt3} & -\frac{1}{\sqrt2} & \frac{1}{\sqrt6} \end{bmatrix}} \underset{\mathbb{R}}{\begin{bmatrix} \sqrt3 & 2\sqrt3 & -2\sqrt3 \\ 0 & 2\sqrt2 & 6\sqrt2 \\ 0 & 0 & \sqrt6 \end{bmatrix}}$$

check $\mathbb{A} = \mathbb{Q}\mathbb{R}$ works

check $\mathbb{R} = \mathbb{Q}^T\mathbb{A}$

Find $\mathbb{R}$ by either computing inner products $\hat q_i \vec a_j,\ i \le j$

or $\mathbb{R} = \mathbb{Q}^T\mathbb{A}$ ↳ do this!

E16dp1

$$\underset{m\times n}{\mathbb{A}}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}} \quad \& \quad \underset{m\times n}{\mathbb{A}} = \underset{m\times n}{\mathbb{Q}}\,\underset{n\times n}{\mathbb{R}}:$$

presume $r=n$

Solve the normal equation using $\mathbb{A}=\mathbb{Q}\mathbb{R}$:

$\vec{b} \notin C(\mathbb{A})$

$$\boxed{\mathbb{A}^T\mathbb{A}\vec{x}_* = \mathbb{A}^T\vec{b}}$$

$\mathbb{R}$ is upper triangular

$$\Rightarrow (\mathbb{Q}\mathbb{R})^T(\mathbb{Q}\mathbb{R})\vec{x}_* = (\mathbb{Q}\mathbb{R})^T\vec{b}$$

$$\Rightarrow \underset{n\times n}{\mathbb{R}^T}\,\underbrace{\overset{n\times m}{\mathbb{Q}^T}\,\overset{m\times n}{\mathbb{Q}}}_{\underset{n\times n}{\mathbb{I}}}\,\underset{n\times n}{\mathbb{R}}\,\underbrace{\vec{x}_*}_{n\times 1} = \underbrace{\underset{n\times n}{\mathbb{R}^T}\,\underset{n\times m}{\mathbb{Q}^T}\,\underset{m\times 1}{\vec{b}}}_{n\times 1}$$

$$\Rightarrow \underset{(\mathbb{R}^T)^{-1}\times}{\mathbb{R}^T}\,\mathbb{R}\vec{x}_* = \underset{(\mathbb{R}^T)^{-1}}{\mathbb{R}^T}\,\mathbb{Q}^T\vec{b}$$

$$\Rightarrow \mathbb{R}\vec{x}_* = \mathbb{Q}^T\vec{b} \quad \text{because } (\mathbb{R}^T)^{-1} \text{ exists}$$

square full rank all n pivots exist

$$\boxed{\underset{n\times n}{\mathbb{R}}\,\underset{n\times 1}{\vec{x}_*} = \underbrace{\mathbb{Q}^T}_{n\times m}\,\underset{m\times 1}{\vec{b}}}$$

$$\mathbb{A}'\vec{x}_* = \underset{n\times 1}{\vec{b}'}$$

upper triangular system! Easy to solve!

c.f. $\mathbb{A}=\mathbb{L}\mathbb{U}$

But

$$\mathbb{A}\vec{x}=\vec{b}$$

$$\Rightarrow (\mathbb{Q}\mathbb{R})\vec{x} = \vec{b}$$

$$\Rightarrow \mathbb{Q}^T\mathbb{Q}\mathbb{R}\vec{x} = \mathbb{Q}^T\vec{b}$$

← premultiply both sides

↑ left inverse of $\mathbb{Q}$

$$\Rightarrow \underset{n\times n}{\mathbb{I}}\,\mathbb{R}\vec{x} = \mathbb{Q}^T\vec{b}$$

$$\Rightarrow \mathbb{R}\vec{x} = \mathbb{Q}^T\vec{b}$$

↑ not $\vec{x}_*$ eek? ??

really $\mathbb{R}\vec{x}_* = \mathbb{Q}^T\vec{b}$.

We are really solving the normal equation... because $\mathbb{Q}^T\vec{b} = \mathbb{Q}^T\vec{p}$

projection of $\vec{b}$ onto $C(\mathbb{A})$

$$\mathbb{Q}^T\vec{b} = \begin{bmatrix} - & \hat{q}_1^T & - \\ - & \hat{q}_2^T & - \\ & \vdots & \\ - & \hat{q}_n^T & - \end{bmatrix}\begin{bmatrix} | \\ \vec{b} \\ | \end{bmatrix}$$

$\hat{q}_i$'s span $C(\mathbb{A})$

$$\underset{m\times m}{\mathbb{P}_{\hat{q}_i}} = \underset{m\times 1}{\hat{q}_i}\,\underset{1\times m}{\hat{q}_i^T}$$

← projection operator for direction $\hat{q}_i$

What's going on: $\mathbb{Q}^T\vec{b} = \mathbb{Q}^T(\underset{\in C(\mathbb{A})}{\vec{p}} + \underset{\in N(\mathbb{A}^T)}{\vec{e}})$

$$= \mathbb{Q}^T\vec{p} + \underbrace{\cancel{\underset{C(\mathbb{A})}{\mathbb{Q}^T}\vec{e}}}_{=\vec{0}}$$

really "orthonormal"

Orthogonal Matrices:

The Gram-Schmidt Process gave us $\underset{m\times n}{A} = \underset{m\times n}{Q}\ \underset{n\times n}{R}$ ← upper triangular combining matrix

$$[\ \ ] \qquad [\ \ ][\ ]$$

Q's columns $\{\hat q_i\}$ form an orthonormal basis for A's column space; $\hat q_i^T \hat q_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

[Note: A's columns are ideally linearly independent ($\underline{n=r}$)

More on what Q type matrices can do for you:

left inverse for Q

$$Q^T Q = I$$

$$\underbrace{\begin{bmatrix} — \hat q_1^T — \\ — \hat q_2^T — \\ \vdots \\ — \hat q_n^T — \end{bmatrix}}_{n\times m} \underbrace{\begin{bmatrix} | & | & & | \\ \hat q_1 & \hat q_2 & \cdots & \hat q_n \\ | & | & & | \end{bmatrix}}_{m\times n} = \underbrace{\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}}_{n\times n} = I$$

If Q is square, then m=n=r

So inverse exists $(N(Q) = \{\vec 0\})$

then

$$\left.\begin{aligned} Q^TQ &= I = QQ^T \\ Q^{-1}Q &= I = QQ^{-1} \end{aligned}\right\} \quad \boxed{Q^{-1} = Q^T}$$

Say Q is an orthogonal matrix

Many other groovy properties:

$$\|Q\vec x\| = \|\vec x\|$$ ← length is preserved under transformation by Q

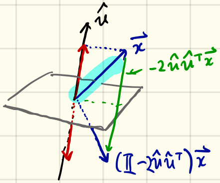
length

$$(Q\vec x)^T(Q\vec y) = \vec x^T \vec y$$ ← Q preserves angles

ex 1 $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ← rotation by θ ↺

ex 2 $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ← permutat.

ex 3 $Q = I - \underbrace{2\overbrace{\hat u \hat u^T}^{P_{\hat u}}}_{\text{project onto } \hat u \text{ and flip}}$



Three reasons to love arbitrary powers of square matrices

In our journey so far, we've spent a lot of time thinking about one of the Monks' favorite equations: $\underset{m\times n}{\mathbb{A}}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$

Now: The Monks tell us to think about $\underset{n\times n}{\text{square matrices}}$ as gadgets, things that transform vectors into new vectors

$$\boxed{\underset{n\times 1}{\vec{x}'} = \overset{\checkmark}{\underset{n\times n}{\mathbb{A}}}\,\overset{\checkmark}{\underset{n\times 1}{\vec{x}}}}$$

$$\mathbb{A} \text{ might } \left\{\begin{matrix}\text{flip}\\ \text{rotate}\\ \text{stretch}\\ \text{project}\end{matrix}\right\} \vec{x}$$

Big Question: what happens if we use $\mathbb{A}$ to repeatedly transform a vector?

E17ap1

Start with $\vec{x}_0$

$\vec{x}_1 = \mathbb{A}\vec{x}_0$, $\vec{x}_2 = \mathbb{A}\vec{x}_1$, ..., $\vec{x}_k = \mathbb{A}\vec{x}_{k-1}$, ...

$$\Rightarrow \boxed{\vec{x}_k = \mathbb{A}^k \vec{x}_0}$$

$$\underset{n\times n}{\mathbb{A}^k} = \overbrace{\underset{n\times n}{\mathbb{A}} \cdot \underset{n\times n}{\mathbb{A}} \times \cdots \times \underset{n\times n}{\mathbb{A}}}^{k}$$

k th power of A

Difficulty: Mindless multiplication of many matrices works but is

(1) computationally expensive;

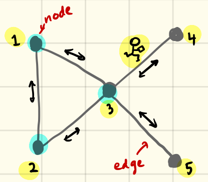
(2) doesn't give us any understanding of $\underbrace{\text{how } \mathbb{A}^k \text{ behaves}}_{\text{deep story}}$

The Monks whisper that we must understand eigenthings...

vast and wonderful

But first: three example areas showing the excellence of $\underset{n\times n}{\mathbb{A}^k}$....

(1) The distracted texter wandering randomly on a network:



columns must sum to 1

$$\underbrace{\begin{bmatrix} P_{t+1,1} \\ P_{t+1,2} \\ P_{t+1,3} \\ P_{t+1,4} \\ P_{t+1,5} \end{bmatrix}}_{\vec{P}_{t+1}} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 & 1 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \end{bmatrix}}_{\mathbb{A}} \underbrace{\begin{bmatrix} P_{t,1} \\ P_{t,2} \\ P_{t,3} \\ P_{t,4} \\ P_{t,5} \end{bmatrix}}_{\vec{P}_t}$$

probability texter is at nodes 1,..,5 at time t+1

transition matrix

Natural question:

where is our texter likely to be as time goes on?

or, what is $\vec{P}_\infty$?

or what is $\mathbb{A}^k$ as $k \to \infty$?

Monks (and soon you) tell us that

eigenvalue (later)

$$\mathbb{A} \begin{bmatrix} 2 \\ 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \vec{P}_\infty = \frac{1}{10} \begin{bmatrix} 2 \\ 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

no change

eigenvector

normalized

great result: $P_{\infty,i}$ is proportional to the degree of node i

---

E17ap2

(2) Solving coupled linear differential equations

simple $$\frac{dx}{dt} = 3x \Rightarrow x(t) = x(0)e^{3t}$$

initial value at $t=0$

check this works

Coupled
$$\frac{dx_1}{dt} = 2x_1 - x_2$$
$$\frac{dx_2}{dt} = 3x_1 + 2x_2$$

change depends on current position

continuous & discrete math crushing it together

Rewrite with matrices:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \frac{d\vec{x}}{dt} = \underset{2\times 2}{\mathbb{A}}\vec{x}$$

Solution is of the form: $$\underset{2\times1}{\vec{x}(t)} = \underbrace{e^{\mathbb{A}t}}_{2\times 2} \underset{2\times1}{\vec{x}(0)}$$

what??

It's true!:

you can exponentiate matrices!!

Taylor expansion

$$e^{\mathbb{A}} = \mathbb{I} + \mathbb{A} + \frac{1}{2!}\mathbb{A}^2 + \frac{1}{3!}\mathbb{A}^3 + \dots + \frac{1}{k!}\mathbb{A}^k + \dots$$

we need to really understand all powers of A to cope with this...

scalar

$$e^{\mathbb{A}t} = \mathbb{I} + \mathbb{A}t + \frac{1}{2!}\mathbb{A}^2t^2 + \frac{1}{3!}\mathbb{A}^3t^3 + \dots$$

$$e^{\varpi} = \mathbb{I}_{\varpi} + \varpi + \frac{1}{2!}\varpi^2 + \frac{1}{3!}\varpi^3 + \dots$$

#awesome

(3) Solving super fun difference equations

ex $F_{k+2} = F_{k+1} + F_k$ with $\overbrace{F_0 = F_1 = 1}^{\text{initial conditions}}$

↖Fibonacci sequence

Monks say try:

$$\vec{f}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

↖ stack consecutive Fibonacci numbers

$$\vec{f}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{unbelievable matrix...}} \underbrace{\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}}_{\vec{f}_{k-1}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \vec{f}_{k-2}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \vec{f}_0$$

So if we can compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k$ for all k in a clever way, we'll have a formula for the Fibonacci numbers.

↘ later.

E17ap3

Again, understanding and calculating $\mathbb{A}^k$ is made possible through the magic of eigenthings

values

vectors

spaces

functions

The Magic of Eigenthings

an introduction to happiness

Scene:

A Monk hands us a parchment with

$$\mathbb{A} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and other strange symbols written on it, smiles, and then mysteriously disappears...

note $\vec{v}_1 \perp \vec{v}_2$ ....

Let's try some things...

2×2

$$\mathbb{A}\vec{v}_1 = \underbrace{\begin{bmatrix} 1 & 1/2 \\ \frac{1}{2} & 1 \end{bmatrix}}_{\mathbb{A}} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}_1} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{3}{2}\vec{v}_1$$

symmetric

only direction matters

$\vec{v}_1$ comes back with a 3/2 stretch factor

$$\mathbb{A}^2\vec{v}_1 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{3}{2}\right)^2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{3}{2}\right)^2\vec{v}_1$$

$$\mathbb{A}^k\vec{v}_1 = \begin{bmatrix} 1 & 1/2 \\ \frac{1}{2} & 1 \end{bmatrix}^k\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{3}{2}\right)^k\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{3}{2}\right)^k\vec{v}_1$$

grows

$\mathbb{A}$ "likes" the direction of $\vec{v}_1$

↳ "eigenvector"

German for "own"

And we'll call $\lambda_1 = 3/2$ the "eigenvalue" associated with $\vec{v}_1$

E17bp1

$$\mathbb{A}\vec{v}_2 = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}\vec{v}_2$$

$\vec{v}_2$ is also an eigenvector of $\mathbb{A}$

$\lambda_2 = \frac{1}{2}$ is the associated eigenvalue

- Note again: only direction of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ matters

$$\mathbb{A}\underbrace{\begin{bmatrix} 4 \\ 4 \end{bmatrix}}_{4\vec{v}_1} = \frac{3}{2}\begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad \& \quad \mathbb{A}\underbrace{\begin{bmatrix} 17 \\ -17 \end{bmatrix}}_{17\vec{v}_2} = \frac{1}{2}\begin{bmatrix} 17 \\ -17 \end{bmatrix}$$

- Also: $\mathbb{A}^k\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \left(\frac{1}{2}\right)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

vanishes

One more thing:

$$\underbrace{\mathbb{A}\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\text{not an eigenvector}} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

different direct

$$\mathbb{A}^2\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \mathbb{A}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 2 \end{bmatrix}$$

different again...

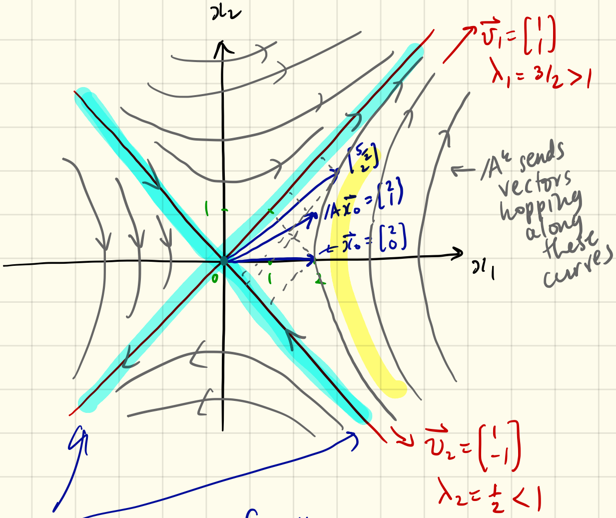
Better:

$$\mathbb{A}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \mathbb{A}\left(\overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}\right) = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A is simple in eigenvector basis

huge: $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ is a great basis for $\mathbb{A}$

picture for A^k:



Eigenspaces of A
(1-d subspaces of $\mathbb{R}^2$)

possibilities for $A\vec{v} = \lambda\vec{v}$

$\lambda > 1$: growth

$\lambda = 1$: stays the same

$0 < \lambda < 1$: shrinkage

$\lambda < 0$: jumping back and forth across origin, $|\lambda|$ governs growth

λ complex: rotation

Big question: If monks aren't around, how do we find $\vec{v}$'s and λ's? How many are possible if A is $n \times n$?

Game is to solve the Eigenvalue Equation:

$$\boxed{\underset{n\times n}{\overset{\checkmark}{A}}\ \underset{n\times 1}{\overset{?}{\vec{v}}} = \underset{\text{scalar}}{\overset{?}{\lambda}}\ \underset{n\times 1}{\overset{?}{\vec{v}}}}$$

$$\underset{\checkmark}{A}\,\underset{?}{\vec{x}} = \underset{\checkmark}{\vec{b}}$$

$$A\vec{v} - \lambda\vec{v} = \underset{n\times 1}{\overset{\text{vector!!}}{\vec{0}}}$$

sneakiness (Monks)

$$\underset{n\times n}{A}\ \underset{n\times 1}{\vec{v}} - \underset{\text{scalar}}{\lambda}\ \underset{n\times n}{I}\ \overset{n\times 1}{\vec{v}} = \underset{n\times 1}{\vec{0}}$$

temptation

shame

$$\cancel{(\underset{}{\overset{n\times n}{A}} - \overset{1\times 1}{\lambda})}\,\vec{v} = \vec{0}$$

no!!

$$\boxed{(\underbrace{A - \lambda I}_{n\times n})\ \underset{n\times 1}{\vec{v}} = \underset{n\times 1}{\vec{0}}}$$

Nullspace Equation!!

λ ← lambda

λ must be such that:

- rank$(A - \lambda I) < n$
- $N(A - \lambda I) \neq \{\vec{0}\}$
- $A - \lambda I$ has no inverse (A is singular)
- $\det(A - \lambda I) = |A - \lambda I| \equiv 0$

new thing

Solve $(\mathbb{A} - \lambda \mathbb{I})\vec{v} = \vec{0}$ for $\mathbb{A} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

$\lambda\mathbb{I} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

Usual way:

$$\begin{bmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

subtracting λ from diagonal entries of $\mathbb{A}$

Augmented matrix

$$\left[\begin{array}{cc|c} 1-\lambda & \frac{1}{2} & 0 \\ \frac{1}{2} & 1-\lambda & 0 \end{array}\right] \underset{R_2' = R_2 - \left(\frac{1/2}{1-\lambda}\right)R_1}{\sim} \left[\begin{array}{cc|c} 1-\lambda & \frac{1}{2} & 0 \\ 0 & 1-\lambda - \frac{(\frac{1}{2})^2}{1-\lambda} & 0 \end{array}\right]$$

eek!!

See $(1-\lambda) - \frac{(\frac{1}{2})^2}{1-\lambda} = 0$

for $r = 1$, $N(\mathbb{A} - \lambda\mathbb{I}) \neq \{\vec{0}\}$

rank

$\Rightarrow \vec{v}$ is healthy

$(1-\lambda)^2 = (\frac{1}{2})^2$

$\Rightarrow 1 - \lambda = \pm \frac{1}{2}$

$\Rightarrow \lambda = 1 \pm \frac{1}{2}$

$\lambda_1 = 3/2$, $\lambda_2 = \frac{1}{2}$

just as we found...

$\Rightarrow$ next step: find $\vec{v}$ as nullspace vectors for $\mathbb{A} - \frac{3}{2}\mathbb{I}$ & $\mathbb{A} - \frac{1}{2}\mathbb{I}$

really: basis vectors

Unfortunately, preceding is a very messy way to handle $(\mathbb{A} - \lambda\mathbb{I})\vec{v} = \vec{0}$... There's a better, more illuminating way.

=

Set $\det(\mathbb{A} - \lambda\mathbb{I}) = 0$ and find λ... ← reason is coming...

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ for 2×2s

$$\begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (\tfrac{1}{2})^2 = 0$$

same equation as before

- see next episodes for all things determinants
- we return to eigenthings after this strange excursion

$\lambda_1 = 3/2$, $\lambda_2 = 1/2$

Find $\vec{v}_1$ & $\vec{v}_2$

$\lambda_1 = 3/2$:

$$(A - \tfrac{3}{2}I)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ ← standard Nullspace Equation

$$\Downarrow$$

$$\left[\begin{array}{cc|c} 1-3/2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1-3/2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right] \underset{R_2' = R_2 - \left(\frac{+\frac{1}{2}}{-\frac{1}{2}}\right)R_1}{\sim} \left[\begin{array}{cc|c} P & F & \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-\tfrac{1}{2}v_1 + \tfrac{1}{2}v_2 = 0$$

$$\Rightarrow v_1 = v_2$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = v_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ ← Eigenspace for $\lambda_1 = 3/2$

Say $\lambda_1 = 3/2$ with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

↑ basis for eigenspace

$\lambda_2 = \frac{1}{2}$:

$$(A - \tfrac{1}{2}I)\vec{v}_2 = \vec{0}$$

$$\left[\begin{array}{cc|c} 1-\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1-\frac{1}{2} & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} P & F & \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \vec{v}_2 = c\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad c \in \mathbb{R}$$ ← Eigenspace

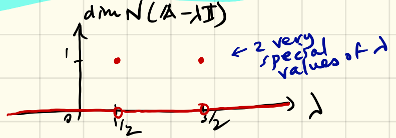
$\lambda_1 = \frac{1}{2}$ has eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Often, unit vectors are best

$$\hat{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \& \quad \hat{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_1 = 3/2$ $\lambda_2 = 1/2$

↗ A's natural basis



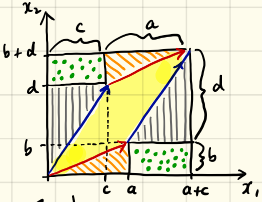
E18ap1

Determinants from the ground up:

$\mathbb{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

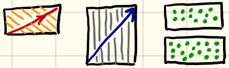
2x2's first then nxn's

idea: Consider area of parallelogram formed by row vectors of $\mathbb{A}$: $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$



Area of  $= (a+c)(b+d)$

$- ab \quad - dc \quad - 2bc$



$= ab + ad + cb + cd$
$\quad - ab - dc - 2bc$

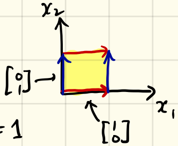
$= ad - bc \longrightarrow$ call this $\mathbb{A}$'s determinant: $\det(\mathbb{A})$ or $|\mathbb{A}|$

Three Observations about this determinant thing for 2x2's:

① $|\mathbb{I}| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

anchor

area = 1



② If we swap $\mathbb{A}$'s rows, we flip the sign of the determinant

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ but $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$
$= -(ad - bc)$

straight lines for determinant

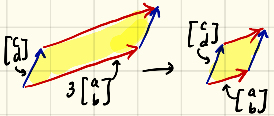
-ve area indicates ordering of vectors

③ $|\mathbb{A}| = ad - bc$ is multilinear in the rows of $\mathbb{A}$

Two pieces:

3.1 Area Scales:

ex $\begin{vmatrix} 3a & 3b \\ c & d \end{vmatrix} = 3 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$



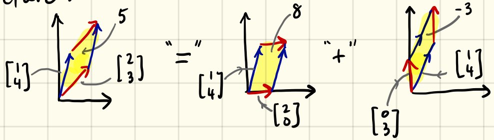
ex $\begin{vmatrix} 2a & 2b \\ 4c & 4d \end{vmatrix} = 2 \cdot 4 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

3.2 Areas add when single rows add:

ex $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix}$

formula: $\underbrace{2 \cdot 4 - 3 \cdot 1}_{5} = (\underbrace{2 \cdot 4 - 1 \cdot 0}_{8}) + (\underbrace{0 \cdot 4 - 1 \cdot 3}_{-3})$

pictures:



Determinants from the ground up:

The plan: We assert Three Properties for Determinants of nxn matrices

① $|\mathbb{I}| = 1.$ ← nxn unit hypercube

[Determinant = ± volume of parallelipiped created by row vectors of $\mathbb{A}$

② Swapping any two rows of $\mathbb{A}$ changes the sign of the determinant.

③ Determinants are multilinear in their rows.

Big Deal: Can now connect $|\mathbb{A}|$ to $|\mathbb{I}| = 1$ and many, many good things will follow

---

ok: Let's fully connect $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$ to $|\mathbb{I}|$

$$|\mathbb{A}| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix}$$

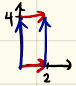
← linear in row 1 ← ③

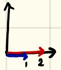
need to introduce 0's to get to $\mathbb{I}$

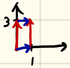
$$= \overbrace{\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix}} + \overbrace{\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix}}$$

linear in row 2 ← ③

linear in row 2 ← ③





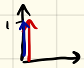


$$= 2 \cdot 4 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + 3 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 3.4 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$$

↑ 0 volume

↑ 0 volume

row swap



$$= 8 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 3(-1) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= (8-3) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 5\,|\mathbb{I}| = 5$$

Next: Show how our standard row operations lead to many results ← $\mathbb{P}\mathbb{A} = \mathbb{L}\mathbb{U}$ including $|\mathbb{A}| = \pm|\mathbb{U}|$

& $|\mathbb{A}\,\mathbb{B}| = |\mathbb{A}|\,|\mathbb{B}|$

#excitement

[Many results for determinants based on three properties:

① $|\underset{n\times n}{\mathbb{I}}|=1$, ② Row swap $\xrightarrow[\text{sign flip}]{}$ ×(−1), ③ Multilinearity (rows)]

[notation $\det \mathbb{A} = |\mathbb{A}|$]

(a) $|t\overset{n\times n}{\mathbb{A}}| = t^n|\mathbb{A}|$

$t \in \mathbb{R}$ — multilinearity

$$t\begin{bmatrix}3 & 4\\ 2 & 1\end{bmatrix} = \begin{bmatrix}3t & 4t\\ 2t & t\end{bmatrix}$$

(b) If two of $\mathbb{A}$'s rows are the same, then $|\mathbb{A}|=0$.

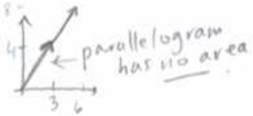
ex
$$\begin{vmatrix}3 & 4\\ 3 & 4\end{vmatrix} \underset{R_2\leftrightarrow R_1}{=} -\begin{vmatrix}3 & 4\\ 3 & 4\end{vmatrix} \quad \text{so} \quad \begin{vmatrix}3 & 4\\ 3 & 4\end{vmatrix} = 0$$

property ②

ex
$$\begin{vmatrix}3 & 6 & 6\\ 3 & 6 & 6\\ 2 & -2 & 1\end{vmatrix} \underset{R_2\leftrightarrow R_1}{=} -\begin{vmatrix}3 & 6 & 6\\ 3 & 6 & 6\\ 2 & -2 & 1\end{vmatrix}$$

property ②

so must be zilch



(c) Performing a step of standard row reduction doesn't change the value of the determinant

\#bigdeal

$$R_i' = 1R_i - (l_{ij})R_j$$

/E186p1

Row reduction for a 2×2:

$$\mathbb{A} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \underset{R_2' = R_2 - l_{21}R_1}{\sim} \begin{bmatrix}a_{11} & a_{12}\\ a_{21}-l_{21}a_{11} & a_{22}-l_{21}a_{12}\end{bmatrix}$$

tilde

zero this out.

$\left(\frac{a_{21}}{a_{11}}\right)$

now find determinant of this new matrix

$$\begin{vmatrix}a_{11} & a_{12}\\ a_{21}-l_{21}a_{11} & a_{22}-l_{21}a_{22}\end{vmatrix} =$$

multilinearity

$$\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} - l_{21}\begin{vmatrix}a_{11} & a_{12}\\ a_{11} & a_{12}\end{vmatrix}$$

0

original matrix.

same rows (b)

- Above generalizes to n×n
- "∼" for solving $\mathbb{A}\vec{x}=\vec{b}$ ⇒ "=" for determinants

(d) If $\underset{n\times n}{\mathbb{A}}$'s rows are linearly dependent, then $|\mathbb{A}|=0$

\#bigdeal

Reason:

* We can use row row ops to make one or more rows of zeros

* Now add one non-zero row to a zero row using one more row op

⇒ 2 rows the same (b) ⇒ $|\mathbb{A}|=0$

(d) cont.

Or multiply zero row by any number c

$\Rightarrow$ Multilinearity means determinant should scale by a factor of c

$\Rightarrow$ But row was unchanged $c\vec{0}^T = \vec{0}^T$

so $|\mathbb{A}| = 0$.

ex $$\begin{vmatrix} 2 & -2 & 1 \\ 3 & 6 & 6 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -2 & 1 \\ 3 & 6 & 6 \\ c[0 & 0 & 0] \end{vmatrix} = c\begin{vmatrix} 2 & -2 & 1 \\ 3 & 6 & 6 \\ 0 & 0 & 0 \end{vmatrix}$$

only possible if $\begin{vmatrix} 2 & -2 & 1 \\ 3 & 6 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$

Connections:

Applies if $\mathbb{A}$ is square only

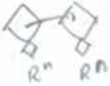
$\underset{n\times n}{|\mathbb{A}|} = 0$

$\Leftrightarrow$ $\mathbb{A}$'s rows are linearly dependent

$\Leftrightarrow$ Rank of $\mathbb{A}$, r, is less than n

$\Leftrightarrow$ $N(\mathbb{A})$ & $N(\mathbb{A}^T)$ have dim $n - r \geq 1$

$\Leftrightarrow$ $\mathbb{A}$ has no inverse



$\Leftrightarrow$ If $\mathbb{A}\vec{x} = \vec{b}$ has a solution, then there are ∞ly many solutions

$\Leftrightarrow$ One or more sides of $\mathbb{A}$'s parallelipiped have 0 length.

Example calculation of $|\mathbb{A}|$ using row ops:

$$|\mathbb{A}| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{vmatrix}$$

freestyling

$$\underset{R_2 \Leftrightarrow R_3}{=} -\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix}$$

$$\underset{R_2' = R_2 - (\frac{2}{1})R_1}{=} -\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 3 & 2 \end{vmatrix}$$

$$\underset{R_3' = R_3 - (\frac{3}{-1})R_2}{=} -\begin{vmatrix} \overset{d_1}{1} & 1 & 1 \\ 0 & \overset{d_2}{-1} & 0 \\ 0 & 0 & \boxed{2}_{d_3} \end{vmatrix}$$

pivots!

$\mathbb{U}$ from $\mathbb{P}\mathbb{A} = \mathbb{L}\mathbb{U}$

$$\underset{\substack{R_1' = R_1 - (\frac{-1}{1})R_2 \\ R_1'' = R_1' - (\frac{1}{2})R_3}}{=} -\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = \underbrace{-(1)(-1)(2)}_{2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\mathbb{R}^{echelon}_{\mathbb{A}}$ new thing

$\mathbb{R}_{\mathbb{A}} = \mathbb{I}$

$$= 2\,|\mathbb{I}| = 2 \times 1 = 2$$

$\Leftrightarrow$ one or more pivots $= 0$

(e) $|\mathbb{A}| = \pm |\mathbb{U}|$ ← as in $\mathbb{P}\mathbb{A} = \mathbb{L}\mathbb{U}$

↑ depends on number of row swaps

↑ permutation

(f) $|\mathbb{A}| = \pm \prod_{i=1}^{n} d_i$ ← pivots of $\mathbb{A}$

follows from row op goodness (c)

if one or more pivots = 0

$\Rightarrow |\mathbb{A}| = 0.$

(g) $\underset{n\times n}{|\mathbb{A}} \underset{n\times n}{\mathbb{B}|} = \underset{n\times n}{|\mathbb{A}|}\,\underset{n\times n}{|\mathbb{B}|}$ ← #bigdeal

Various proofs exist.

Monks say use row reduction:

Know $\mathbb{E}\,\mathbb{P}\,\mathbb{A} = \mathbb{R}^{e}_{\mathbb{A}} = \mathbb{D}_{\mathbb{A}}$ ← reduced row echelon form with pivots still in

↑ all elimination matrices

↑ permutation matrix

↑ pivot matrix for $\mathbb{A}$

(first presume all pivots ≠ 0).

ok $|\mathbb{A}\,\mathbb{B}|$

$$= \pm|\mathbb{E}\,\mathbb{P}\,\mathbb{A}\,\mathbb{B}| = \pm|\underline{\mathbb{D}_{\mathbb{A}}}\,\mathbb{B}|$$

det unchanged with row ops

↑ depends on $\mathbb{P}$

$$= \pm\left|\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}\begin{bmatrix} — \vec{b}_{1*} — \\ — \vec{b}_{2*} — \\ \vdots \\ — \vec{b}_{n*} — \end{bmatrix}\right|$$

$$= \pm\left|\begin{matrix} — d_1\vec{b}_{1*} — \\ — d_2\vec{b}_{2*} — \\ \vdots \\ — d_n\vec{b}_{n*} — \end{matrix}\right|$$

$$= \pm\left(\prod_{i=1}^{n} d_i\right)|\mathbb{B}| = |\mathbb{A}|\,|\mathbb{B}|.$$

↑ multilinearity

Now if one or more pivots = 0, ← $|\mathbb{A}| = 0$
can see same row reductions leads to
row of 0's $\Rightarrow |\mathbb{A}\mathbb{B}| = 0$ ✓

(h) $|\mathbb{A}^{-1}| = \dfrac{1}{|\mathbb{A}|}$ ← from (g)

reason $|\mathbb{A}\mathbb{A}^{-1}| = |\mathbb{A}|\,|\mathbb{A}^{-1}|$

$\overset{\shortparallel}{|\mathbb{I}|} = 1$

Note $|\mathbb{A}| = 0 \Rightarrow |\mathbb{A}^{-1}| = \infty$ ← ouch!

(i) If $\mathbb{A}$ is upper or lower triangular then $|\mathbb{A}|$ = product of entries on $\mathbb{A}$'s main diagonal.

Reason: row reduction on a triangular matrix requires no row swaps and does not change entries on main diag.
Plus: zero leads to a zero row $\Rightarrow |\mathbb{A}| = 0$.

ex

$$\begin{vmatrix} 4 & 77 & 16 \\ 0 & -3 & 17 \\ 0 & 0 & 2 \end{vmatrix} = (4)(-3)(2) = -24 \; ; \; \begin{vmatrix} 4 & 0 & 0 \\ 77 & 1 & 0 \\ 13 & 99 & 2 \end{vmatrix} = (4)(1)(2) = 8.$$

$$\begin{vmatrix} 4 & 77 & 16 \\ 0 & 0 & 17 \\ 0 & 0 & 2 \end{vmatrix} = (4)(0)(2) = 0.$$

last

(j) $$\boxed{|\mathbb{A}| = |\mathbb{A}^T|}$$ ← #groovy ← #bigdeal

Means: can use "column ops" in the same way as row ops
& all results for rows work for columns too.

Reason

Use monks

$$\underset{n\times n}{\mathbb{P}}\,\underset{n\times n}{\mathbb{A}} = \underset{n\times n}{\mathbb{L}}\,\underset{n\times n}{\mathbb{U}} \longrightarrow \boxed{|\mathbb{P}|\,|\mathbb{A}| = |\mathbb{L}|\,|\mathbb{U}|}$$

G*

E18bp4

$$(\mathbb{P}\mathbb{A})^T = (\mathbb{L}\mathbb{U})^T$$

$$\mathbb{A}^T\mathbb{P}^T = \mathbb{U}^T\mathbb{L}^T$$

Take determinants of both sides

$$|\mathbb{A}^T|\,|\mathbb{P}^T| = |\mathbb{U}^T|\,|\mathbb{L}^T|$$

triangular determinant is unchanged by transpose

$$|\mathbb{P}^T|\,|\mathbb{A}^T| = |\mathbb{L}|\,|\mathbb{U}|$$

↖ handle this

$\mathbb{P}$ is a permutation matrix, a shuffling of the identity matrix.

$\Rightarrow |\mathbb{P}| = \pm 1$ ↖ # row swaps

Also know

$$\mathbb{P}^{-1} = \mathbb{P}^T \text{ so } |\mathbb{P}^T\mathbb{P}| = |\mathbb{I}| = 1.$$

$$\overset{(g)}{=} |\mathbb{P}^T|\,|\mathbb{P}|$$

$\Rightarrow$ either $|\mathbb{P}| = |\mathbb{P}^T| = 1$, or $|\mathbb{P}| = |\mathbb{P}^T| = -1$ ← they match.

G*

$$\Rightarrow \boxed{|\mathbb{P}|\,|\mathbb{A}^T| = |\mathbb{L}|\,|\mathbb{U}|} \Rightarrow |\mathbb{A}^T| = |\mathbb{A}|.$$

Computing determinants: The way of the cofactor

- Recipe first, understanding later.
- Need a clean way to find determinants for eigenvalue problem $\mathbb{A}\vec{v} = \lambda\vec{v}$
- Row operations helped us with results about determinants but are messy...

The story:

- $n \times n$ determinants are sums of $\underline{n}$ $(n-1)\times(n-1)$ determinants
- 3×3 determinants are sums of 3 2×2 determinants. recursive.

Example to work with:

$$\mathbb{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}.$$

Defn: $\mathbb{M}_{ij}$ is the $(n-1)\times(n-1)$ matrix formed by deleting the i^{th} row and j^{th} column of $\mathbb{A}$; there are n^2 of these **"minor matrices"**

exs

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \quad \mathbb{M}_{11} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & \textcircled{1}\cdots \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \quad \mathbb{M}_{13} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \quad \mathbb{M}_{\underline{22}} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

← $\underset{3\times3}{\mathbb{A}}$ has $3\times3=9$ minor matrices

Defn:

$|\mathbb{M}_{ij}|$ ← det is the ij^{th} **minor** of $\mathbb{A}$

Defn:

$C_{ij} = (-1)^{i+j}|\mathbb{M}_{ij}|$ is the ij^{th} **cofactor** of $\mathbb{A}$

$(-1)^{i+j} \Rightarrow$ checkerboard of +'s & −'s

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \\ + & - & + & - & \\ - & + & - & + & \\ \vdots & & & & \ddots \end{bmatrix}$$

Theorem:

The Determinant of $\mathbb{A}$ is given by the dot product of $\mathbb{A}$'s cofactors and $\mathbb{A}$'s entries along <u>any</u> <u>one</u> row or column.

ex Using row 1

$$|\mathbb{A}| \overset{!}{=} \sum_{j=1}^{n} C_{1j} a_{1j}$$

$$\underset{\text{column 2}}{=} \sum_{i=1}^{n} C_{i2} a_{i2}$$

#crazytown bananapants

Ex $\mathbb{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$

Let's first try row 1

$\Rightarrow$ Compute C_{11}, C_{12}, C_{13} using $C_{ij} = (-1)^{i+j} |\mathbb{M}_{ij}|$

$\mathbb{M}_{11} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$; $|\mathbb{M}_{11}| = 3\cdot2 - 2\cdot1 = 4$; $C_{11} = (-1)^{1+1}\cdot 4 = 4$

$\mathbb{M}_{12} = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$; $|\mathbb{M}_{12}| = 0\cdot2 - 2\cdot2 = -4$; $C_{12} = (-1)^{1+2}\cdot(-4) = 4$

$\mathbb{M}_{13} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$; $|\mathbb{M}_{13}| = 0\cdot1 - 3\cdot2 = -6$; $C_{13} = (-1)^{1+3}(-6) = -6$

E18cp2

D8

$$|\mathbb{A}| = C_{11}a_{11} + C_{12}a_{12} + C_{13}a_{13} \quad \leftarrow \text{dot product}$$

$$= (4)(1) + (4)(1) + (-6)(1) = 2 \quad // \quad \text{(as for row ops)}$$

Now We can choose any row or column so let's do all <u>of them at</u> once. #crazy

Create cofactor matrix $\mathbb{C}$:

$$\mathbb{C} = \begin{bmatrix} 4 & 4 & -6 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix} \begin{matrix} \leftarrow \text{first row as above} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{exercise} \end{matrix}$$

$$\underbrace{[a_{ij} C_{ij}]}_{\text{direct product of elements}} = \begin{bmatrix} 1\times4 & 1\times4 & 1\times(-6) \\ 0\times(-1) & 3\times0 & 2\times1 \\ 2\times(-1) & 1\times(-2) & 2\times3 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -6 \\ 0 & 0 & 2 \\ -2 & -2 & 6 \end{bmatrix}$$

Magic:

$$\begin{array}{c|ccc|c} & & & & \text{row sums} \\ & 4 & 4 & -6 & ② \text{ as above} \\ & 0 & 0 & 2 & ② \\ & -2 & -2 & 6 & ② \\ \text{column sums} \rightarrow & ② & ② & ② & \end{array}$$

#inconceivable

E18cp3

excellent.

Cofactor method enables sneakiness:

Choose row or column with most zeros:

ex

$$\left|\begin{matrix}1&0&0\\0&2&-1\\2&1&3\end{matrix}\right| = \underset{a_{11}}{1}\,\underbrace{(-1)^{1+1}\left|\begin{matrix}2&-1\\1&3\end{matrix}\right|}_{C_{11}} = 1\cdot1\cdot(6+1)=7$$

← best choice

$a_{12}=a_{13}=0$

ex row 2 col 3

$$\left|\begin{matrix}2&7&0\\0&3&2\\-1&4&0\end{matrix}\right| = \underset{a_{23}}{2}\,\underbrace{(-1)^{2+3}\left|\begin{matrix}2&7\\-1&4\end{matrix}\right|}_{C_{23}} = 2\cdot(-1)\cdot(8+7) = -30.$$

No need to compute cofactors associated with 0's in A.

avoid trauma

//

Fun:

$$\left|\begin{matrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{matrix}\right|$$

← go along top row

$$= a_{11}\underbrace{(-1)^{1+1}|a_{22}|}_{C_{11}} + a_{12}\underbrace{(-1)^{1+2}|a_{21}|}_{C_{12}}$$

det of a 1×1 = length

$$= a_{11}a_{22} - a_{12}a_{21} \checkmark$$

everything works

One more example: 4×4

$$\left|\begin{matrix}2&2&3&-1\\0&7&0&0\\-1&2&1&0\\0&4&0&3\end{matrix}\right|$$

← most 0's

$$= \underset{a_{22}}{7}\cdot(-1)^{2+2}\left|\begin{matrix}2&3&-1\\-1&1&0\\0&0&3\end{matrix}\right|$$

← most 0's

$$= 7\cdot\left(3\cdot(-1)^{3+3}\left|\begin{matrix}2&3\\-1&1\end{matrix}\right|\right)$$

$$= 7\cdot3\,(\underbrace{2+3}_{5}) = 7\cdot3\cdot5 = 105$$

* Starting with, say, row 1 would have really hurt...

Determinants & $\underset{n\times n}{\mathbb{A}}\vec{x}=\vec{b}$
Cramer's rule and a formula for the inverse of $\mathbb{A}$

#inconceivable

Monks say try this for 3×3 matrices:

$$\mathbb{A}\begin{bmatrix}|&0&0\\ \vec{x}&1&0\\ |&0&1\end{bmatrix} = \begin{bmatrix}| & & \\ \mathbb{A}\vec{x} & \mathbb{A}\begin{bmatrix}0\\1\\0\end{bmatrix} & \mathbb{A}\begin{bmatrix}0\\0\\1\end{bmatrix}\\ | & & \end{bmatrix}$$

$\mathbb{I}$ with $\vec{x}$ in first column

$$= \begin{bmatrix}|&|&|\\ \vec{b}&\vec{a}_2&\vec{a}_3\\ |&|&|\end{bmatrix} = \mathbb{B}_1$$

call this

$\mathbb{A}$ with first column replaced by $\vec{b}$.

Similarly

$$\mathbb{A}\begin{bmatrix}1&|&0\\0&\vec{x}&0\\0&|&1\end{bmatrix} = \underbrace{\begin{bmatrix}|&|&|\\ \vec{a}_1&\vec{b}&\vec{a}_3\\ |&|&|\end{bmatrix}}_{\mathbb{B}_2};\quad \mathbb{A}\begin{bmatrix}1&0&|\\0&1&\vec{x}\\0&0&|\end{bmatrix}=\underbrace{\begin{bmatrix}|&|&|\\ \vec{a}_1&\vec{a}_2&\vec{b}\\|&|&|\end{bmatrix}}_{\mathbb{B}_3}$$

Monks whisper "take determinants"....

$$|\mathbb{A}|\begin{vmatrix}|&0&0\\ \vec{x}&1&0\\ |&0&1\end{vmatrix} = |\mathbb{B}_1|$$

$= x_1$ (use row reduction on transpose)

$$\Rightarrow x_1 = \frac{|\mathbb{B}_1|}{|\mathbb{A}|},\quad x_2 = \frac{|\mathbb{B}_2|}{|\mathbb{A}|},\quad x_3=\frac{|\mathbb{B}_3|}{|\mathbb{A}|}$$

wait!

we just solved $\underset{n\times n}{\mathbb{A}}\vec{x}=\vec{b}$!!

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = \frac{1}{|\mathbb{A}|}\begin{bmatrix}|\mathbb{B}_1|\\|\mathbb{B}_2|\\\vdots\\|\mathbb{B}_n|\end{bmatrix} \quad !!!$$

Problems: ① only works for n×n ← ok for normal equations

② computing determinants is horribly slow

③ must recompute for new $\vec{b}$

Main utility: theoretical

E18dp1

Let's use Cramer's rule to find $\mathbb{A}^{-1}$:

Monks say solve these special $\mathbb{A}\vec{x}=\vec{b}$ problems:

$$\mathbb{A}\vec{x}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \leftarrow \vec{b}$$

$$\mathbb{A}\vec{x}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\mathbb{A}\vec{x}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

← generalizes to n×n

$$\Rightarrow \mathbb{A}\begin{bmatrix}| & | & |\\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3\\ | & | & |\end{bmatrix} \leftarrow \text{so must have } \mathbb{A}^{-1} \text{ here}$$

$$= \begin{bmatrix}| & | & |\\ \mathbb{A}\vec{x}_1 & \mathbb{A}\vec{x}_2 & \mathbb{A}\vec{x}_3\\ | & | & |\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \mathbb{I}.$$

Use Cramer's rule and work with example $\mathbb{A} = \begin{bmatrix}1&1&1\\0&3&2\\2&1&2\end{bmatrix}$:

Solve $\mathbb{A}\vec{x} = \overset{\vec{b}}{\begin{bmatrix}1\\0\\0\end{bmatrix}}$ first with $\vec{x} = \frac{1}{|\mathbb{A}|}\begin{bmatrix}|\mathbb{B}_1|\\|\mathbb{B}_2|\\|\mathbb{B}_3|\end{bmatrix}$

$$\mathbb{B}_1 = \begin{bmatrix}1&1&1\\0&3&2\\0&1&2\end{bmatrix}, \quad \mathbb{B}_2 = \begin{bmatrix}1&1&1\\0&0&2\\2&0&2\end{bmatrix}, \quad \mathbb{B}_3 = \begin{bmatrix}1&1&1\\0&3&0\\2&1&0\end{bmatrix}$$

$\uparrow \mathbb{M}_{11}$ $\quad$ $\uparrow \mathbb{M}_{12}$ $\quad$ $\uparrow \mathbb{M}_{13}$

$|\mathbb{B}_1| = C_{11}$ ← cofactors of $\mathbb{A}$! → $|\mathbb{B}_2| = C_{12}$ $\quad$ $|\mathbb{B}_3| = C_{13}$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}: \quad \vec{x}_1 = \frac{1}{|\mathbb{A}|}\begin{bmatrix}C_{11}\\C_{12}\\C_{13}\end{bmatrix} \leftarrow \text{top row of } \mathbb{C}$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}: \quad \vec{x}_2 = \frac{1}{|\mathbb{A}|}\begin{bmatrix}C_{21}\\C_{22}\\C_{23}\end{bmatrix} \leftarrow \text{middle row of } \mathbb{C}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix}: \quad \vec{x}_3 = \frac{1}{|\mathbb{A}|}\begin{bmatrix}C_{31}\\C_{32}\\C_{33}\end{bmatrix} \leftarrow \text{bottom row of } \mathbb{C}$$

see transpose of $\mathbb{C}$

Combine: $\mathbb{A}^{-1} = \begin{bmatrix}| & | & |\\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3\\ | & | & |\end{bmatrix} = \frac{1}{|\mathbb{A}|}\begin{bmatrix}C_{11}&C_{21}&C_{31}\\C_{12}&C_{22}&C_{32}\\C_{13}&C_{23}&C_{33}\end{bmatrix}$

$$\Rightarrow \boxed{\mathbb{A}^{-1} = \frac{1}{|\mathbb{A}|}\mathbb{C}^T}$$

#inconceivable

Using earlier calculations

$$\mathbb{A}^{-1} = \underset{|\mathbb{A}|}{\frac{1}{2}}\begin{bmatrix}4&-1&-1\\4&0&-2\\-6&1&3\end{bmatrix} \leftarrow \mathbb{C}^T$$

Check

$$\frac{1}{2}\begin{bmatrix}4&-1&-1\\4&0&-2\\-6&1&3\end{bmatrix}\begin{bmatrix}1&1&1\\0&3&2\\2&1&2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix} = \mathbb{I} \quad ✓$$

E18ep1

Algebraic & Geometric Multiplicity of eigenvalues

"Some matrices are bad matrices"
— traditional matrix-fu saying.

ex
$\mathbb{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ Solve $\mathbb{A}\vec{v} = \lambda\vec{v}$:

$(4-\lambda)(7-\lambda)^2 = 0$

$\lambda_1 = 4, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\begin{matrix} \lambda_2 = 7 \\ \lambda_3 = 7 \end{matrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

↑ y-z plane

$(\mathbb{A} - 7\mathbb{I})\vec{v} = \vec{0}$ gives a plane of vectors.

↑
$\dim N(\mathbb{A} - 7\mathbb{I}) = 2$

Defn: Algebraic Multiplicity is # times an eigenvalue appears as a root of $|\mathbb{A} - \lambda\mathbb{I}| = 0$

↖ characteristic equation of $\mathbb{A}$

Defn: Geometric Multiplicity is the dimension of the eigenspace associated with an eigenvalue λ

$\Leftrightarrow \dim N(\mathbb{A} - \lambda\mathbb{I})$

ex a.m. of $\lambda = 4$ is 1, g.m. is 1

a.m. of $\lambda = 7$ is 2, g.m. is 2 ← healthy

Observation $1 \leq$ g.m. $\leq$ a.m.

ex $\mathbb{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

Find eigenvalues: Solve $|\mathbb{A} - \lambda\mathbb{I}| = 0$

$0 = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{vmatrix} = \underline{(1-\lambda)^3} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1$

$\Rightarrow \lambda = 1$ has algebraic multiplicity of $\underline{3}$.

Find evectors: Solve $(\mathbb{A} - \overset{\lambda}{(1)}\mathbb{I})\underline{\underline{\vec{v}}} = \vec{0}$.

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ P & P & F & \end{array}\right] \Rightarrow \vec{v} = c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ only

↑ 1-d $c \in \mathbb{R}$.

$\Rightarrow \lambda = 1$ has geometric multiplicity of 1

* $N(\mathbb{A} - \lambda\mathbb{I})$ not big enough.. $\underline{\dim = 1}$

* $\mathbb{A}$ is a bad matrix...
and does not have a full complement of eigenvectors

↑ basis for eigenspace

Sneaky Monk Tricks (SMTs) for Eigen Stuff:

All about $\underset{n\times n}{A}\,\underset{n\times 1}{\vec{v}} = \underset{\#}{\lambda}\,\underset{n\times 1}{\vec{v}}$

Recap: Solve by

① Finding λs as roots of $|A-\lambda I| = 0$ Characteristic Equation

use cofactor method

&

② For each distinct λ, solving the nullspace equation $(A-\lambda I)\vec{v} = \vec{0}$ for λ's eigenspace

Our helper example:

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \qquad \lambda_1 = \tfrac{3}{2} > 1 \qquad \vec{v}_1 \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \tfrac{1}{2} < 1 \qquad \vec{v}_2 \propto \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

symmetry will be meaningful

SMT #1

$$|A| = \prod_{i=1}^{n} \lambda_i$$

The determinant of A is equal to the product of its eigenvalues

E19ap1

Check:

$$|A| = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = 1\cdot 1 - \tfrac{1}{2}\cdot\tfrac{1}{2} = 3/4$$

$$\lambda_1 \cdot \lambda_2 = \tfrac{3}{2}\cdot\tfrac{1}{4} = 3/4 \quad ✓$$

general characteristic equation

ex $\left(\frac{3}{2}-\lambda\right)\left(\frac{1}{2}-\lambda\right)$

Why? $$|A - \cancel{\lambda I}| = (\underset{\#}{\lambda_1} - \cancel{\lambda})(\underset{\#}{\lambda_2} - \cancel{\lambda}) \cdots (\lambda_n - \cancel{\lambda})$$

set $\lambda = 0$

$$\Rightarrow |A| = \prod_{i=1}^{n} \lambda_i$$

from before: $|A| = \pm \prod_{i=1}^{n} d_i$ ←pivots

SMT #2

Defn Trace of $A = \mathrm{Tr}(A)$
= Sum A's main diagonal elements $= \sum_{i=1}^{n} a_{ii}$

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$$

Our example: $\mathrm{Tr}\left(\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\right) = 1 + 1 = 2$ ✓

$$\sum_{i=1}^{n} \lambda_i = \tfrac{3}{2} + \tfrac{1}{2} = 2$$

General 2x2

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (\lambda_1 - \lambda)(\lambda_2 - \lambda)$$

$$= (a-\lambda)(d-\lambda) - b.c$$

$$(-\lambda)^2 + (a+d)(-\lambda) + ad - bc$$

$$= (-\lambda)^2 + (\lambda_1 + \lambda_2)(-\lambda) + \lambda_1\lambda_2$$

matching

$\lambda_1 + \lambda_2 = a + d = \mathrm{Tr}(A)$

$\lambda_1 \lambda_2 = ad - bc = |A|$

General $n \times n$:

$$|\mathbb{A} - \lambda \mathbb{I}| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

$$/\!/\!/ \qquad\qquad (-\lambda)^n + \left(\sum_{i=1}^{n} \lambda_i\right)(-\lambda)^{n-1} + \cdots$$

$$(-\lambda)^n + (\mathrm{Tr}\,\mathbb{A})(-\lambda)^{n-1} + \cdots$$

One thing: Check $\underbrace{\mathrm{Tr}(\mathbb{A})}_{\text{easy}} = \sum_{i=1}^{n} \lambda_i$

SMT3

$$|\mathbb{A}| = \underset{\substack{\uparrow \\ \text{depends on row swaps required to uncover } \mathbb{U}}}{\pm} \prod_{i=1}^{n} d_i = \prod_{i=1}^{n} \lambda_i$$

SMT4

If $\overset{n\times n}{\mathbb{A}} \vec{v} = \lambda \vec{v}$ then $\mathbb{A}^k \vec{v} = \lambda^k \vec{v}$

$$\mathbb{A}^k \vec{v} = \mathbb{A}^{k-1}(\mathbb{A}\vec{v}) = \lambda \mathbb{A}^{k-1} \vec{v} = \cdots = \lambda^k \vec{v}$$

SMT5

If $\mathbb{A}\vec{v} = \lambda \vec{v}$ then $(\mathbb{A} + t\mathbb{I})\vec{v} = (\underbrace{\lambda + t}_{\text{shift}})\vec{v}$

$$(\mathbb{A} + t\mathbb{I})\vec{v} = \mathbb{A}\vec{v} + t\mathbb{I}\vec{v} = \lambda\vec{v} + t\vec{v} = (\lambda + t)\vec{v}$$

SMT6

If $\mathbb{A}\vec{v} = \lambda \vec{v}$ then $\mathbb{A}^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$

if $\mathbb{A}^{-1}$ exists

$$\overset{\mathbb{I}}{\cancel{\mathbb{A}^{-1}\mathbb{A}}}\,\vec{v} = \lambda \mathbb{A}^{-1}\vec{v}$$

$$\mathbb{A}^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$$

matches SMT4 for $k = -1$

E19ap2

SMT7

If $\mathbb{A}$'s eigenvalues are all different from each other then $\mathbb{A}$'s eigenvectors are linearly independent and form a basis for $\mathbb{R}^n$.

Reason:

Assume λ's are distinct and look at $\vec{v}_1$ & $\vec{v}_2$

If dependent, $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ for some $c_1, c_2 \neq 0$

(a) $\mathbb{A}(c_1\vec{v}_1 + c_2\vec{v}_2) = \mathbb{A}\vec{0}$

$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$ ------ (1)

(b) $\lambda_2 \times (c_1\vec{v}_1 + c_2\vec{v}_2) = \lambda_2 \times \vec{0}$

$c_1\lambda_2\vec{v}_1 + c_2\lambda_2\vec{v}_2 = 0$ ---- (2)

(2) − (1): $c_1(\lambda_2 - \lambda_1)\vec{v}_1 = \vec{0}$

$\Rightarrow \lambda_1 \neq \lambda_2$

] build up from here for $n \times n$'s

SMT8

eigenvalues & eigenvectors of $\mathbb{A}$ & $\mathbb{B}$ are not simply related to those of $\mathbb{A}\mathbb{B}$ & $\mathbb{A} + \mathbb{B}$

- If $\mathbb{A}$ & $\mathbb{B}$ share an eigenvector $\vec{v}$ with eigenvalues λ_A & λ_B then $(\mathbb{A} + \mathbb{B})\vec{v} = (\lambda_A + \lambda_B)\vec{v}$ & $(\mathbb{A}\mathbb{B})\vec{v} = \lambda_A\lambda_B\vec{v}$

but this is generally not the case.

Why diagonal matrices make us happy

E20ap1

ex

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 17 \end{bmatrix} \Rightarrow A^k = \begin{bmatrix} 3^k & 0 & 0 \\ 0 & (-7)^k & 0 \\ 0 & 0 & (17)^k \end{bmatrix}$$

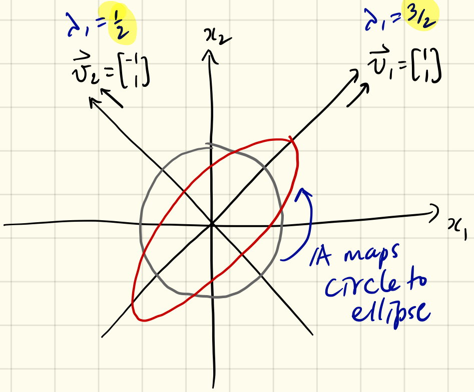
$$A\vec{x} = \begin{bmatrix} 3x_1 \\ -7x_2 \\ 17x_3 \end{bmatrix}$$

how A changes $\vec{x}$ is simple

$\lambda_1 = 3, \lambda_2 = -7, \lambda_3 = 17$

$$\underbrace{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{natural or standard basis for } R^3}$$

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$



- If we could rotate the axes, A's action would be simple
- Big idea: change from standard basis to eigenvector basis and find happiness

Diagonalization is just the best

- Let's assume $\underset{n\times n}{A}$ has n linearly independent eigenvectors

  form a basis for $\mathbb{R}^n$

- know $A\vec{v}_i = \lambda_i \vec{v}_i$ for $i = 1, \ldots, n$

Monks whisper

Create a new matrix with A's eigenvectors:

$$\underset{n\times n}{S} = \underset{n\times n.}{\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}}$$

Consider:

$$\underset{n\times n}{A}\,\underset{n\times n}{S} \overset{\text{matrix-fu}}{=} \begin{bmatrix} | & | & & | \\ \underbrace{\underset{n\times n}{A}\,\underset{n\times 1}{\vec{v}_1}}_{n\times 1} & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix} \overset{\text{sneaky}}{=} \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

$$= \underset{n\times n}{S}\,\underset{n\times n}{\Lambda} \leftarrow \text{capital } \lambda$$

Let's assume A is a good matrix meaning its eigenvectors form a basis for $\mathbb{R}^n$ $\iff S^{-1}$ exists

E20bp1

$$\Rightarrow AS = S\Lambda \Rightarrow$$

postmultiply by S^{-1}

Diagonalization:

$$\boxed{\underset{n\times n}{A} = \underset{n\times n}{S}\,\underset{n\times n}{\Lambda}\,\underset{n\times n}{S^{-1}}}$$

↑ an amazing factorization

- We say A and Λ are similar ↑ similarity transformat

- We begin to see how $A\vec{x}$ works:

more soon

$$A\vec{x} = S\Lambda S^{-1}\vec{x}$$

① changes representation of $\vec{x}$ from standard basis to A's eigenvector basis

② simple multiplication because Λ is diagonal

③ changes back to standard basis representation

Big deal: If A is diagonalizable, then A is really a diagonal matrix when viewed in the right way.

Example: $\mathbb{A} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$ $\quad \lambda_1 = \frac{3}{2} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = \frac{1}{2} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

note: symmetry

our choice (any multiple would work)

$$\Rightarrow S = \begin{bmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

note: $S^{-1} = S^T$

$$\Rightarrow S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$|S| = 2$

use $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\Rightarrow \mathbb{A} = S \Lambda S^{-1}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Let's see how this is super useful for ① $\mathbb{A}\vec{x}$ & ② $\mathbb{A}^k$

① Examine what happens for $\vec{x} = 2\vec{v}_1 + 2\vec{v}_2$

$$\vec{x} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$\mathbb{A}\vec{x}$ in 3 ways ⇓

(i) $\mathbb{A}\vec{x} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ ← no great understanding

(ii) $\mathbb{A}\vec{x} = \mathbb{A}\Big(\underbrace{2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}}\Big) = \frac{3}{2} \cdot 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \cdot 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ ✓

may not know this

(iii) $\mathbb{A}\vec{x} = S \Lambda S^{-1} \vec{x} = S \Lambda \left(\frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right) = S \Lambda \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

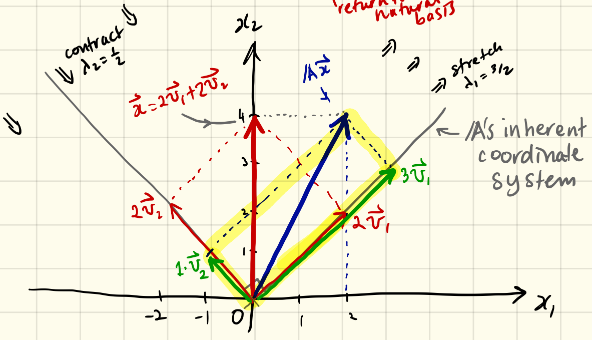
$= S \begin{bmatrix} 3/2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = S \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ ✓

↑ representat of $\vec{x}$ in eigenvector basis

↑ in eigenvector basis still

↑ return to natural basis ↗



② A^k for $k = 0, \pm 1, \pm 2, \ldots$

$$A^2 = (S\Lambda \underbrace{S^{-1})(S}_{I}\Lambda S^{-1}) = S\Lambda^2 S^{-1}$$

$$A^3 = (S\Lambda \underbrace{\cancel{S^{-1})(S}}_{I}\Lambda \underbrace{\cancel{S^{-1})(S}}_{I}\Lambda S^{-1})$$

$$= S\Lambda^3 S^{-1}$$

$$\boxed{A^k = S\Lambda^k S^{-1}}$$

super easy!!

Super easy to compute!

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{bmatrix}$$

→ clearly important

$|\lambda_i| < 1$

$|\lambda_i| = 1$

$|\lambda_i| > 1$

can see that largest eigen value will dominate Λ^k and A^k

E206p3

Ex/ $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}^{523}$

$$= \overset{S}{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}} \overset{\Lambda^{523}}{\begin{bmatrix} \left(\frac{3}{2}\right)^{523} & 0 \\ 0 & \left(\frac{1}{2}\right)^{523} \end{bmatrix}} \overset{S^{-1}}{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}$$

blows up

→ 0

$$\simeq \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^{523} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

More goodness;

$2^0 = 1$

- $A^0 = S\Lambda^0 S^{-1} = S I S^{-1} = I$
- $A^{-1} = S\Lambda^{-1}S^{-1}$ works:

$$(\overset{A^{-1}}{S\Lambda^{-1}S^{-1}})(\overset{A}{S\Lambda S^{-1}}) = I$$

I

I

I

$A^{\frac{1}{2}} = S\Lambda^{1/2}S^{-1}$ works too!!!

$A^{1/2}A^{1/2} = A^1$ ✓

Fibonacci number finder

note: clearly a monk

From before:

$$F_{k+2} = F_{k+1} + F_k \quad \text{with} \quad F_0 = F_1 = 1$$

← Fibonacci sequence

$$\vec{f}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}}_{\vec{f}_{k-1}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \vec{f}_0$$

note symmetry...

$$\vec{f}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Mission: Diagonalize $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

usual thing

$$|A - \lambda I| = 0 \Rightarrow \lambda^2 - \lambda - 1 = 0$$

← very famous

Find $\lambda_1 = \frac{1+\sqrt{5}}{2} = \varphi$ ← golden ratio $\Rightarrow \vec{v}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ ← good hygiene

$\lambda_2 = \frac{1-\sqrt{5}}{2}$ $\quad \vec{v}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ ← keep $\sqrt{5}$'s at bay

Note: $\lambda_1 + \lambda_2 = 1 = \mathrm{Tr}(A)$

$\lambda_1 \lambda_2 = -1 = |A|$

E20cp1

Three pieces:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

aside

$$\varphi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}$$

← most irrational number

continued fraction

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k}_{A} = \underbrace{\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}}_{\Lambda^k} \underbrace{\frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}}_{S^{-1}}$$

← grows

decays

So:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} (\lambda_1^{k+1} - \lambda_2^{k+1}) \\ (\lambda_1^k - \lambda_2^k) \end{bmatrix}$$

2x1 (not a 2x2)

$$F_k = \frac{1}{\lambda_1 - \lambda_2} \left(\lambda_1^k - \lambda_2^k \right)$$

$$= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right)$$

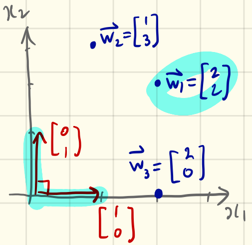
← disappears

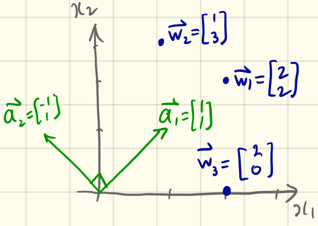
← dominates

#delicious

Cool beans; $\frac{F_{k+1}}{F_k} \rightarrow \overset{=\lambda_1}{\frac{1+\sqrt{5}}{2}} = \varphi$ as $k \to \infty$

The gentle art of changing basis:





So far, we've expressed all vectors in terms of the standard (or natural) basis.

ex $\vec{w}_1 = \begin{bmatrix}2\\2\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$

- What's the representation of $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ in terms of the new basis $\{\vec{a}_1, \vec{a}_2\}$?
- How do we do this systematically?

By solving an $\mathbb{A}\vec{x} = \vec{b}$ problem!!!

The set up for $\vec{w}_1$:

column picture

coordinate of $\vec{w}_1$ in $\{\vec{a}_1, \vec{a}_2\}$ basis

$$\vec{w}_1 = \begin{bmatrix}2\\2\end{bmatrix} = c_1\vec{a}_1 + c_2\vec{a}_2 = \begin{bmatrix}| & |\\ a_1 & a_2\\ | & |\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

in natural basis

often used for basis transformation

$$= \mathbb{A}\begin{bmatrix}c_1\\c_2\end{bmatrix} \equiv \mathbb{M}\vec{w}_1^{(a)}$$

E21ap1

inverse takes us from natural to new basis

$$\Rightarrow \vec{w}_1^{(a)} = \mathbb{M}^{-1}\vec{w}_1$$

$$= \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}^{-1}\begin{bmatrix}2\\2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

$\vec{w}_1$ in $\{\vec{a}_1, \vec{a}_2\}$ basis.

Similarly: $\vec{w}_2^{(a)} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\vec{w}_3^{(a)} = \begin{bmatrix}1\\1\end{bmatrix}$

To change back: $\vec{w}_i = \mathbb{M}\vec{w}_i^{(a)}$

We say:

In basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$, $\vec{w}_1$ is represented as $\begin{bmatrix}2\\2\end{bmatrix}$

In basis $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$, $\vec{w}_1$ is represented as $\begin{bmatrix}2\\0\end{bmatrix}$

specify same point/vector in space

Big deal:

The vector $\vec{w}_1$ never changes but our representation does.

Big deal:

$\mathbb{M}$ does the real work $\mathbb{M}^{-1}$

$$\mathbb{A} = \mathbb{S}\Lambda\mathbb{S}^{-1}$$

change basis only (from eigenvector to normal)

"change" vector

change basis only (from normal to eigenvector)

Symmetry and the Spectral Theorem

We know:

- Diagonalization is joyous and empowering
- A ($n \times n$) can only be diagonalized if it has n linearly independent eigenvectors
- Trouble potentially arises when eigenvalues are repeated (algebraic multiplicity > 1)
  - ↳ May not end up with a full eigenspace

Bonus truths:

- If one or more eigenvalues $= 0$, A^{-1} does not exist
  - ↳ $|A| = 0$
- But A may still be diagonalizable
  - ↳ depends on eigenvectors

An amazing matrix truth:

If A ($n \times n$) is real & symmetric, i.e. $A = A^T$, then (+ a_{ij} is real for all i,j)

A always has n linearly independent eigenvectors and is therefore always diagonalizable

even better

① All of A's eigenvalues are real (no complex numbers ⇒ no rotations)

② A's eigenvectors form an orthogonal basis for R^n !!!

proofs later

Ezzap1

We get so excited, we replace $S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$ with $Q = \begin{bmatrix} | & | & & | \\ \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \\ | & | & & | \end{bmatrix}$

↑ unit vectors

because we realize we have an orthogonal matrix.

And because $Q^{-1} = Q^T$ ← saves a lot of trouble, our diagonalization takes on a new level of majesty:

$$\boxed{A = Q \Lambda Q^T}$$

Wow!!

More amazingness:

$$A = Q \Lambda Q^T = \begin{bmatrix} | & | & & | \\ \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} - & \hat{v}_1^T & - \\ - & \hat{v}_2^T & - \\ & \vdots & \\ - & \hat{v}_n^T & - \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & \lambda_1 \hat{v}_1^T & - \\ - & \lambda_2 \hat{v}_2^T & - \\ & \vdots & \\ - & \lambda_n \hat{v}_n^T & - \end{bmatrix}$$

$A\vec{x}$ broken into clean pieces

$$= \lambda_1 \hat{v}_1 \hat{v}_1^T + \lambda_2 \hat{v}_2 \hat{v}_2^T + \ldots + \lambda_n \hat{v}_n \hat{v}_n^T = \sum_{i=1}^{n} \lambda_i \hat{v}_i \hat{v}_i^T$$

outer products

projection operators!!

for $A\vec{x}$, each one chops out a piece of $\vec{x}$ and then scales by λ.

⇑ Spectral Theorem for Symmetric Matrices

E22ap2

Example: $A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} = A^T$ ← symmetric!

Use unit vectors for eigenvectors

$$S = Q = \begin{bmatrix} | & | \\ \hat{v}_1 & \hat{v}_2 \\ | & | \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$S^{-1} = Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

take transpose! easy!

$$A = \overset{S}{Q}\,\overset{\Lambda}{\Lambda}\,\overset{S^{-1}}{Q^T} =$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$A = \sum_{i=1}^{n} \lambda_i \hat{v}_i \hat{v}_i^T$$

$$= \underset{\lambda_1}{\left(\frac{3}{2}\right)} \underset{\hat{v}_1}{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \underset{\hat{v}_1^T}{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}} + \underset{\lambda_2}{\left(\frac{1}{2}\right)} \underset{\hat{v}_2}{\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}} \underset{\hat{v}_2^T}{\frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \end{bmatrix}}$$

$A\vec{x}$: breaks $\vec{x}$ into two orthogonal pieces for which $A\hat{v}_1$ & $A\hat{v}_2$ are very simple. and then recombines.

Why the spectral theorem works

no rotations

① All of A's eigenvalues are real

Assume $A = A^T$ and A's entries are real

Given $A\vec{v} = \lambda\vec{v}$, we test to see if λ can be complex: $\lambda = a+bi$ $b \neq 0$

$i = \sqrt{-1}$

Denote complex conjugate by over bar:

Result: $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

$\overline{a+bi} = a-bi$

$$\underset{\substack{[\ ]\\ n\times n}}{A}\,\underset{[\,]}{\vec{v}} = \underset{\#}{\lambda}\,\underset{[\,]}{\vec{v}} \xrightarrow{\text{monks}} \overline{A}\,\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$$

↓ real

$$A\,\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$$

take transpose

$$\overline{\vec{v}}^T A^T = \overline{\lambda}\,\overline{\vec{v}}^T$$

↓ symmetry

$$\overline{\vec{v}}^T A = \overline{\lambda}\,\overline{\vec{v}}^T$$

↓ postmultiply by $\vec{v}$

$$\overline{\vec{v}}^T A\vec{v} = \overline{\lambda}\,\overline{\vec{v}}^T\vec{v}$$

Monk sneakiness

monks ↓ premultiply by $\overline{\vec{v}}^T$

$$\overline{\vec{v}}^T A\vec{v} = \lambda\,\overline{\vec{v}}^T\vec{v}$$

See everything matches except λ & $\overline{\lambda}$

$\Rightarrow \lambda = \overline{\lambda}$ so λ is real

E22 bp1

crazy!

② A's eigenvectors form an orthogonal basis for R^n

- Again have $A = A^T$ and A is real
- We want to show $\vec{v}_i^T\vec{v}_j = 0$ if $i \neq j$
- Work up to full story...

First If A's eigenvalues are all distinct (i.e., each has algebraic multiplicity 1):

$$\underset{1\times n}{\vec{v}_1}^{\textcircled{T}}\,\underset{n\times n}{A}\,\underset{n\times 1}{\vec{v}_2} = \vec{v}_1^T\left(A\vec{v}_2\right)$$

a number

More Monk Sneakiness

$$\vec{v}_1^T A\vec{v}_2 = \left(A^T\vec{v}_1\right)^T\vec{v}_2 = \lambda_1\underbrace{\vec{v}_1^T\vec{v}_2}$$

$A^T = A$

$$\vec{v}_1^T\left(A\vec{v}_2\right) = \vec{v}_1^T\left(\lambda_2\vec{v}_2\right) = \lambda_2\underbrace{\vec{v}_1^T\vec{v}_2}$$

what we're interested in...

$\lambda_1\vec{v}_1^T\vec{v}_2 = \lambda_2\vec{v}_1^T\vec{v}_2$

but $\lambda_1 \neq \lambda_2$ so these can only be equal if $\vec{v}_1^T\vec{v}_2 = 0$

OK

- What if an eigenvalue is repeated?
- We're worried we won't have enough eigenvectors ....

A suggestive pair of examples:

Not symmetric:

$$\mathbb{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq \mathbb{A}^T$$

$\lambda_1 = \lambda_2 = 1$ repeated

only one dimension for eigenspace

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ #sadness

Symmetric:

$$\mathbb{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{A}^T$$

$\lambda_1 = \lambda_2 = 1$ repeated

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2-d eigenspace healthy

Tweaks:

$$\mathbb{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1+\epsilon \end{bmatrix} \neq \mathbb{A}^T$$

small

now distinct

$\lambda_1 = 1, \lambda_2 = 1+\epsilon$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✓, $\vec{v}_2 = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$ ✓

$\Uparrow$

See as $\epsilon \to 0$, eigenvectors become the same

$$\mathbb{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1+\epsilon \end{bmatrix} = \mathbb{A}^T$$

distinct

$\lambda_1 = 1, \lambda_2 = 1+\epsilon$

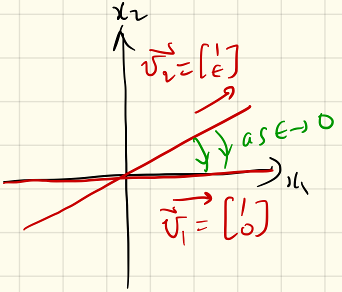
$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✓, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ✓

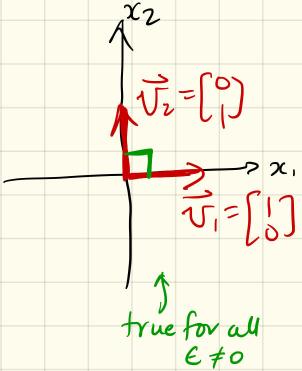
as before

$\vec{v}_1^T \vec{v}_2 = 0$

$\Uparrow$

Eigenvectors do not budge as $\epsilon \to 0$





Idea: smooth change in tweaks ($\epsilon \to 0$) cannot lead to eigenvectors snapping into orthogonal directions

$\Rightarrow$ orthogonality is preserved //.

Requires more work to show in general but we have the basic story here.

Surprising things about traces

Defn Trace of $\underset{n\times n}{\mathbb{A}} = \text{Tr}(\mathbb{A})$

= Sum of the entries of $\mathbb{A}$'s main diagonal

$= \sum_{i=1}^{n} a_{ii}$

ex $\text{Tr}\left(\begin{bmatrix} 3 & 0 & 2 \\ 2 & -1 & -1 \\ 1 & 2 & 4 \end{bmatrix}\right) = 3 + (-1) + 4 = 6$

From earlier: $\text{Tr}(\mathbb{A}) = \sum_{i=1}^{n} \lambda_i$

Now, two more things:

(1) $\text{Tr}(\underset{n\times n}{\mathbb{A}}\ \underset{n\times n}{\mathbb{B}}) = \text{Tr}(\underset{n\times n}{\mathbb{B}}\ \underset{n\times n}{\mathbb{A}})$

Reason: $\text{Tr}(\mathbb{A}\mathbb{B}) = \sum_{i=1}^{n} (\mathbb{A}\mathbb{B})_{ii}$

$= \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} a_{ij} b_{ji}}$ ← from defn of multiplication

inner product of ith row of A & jth column of $\mathbb{B}$

swap everything →

$= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} = \text{Tr}(\mathbb{B}\mathbb{A})$

E22C p1

Generalizes:

two matrices

$\text{Tr}(\mathbb{A}\mathbb{B}\mathbb{C}) = \text{Tr}((\mathbb{A}\mathbb{B})\mathbb{C}) = \text{Tr}(\mathbb{C}(\mathbb{A}\mathbb{B}))$

$= \text{Tr}((\mathbb{C}\mathbb{A})\mathbb{B}) = \text{Tr}(\mathbb{B}\mathbb{C}\mathbb{A})$

↑ any cycling leaves Trace unchanged

(2) If $\mathbb{A} = \mathbb{S}\Lambda\mathbb{S}^{-1}$ ← sadly not possible for all matrices

then $\text{Tr}(\mathbb{A}) = \text{Tr}(\mathbb{S}\Lambda\mathbb{S}^{-1})$ — cycle to front

$= \text{Tr}(\underbrace{\mathbb{S}^{-1}\mathbb{S}}_{\mathbb{I}}\Lambda) = \text{Tr}(\Lambda)$

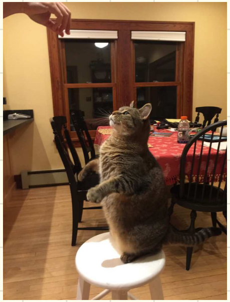
$= \sum_{i=1}^{n} \lambda_i$

#delicious

So: a very enjoyable proof of $\text{Tr}(\mathbb{A}) = \sum_{i=1}^{n} \lambda_i$

(but does not work if $\mathbb{A}$ is not diagonalizable)

Pratchett does tricks for treats...



Positive Definite Matrices

matrices that are really sure about themselves

defn: A Positive Definite Matrix is a real, symmetric matrix with positive eigenvalues, i.e., $\lambda_i > 0, i = 1, \ldots, n$

If a matrix is real and symmetric with $\lambda_i \geq 0$ and at least one eigenvalue equal to zero, then we say it is Semi-Positive Definite

We recall with alacrity that real, symmetric matrices always have

(1) Real eigenvalues ⇐ flipping stretching & shrinking

(2) Eigenvectors that form an orthonormal basis for $\mathbb{R}^n$

$A = Q \Lambda Q^T$

Turns out that also having $\lambda_i > 0$ or $\lambda_i \geq 0$ is an excellent bonus feature ...

Menu

i/ How to spot a PDM

ii/ Why we like PDMs (and SPDMs)

surprising results from monks

pivots ⇔ eigenvalues

E23ap1

Places we'll go, things we'll see:

- * $2x_1^2 + 2x_1x_2 + 2x_2^2 = 1 \Rightarrow$ matrices
- * what elimination really does for symmetric matrices
- * Completing the Square

Three example 2x2 matrices:

$$A_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}; \quad A_3 = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

$\lambda_1 = +3$

$\lambda_2 = +1$

computing happens elsewhere

PDM

$\lambda_1 = \sqrt{5}$

$\lambda_2 = -\sqrt{5}$

$\lambda_1 = -1$

$\lambda_2 = -3$

Problem: Finding eigenvalues can be pretty hard for real matrices

- We only want to know signs of the eigenvalues
- Could there be a sneaky way?

especially one that helps computers

SMT #37

If $\underset{n\times n}{\mathbb{A}} = \mathbb{A}^T$ & $\mathbb{A}$ is real then:

positive eigenvalues = # positive pivots

negative eigenvalues = # negative pivots

zero eigenvalues = zero pivots

↑ #crazytownbananapants

- Very peculiar: eigenvalues and pivots come from very different parts of matrixology
- Recall we already know for general $\underset{n\times n}{\mathbb{A}}$ that $|\mathbb{A}| = \prod_{i=1}^{n} \lambda_i = \pm \prod_{i=1}^{n} d_i$
- SMT#37 says more for real symmetric matrices

Big deal: $\mathbb{A}$ is a PDM if all $d_i > 0$
Pivots are much easier to compute than eigenvalues

E23ap2

Beautiful reason:

consider $\mathbb{A}_2 = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$ $\lambda_1 = \sqrt{5}$ $\lambda_2 = -\sqrt{5}$

find pivots using $\mathbb{L}\mathbb{U}$ decomposition

$R_2' = R_2 - \underset{l_{21}}{\left(-\tfrac{1}{2}\right)} R_1$

$$\mathbb{A}_2 = \mathbb{L}\mathbb{U} = \overset{\mathbb{L}}{\begin{bmatrix} 1 & 0 \\ \underset{l_{21}}{\boxed{-\frac{1}{2}}} & 1 \end{bmatrix}} \overset{\mathbb{U}}{\begin{bmatrix} \overset{d_1}{2} & -1 \\ 0 & \underset{d_2}{-5/2} \end{bmatrix}}$$

Because $\mathbb{A}_2$ is symmetric, we can go further:
$(\mathbb{A}_2 = \mathbb{A}_2^T)$

$$\mathbb{A}_2 = \mathbb{L}\,\mathbb{D}\,\mathbb{L}^T = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Let's think about this parametrized matrix:

$$\mathbb{B}(l_{21}) = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & l_{21} \\ 0 & 1 \end{bmatrix}$$

When $l_{21} = -\frac{1}{2}$, we have $\mathbb{B}(-\frac{1}{2}) = \mathbb{A}_2$

Next: What happens as we move from $l_{21} = -\frac{1}{2}$ to $l_{21} = 0$?

$$\mathbb{B}(0) = \underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{\mathbb{I}}\underbrace{\begin{bmatrix}2 & 0\\ 0 & -\frac{5}{2}\end{bmatrix}}_{\mathbb{D}}\underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{\mathbb{I}} = \begin{bmatrix}\overset{d_1}{2} & 0\\ 0 & \underset{d_2}{-\frac{5}{2}}\end{bmatrix}$$

$$\| \quad \mathbb{D}$$

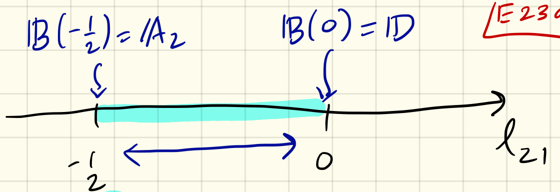
Observations:

- For diagonal matrices, pivots $\equiv$ eigenvalues
- $\mathbb{B}(\ell_{21})$ has pivots $d_1 = 2$ and $d_2 = -\frac{5}{2}$ independent of ℓ_{21}
- $\det(\mathbb{B}(\ell_{21})) = d_1 \cdot d_2 = (2)(-\frac{5}{2}) = -5$ again independent of ℓ_{21}.

Big Connections:

- We also know $\det(\mathbb{B}(\ell_{21})) = \overset{\sqrt{5}}{\lambda_1} \cdot \overset{-\sqrt{5}}{\lambda_2}$ for $\ell_{21} = -\frac{1}{2}$ must $= -5$ for all ℓ_{21}
- As ℓ_{21} changes, the eigenvalues change BUT they cannot pass through 0 as then the determinant would be 0 ($\neq -5$)
- When $\ell_{21} = 0$, $\mathbb{B}(0) = \mathbb{D}$ is diagonal and the pivots and eigenvalues match up: $d_1 \equiv \lambda_1$, $d_2 \equiv \lambda_2$
- Therefore as ℓ_{21} moves away from 0, the eigenvalues must maintain the same signs as the pivots
- Argument assumes all pivots $\neq 0$; proof is tweakable

E23ap3



$\lambda_1 = \sqrt{5} \longleftrightarrow \lambda_1 = 2$

$\lambda_2 = -\sqrt{5} \longleftrightarrow \lambda_2 = -\frac{5}{2}$

} change but signs remain same

$d_1 = 2 \longleftrightarrow d_1 = 2$

$d_2 = -\frac{5}{2} \longleftrightarrow d_2 = -5/2$

} invariant

General argument:

Given $\underset{n\times n}{\mathbb{A}} = \mathbb{L}\mathbb{D}\mathbb{L}^T$ create $\hat{\mathbb{L}}(t) = \mathbb{I} + t(\mathbb{L} - \mathbb{I})$

$\hookrightarrow t = 0: \hat{\mathbb{L}}(0) = \mathbb{I}$

$t = 1: \hat{\mathbb{L}}(1) = \mathbb{L}$

$$\begin{cases}\mathbb{B}(t) = \hat{\mathbb{L}}(t)\,\mathbb{D}\,\hat{\mathbb{L}}^T(t)\\ \mathbb{B}(0) = \mathbb{D} \ \& \ \mathbb{B}(1) = \mathbb{A}\end{cases}$$

- As before, pivots don't change as we vary t from 1 to 0
- Same story: Eigenvalues cannot change sign as t varies
- Signs of eigenvalues must match signs of pivots

Positive Definite Matrices in the Wild:

Menu for 23b,c,d:

- $\vec{x}^T A \vec{x}$ and ellipses and other functions
- Completing the Square
- Cholesky factorization

Idea: re-express polynomial functions using matrices.

← especially PDMs

key construct: $\vec{x}^T A \vec{x}$ where $A = A^T$

$$\underbrace{\underset{1\times n}{[\quad]}\underset{n\times n}{[\quad]}\underset{n\times 1}{[\,]}}_{1\times 1 \leftarrow \text{a scalar}} \quad = (A^T\vec{x})^T\vec{x} = (A\vec{x})^T\vec{x}$$

General 2×2 example:

$$\vec{x}^T A \vec{x} = \underset{1\times 2}{[x_1 \; x_2]} \underset{2\times 2}{\begin{bmatrix} a & b \\ b & c \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \qquad \swarrow A = A^T$$

$$= \overset{1\times 2}{[x_1 \; x_2]} \overset{2\times 1}{\begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}}$$

inner product

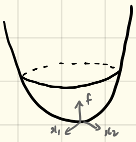
$$= ax_1^2 + bx_1x_2 + bx_1x_2 + cx_2^2$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2 = f(x_1, x_2) \leftarrow \text{height}$$

easy to go back this way

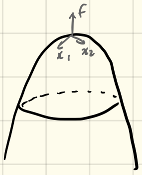
E23bp1

$f(x_1,x_2)$ could be:



min

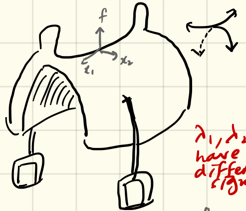
$\lambda_1, \lambda_2 > 0$



max

$\lambda_1, \lambda_2 < 0$

saddle



λ_1, λ_2 have different signs

The Story:
f has a minimum at $x_1 = x_2 = 0$
iff A is Positive Definite

not banana-shaped

Why? (1) $\vec{x}^T A \vec{x} = 0$ at $\vec{x} = 0$

(2) Consider what happens as $\vec{x}$ moves away from $\vec{0}$

Write $\vec{x} = \sum_{i=1}^{n} c_i \hat{v}_i$ ← A's evector basis

possible because $A = A^T$
⇒ eigenvectors form an orthonormal basis for R^n

$$\vec{x}^T A \vec{x} = \left(\sum_{i=1}^{n} c_i \hat{v}_i^T\right) A \left(\sum_{j=1}^{n} c_j \hat{v}_j\right) = \left(\sum_{i=1}^{n} c_i \hat{v}_i^T\right)\left(\sum_{j=1}^{n} c_j \overbrace{A\hat{v}_j}^{\lambda_j \hat{v}_j}\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_j c_i c_j \underbrace{\hat{v}_i^T \hat{v}_j}_{\substack{1 \text{ if } i=j \\ 0 \text{ otherwise}}} = \sum_{i=1}^{n} \lambda_i \underbrace{c_i^2}_{>0} > 0$$

for all $\{c_i\}$
iff
all $\lambda_i > 0$

Ex 1.

Does $f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2$ have a maximum at $x_1 = x_2 = 0$?

Answer: Yes if eigenvalues for f's $\mathbb{A}$ are both positive

$\Leftrightarrow$ $\mathbb{A}$'s pivots are both positive

(1) Construct $\vec{x}^T \mathbb{A} \vec{x}$

$$f(x_1, x_2) = \boxed{2}x_1^2 \boxed{-2}x_1x_2 + \boxed{2}x_2^2$$

split evenly

$$= [x_1 \quad x_2] \begin{bmatrix} \boxed{2} & \boxed{-1} \\ \boxed{-1} & \boxed{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\leftarrow \mathbb{A}_1$ from before

(2) Determine pivots

$$d_1 = 2, \quad d_2 = 3/2 \Rightarrow \lambda_1 > 0, \ \lambda_2 > 0 \Rightarrow f \text{ has a minimum}$$

LE23bp2

Ex 2.

$$f(x_1, x_2) = \boxed{2}x_1^2 \boxed{-2}x_1x_2 \boxed{-2}x_2^2$$

(1) construct $\vec{x}^T \mathbb{A} \vec{x}$

$$f(x_1, x_2) = [x_1 \quad x_2] \begin{bmatrix} \boxed{2} & \boxed{-1} \\ \boxed{-1} & \boxed{-2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\leftarrow \mathbb{A}_2$ from before

(2) Determine pivots:

$$d_1 = 2, \quad \underline{\underline{d_2 = -5/2}} \Rightarrow \lambda_1 > 0, \ \lambda_2 < 0 \Rightarrow \text{saddle}$$

Alternate definition:
$\mathbb{A} = \mathbb{A}^T$ is positive definite iff
$\vec{x}^T \mathbb{A} \vec{x} > 0$ for all $\vec{x} \neq \vec{0}$

Completing the Square = Gaussian Elimination !!

for square symmetric matrices

Idea We could approach question of determining kinds of stationary points by creating clear squares and then looking at signs.

Ex $f(x_1,x_2) = \underbrace{2x_1^2 - 2x_1x_2}_{\text{complete square here}} + 2x_2^2 = [x_1, x_2]\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

leave as x constant

$\Uparrow$ $\mathbb{A}_1$ from before

$$= 2\left(x_1^2 - (x_2)\, x_1\right) + 2x_2^2$$

$$= 2\left(\left(x_1 - \frac{x_2}{2}\right)^2 - \frac{x_2^2}{4}\right) + 2x_2^2$$

$$= 2\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{2}x_2^2$$

d_1 $\quad l_{21}$ $\quad d_2$ $\quad \leftarrow$ what!??

$\Longleftarrow$ from $\mathbb{A}_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

Ex $f(x_1,x_2) = 2x_1^2 - 2x_1x_2 - 2x_2^2 \Leftarrow \mathbb{A}_2 = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$

$$= 2\left(x_1 - \frac{1}{2}x_2\right)^2 - \frac{5}{2}x_2^2$$

d_1 $\quad l_{21}$ $\quad d_2$

same thing

In general for 2×2s:

$$ax_1^2 + 2bx_1x_2 + cx_2^2$$

$$= \underset{d_1}{a}\underbrace{\left(x_1 + \frac{b}{a}x_2\right)^2}_{} + \underbrace{\left(\frac{ac-b^2}{a}\right)}_{d_2}x_2^2$$

l_{21} (for $\frac{b}{a}$)

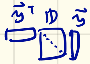
see $x_1 + \frac{b}{a}x_2$ as a new variable...

Does completing the square always work like this?

Yes! for symmetric matrices

$\vec{x}^T \mathbb{A} \vec{x}$ $\leftarrow$ any quadratic in n variables

$\nwarrow \mathbb{A} = \mathbb{A}^T$



$$= \vec{x}^T\left(\mathbb{L}\mathbb{D}\mathbb{L}^T\right)\vec{x}$$ $\leftarrow$ see as a variable transformation

$$= \left(\mathbb{L}^T\vec{x}\right)^T \mathbb{D} \left(\mathbb{L}^T\vec{x}\right)$$

$$= \vec{y}^T \mathbb{D} \vec{y}$$

$$= d_1y_1^2 + d_2y_2^2 + \dots + d_ny_n^2$$ $\leftarrow$ if pivots > 0 then $\mathbb{A}$ is a PDM

Super bonus: if $\mathbb{A}$ is a PDM then

$$\mathbb{A} = \tilde{\mathbb{L}}\tilde{\mathbb{L}}^T \quad \text{with } \tilde{\mathbb{L}} = \mathbb{L}\mathbb{D}^{\frac{1}{2}}$$

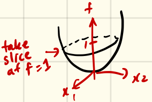
◣ × ◥

$\uparrow$ Cholesky Factorization

$\uparrow$ all real numbers lower triangular

- Even better for $\mathbb{A}\vec{x} = \vec{b}$

Principle Axis theorem



take slice at f=1

Consider $\boxed{2}x_1^2 + \boxed{2}x_1x_2 + \boxed{2}\boxed{x_2^2} = 1$

Equation of an ellipse oriented at an angle to standard axes

Matrixify:

$$[x_1 \;\; x_2]\begin{bmatrix} \boxed{2} & 1 \\ 1 & \boxed{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

↓ use $A = Q\Lambda Q^T$ ← one of our magic friends

$$\underbrace{[x_1 \;\; x_2]\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} \overset{\hat{v}_1}{1} & \overset{\hat{v}_2}{-1} \\ 1 & 1 \end{bmatrix}}_{Q}}_{\vec{y}^T = (Q^T\vec{x})^T}\underbrace{\begin{bmatrix} \overset{\lambda_1}{3} & 0 \\ 0 & \underset{\lambda_2}{1} \end{bmatrix}}_{\Lambda}\underbrace{\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{Q^T}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{y} = Q^T\vec{x}} = 1$$

$\hat{v}_1^T$, $\hat{v}_2^T$

$$\Rightarrow \overset{\vec{y}^T}{[y_1 \;\; y_2]}\overset{\Lambda}{\begin{bmatrix} \boxed{3} & 0 \\ 0 & \boxed{1} \end{bmatrix}}\overset{\vec{y}}{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} = 1$$

$$\Rightarrow \boxed{3}y_1^2 + y_2^2 = 1$$

Completely clear in y_1-y_2 coordinate system

clearly an ellipse

What's this new coordinate system?:

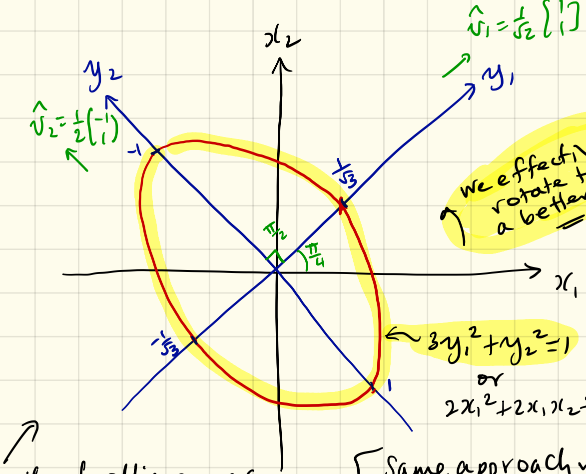
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Q^T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{bmatrix}$$

best

Also: See Q as IM ⇒ basis transformat.

$\left\{ \underset{\hat{v}_1}{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{\hat{v}_2}{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \right\}$ = new basis



we effectively rotate to a better basis

← $3y_1^2 + y_2^2 = 1$

or

$2x_1^2 + 2x_1x_2 + 2x_2^2 = 1$

length of ellipse axes $= \frac{1}{\sqrt{\lambda_i}}$

Same approach works for higher dimensional footballs

E24ap1

Singular Value Decomposition

Insert ominous organ music

Big deal:

- Matrix Factorizations encode our understanding of problems and greatly enable our methods

$\mathbb{P}\mathbb{A} = \mathbb{L}\mathbb{U}$ ⇒ Simultaneous Equations

$\mathbb{A} = \mathbb{Q}\mathbb{R}$ ⇒ $\underset{m \times n}{\mathbb{A}}\underset{n \times 1}{\vec{x}} = \underset{m \times 1}{\vec{b}}$ rectangular

$\mathbb{A} = S \Lambda S^{-1}$

$\mathbb{A} = \mathbb{Q} \Lambda \mathbb{Q}^T$

⇒ $\underset{n \times 1}{\vec{x}}' = \overset{n \times n}{\mathbb{A}}\underset{n \times 1}{\vec{x}}$ & $\mathbb{A}^k$, $\mathbb{A}\vec{v} = \lambda\vec{v}$ square only

- All have limitations
- We love diagonalization for example

but
1. $\mathbb{A}$ must be $n \times n$
2. $\mathbb{A}$ must have n linearly independent eigenvectors
3. Eigenvector basis may not be orthogonal (only guaranteed if $\mathbb{A} = \mathbb{A}^T$)

In attempting to overcome these problems, we'll find a factorization that works for all matrices plus

- helps us identify the most important features of a system (pages on the web, supreme court decisions, data in general, building blocks of images, ...)
- Completes our "Big Picture" story for $\mathbb{A}\vec{x} = \vec{b}$

Fundamental Theorem of Linear Algebra

[Theoretical story first, then some nutritious examples

Eigen Story: $\underset{n \times n}{\mathbb{A}}\vec{v} = \lambda\vec{v}$ ← eigenvectors may not form a basis :(

same direction :)

We give this up to (i) accommodate $m \times n$ matrices & (ii) ensure orthogonality of bases

↑ want this

New plan:

real

unit vector

"singular value"

$$\boxed{\underset{m\times n}{\mathbb{A}}\ \underset{n\times 1}{\hat{v}_i} = \sigma_i\ \underset{m\times 1}{\hat{u}_i}}$$

singular vector

unit vector

where:
- $\hat{v}_i \perp \hat{v}_j$ if $i \neq j$, $\hat{v}_i \in \mathbb{R}^n$ (row space + null space)
- $\hat{u}_i \perp \hat{u}_j$ if $i \neq j$, $\hat{u}_i \in \mathbb{R}^m$ (column space + left nullspace)
- $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_r > 0$ r = rank

- the $\hat{v}_i$ form an orthonormal basis for $\mathbb{R}^n$
- the $\hat{u}_i$ form an orthonormal basis for $\mathbb{R}^m$

How $\mathbb{A}\vec{x}$ works:

(1) Transform $\vec{x}$ to $\{\hat{v}_i\}$ basis in $\mathbb{R}^n$

(2) Send $\hat{v}_i$'s to $\hat{u}_i$'s and multiply by σ_i

(3) Transform from $\{\hat{u}_i\}$ basis to standard basis in $\mathbb{R}^m$

Yet another great moment in Matrixology:

E24ap2

Singular Value Decomposition

$$\boxed{\underset{m\times n}{\mathbb{A}} = \underset{m\times m}{\mathbb{U}}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{\mathbb{V}^T}}$$

Monks tell us to believe

c.f. $\mathbb{A} = \mathbb{Q}\Lambda\mathbb{Q}^T$

$$\underset{m\times m}{\mathbb{U}} = \begin{bmatrix} | & | & & | \\ \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_m \\ | & | & & | \end{bmatrix}$$

← $\vec{b}, \vec{p}, \vec{e}, \vec{y}$

$$\underset{n\times n}{\mathbb{V}} = \begin{bmatrix} | & | & & | \\ \hat{v}_1, & \hat{v}_2 & \cdots & \hat{v}_n \\ | & | & & | \end{bmatrix}$$

← $\vec{x}$, $\vec{x}_r, \vec{x}_n$, $\vec{x}_p, \vec{x}_h$

all zeros

$$\underset{m\times n}{\Sigma} = \begin{bmatrix} \sigma_1 & & & & & 0 \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & 0 & & \sigma_r & & \\ & & & & 0 \cdots & 0 \end{bmatrix}$$

← same shape as $\mathbb{A}$

r = rank

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

Let's see how this all works:

Claim → $\mathbb{A} = \mathbb{U}\Sigma\mathbb{V}^T$ with $\mathbb{A}\hat{v}_i = \sigma_i\hat{u}_i$

Monks: "Try $\underset{\uparrow}{\mathbb{A}^T\mathbb{A}}$, grasshopper"   okay...

symmetric for all $\mathbb{A}$

key: know $(\mathbb{A}^T\mathbb{A})^T = \mathbb{A}^T\mathbb{A}$ for any $\mathbb{A}$

$$\underbrace{\underset{n\times m}{\mathbb{A}^T}\,\underset{m\times n}{\mathbb{A}}}_{\underline{\underline{n\times n}}} = (\mathbb{U}\Sigma\mathbb{V}^T)^T(\mathbb{U}\Sigma\mathbb{V}^T)$$

$$= \mathbb{V}^{TT}\Sigma^T\underbrace{\mathbb{U}^T\mathbb{U}}_{\mathbb{I}}\Sigma\mathbb{V}^T$$

$$= \mathbb{V}\underset{n\times m}{\Sigma^T}\underset{m\times n}{\Sigma}\mathbb{V}^T$$

$\left[\begin{array}{l}\mathbb{U}^{-1} = \mathbb{U}^T \\ \mathbb{V}^{-1} = \mathbb{V}^T\end{array}\right.$

$$= \begin{bmatrix} | & | & & | \\ \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \\ | & | & & | \end{bmatrix} \underset{n\times n}{\overset{\text{square}}{\begin{bmatrix} \sigma_1^2 & & & & & 0 \\ & \sigma_2^2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r^2 & & \\ & & & & 0 \cdots & \\ 0 & & & & & 0 \end{bmatrix}}} \begin{bmatrix} - & \hat{v}_1^T & - \\ - & \hat{v}_2^T & - \\ & \vdots & \\ - & \hat{v}_n^T & - \end{bmatrix}$$

Looks a lot like: $\mathbb{Q}\Lambda\mathbb{Q}^T$ (sneaky monks)

But will $\mathbb{A}^T\mathbb{A}$ always be so wonderfully diagonalizable? ← drama

Monk Joy

$\underbrace{\underset{n\times m}{\mathbb{A}^T}\,\underset{m\times n}{\mathbb{A}}}_{n\times n}$ is real, symmetric

and therefore (1) eigenvalues are real

(2) eigenvectors form an orthonormal basis for R^n

E24ap3

Monk Joy

↑ augmented

$$\vec{x}^T(\mathbb{A}^T\mathbb{A})\vec{x}$$

$$= (\underbrace{\mathbb{A}\vec{x}}_{n\times 1})^T(\underbrace{\underset{n\times n}{\mathbb{A}}\,\underset{n\times 1}{\vec{x}}}_{n\times 1})$$

$$= \|\mathbb{A}\vec{x}\|^2 \geq 0$$

↑length   ↖has to be true

⇒ $\mathbb{A}^T\mathbb{A}$ is Semi-Positive Definite

⇒ $\mathbb{A}^T\mathbb{A}$'s eigenvalues are all ≥ 0

⇒ $\sigma_i = \sqrt{\lambda_i} \geq 0$ is all good

Upshot: Diagonalize $\mathbb{A}^T\mathbb{A}$ to find σ_i's and $\hat{v}_i$'s

Monks chant: "$\mathbb{A}\mathbb{A}^T$! $\mathbb{A}\mathbb{A}^T$! $\mathbb{A}\mathbb{A}^T$!..."

$$\underbrace{\underset{m\times n}{\mathbb{A}}\,\underset{n\times m}{\mathbb{A}^T}}_{m\times m} = (\mathbb{U}\Sigma\mathbb{V}^T)(\mathbb{U}\Sigma\mathbb{V}^T)^{\textcircled{T}}$$

↗ symmetric

$$= \mathbb{U}\Sigma\underbrace{\mathbb{V}^T\mathbb{V}^{\cancel{TT}}}_{\mathbb{I}}\Sigma^T\mathbb{U}^T$$

$$= \mathbb{U}\underbrace{\Sigma\Sigma^T}_{m\times m}\mathbb{U}^T$$

surprising!! ↓ $\mathbb{A}^T\mathbb{A}$ & $\mathbb{A}\mathbb{A}^T$ must have same non-zero eigenvalues

$$= \begin{bmatrix} | & | & & | \\ \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_m \\ | & | & & | \end{bmatrix} \overset{m\times m}{\begin{bmatrix} \sigma_1^2 & & & & & 0 \\ & \sigma_2^2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r^2 & & \\ & & & & 0 \cdots & \\ 0 & & & & & 0 \end{bmatrix}} \begin{bmatrix} - & \hat{u}_1^T & - \\ - & \hat{u}_2^T & - \\ & \vdots & \\ - & \hat{u}_n^T & - \end{bmatrix}$$

More upshot:

Diagonalize $\mathbb{A}\mathbb{A}^T$ to find $\hat{u}_i$'s and, again, σ_i's

E24ap4

How do we know $A\hat{v}_i = \sigma_i \hat{u}_i$?

We have: $A^T A \hat{v}_i = \sigma_i^2 \hat{v}_i$

(1) Monks: $\hat{v}_i^T (A^T A \hat{v}_i)$

$$\underbrace{(A\hat{v}_i)^T(A\hat{v}_i)}_{} = \|\underbrace{A\hat{v}_i}_{m\times 1 \text{ vector}}\|^2$$

$$\hat{v}_i^T(\sigma_i^2 \hat{v}_i) = \sigma_i^2 \underbrace{\hat{v}_i^T \hat{v}_i}_{=1} = \sigma_i^2$$

$$\|A\hat{v}_i\|^2 = \sigma_i^2$$

$$\Rightarrow \|A\hat{v}_i\|^2 = \sigma_i^2$$

$$\Rightarrow \|A\hat{v}_i\| = \sigma_i$$

So we have the right length ✓

(2) Monks again

$$A(A^T A \hat{v}_i) = A(\overset{\text{eigenvalue equat}}{\sigma_i^2 \hat{v}_i}) = \sigma_i^2 \underbrace{(A\hat{v}_i)}_{m\times 1 \text{ vector}}$$

$$A(A^T A \hat{v}_i) = (AA^T)\underbrace{(A\hat{v}_i)}_{m\times 1 \text{ vector}}$$

$\Rightarrow$ $A\hat{v}_i$ is an eigenvector of AA^T with eigenvalue σ_i^2

$\Rightarrow$ $A\hat{v}_i \propto \hat{u}_i$

(1) + (2) $\Rightarrow$ $A\hat{v}_i = \sigma_i \hat{u}_i$

Important details:

- Choose $\hat{u}_i$'s direction to match $A\hat{v}_i$ — makes sure $\sigma_i > 0$
- If we have found $\hat{v}_i$ already, $\boxed{\hat{u}_i = \frac{1}{\sigma_i} A\hat{v}_i}$ is best way to compute $\hat{u}_i$
- $A\hat{v}_i = \vec{0}$ for $\underbrace{i = r+1, r+2, \dots, n}_{\text{null space basis}}$
- $\hat{u}_i$ for $i = r+1, r+2, \dots, m$ = left nullspace basis

One last piece:

For $A = A^T$, we had $Q \Lambda Q^T$ and therefore

$$\underset{n\times n}{A} = \lambda_1 \hat{u}_1 \hat{u}_1^T + \lambda_2 \hat{u}_2 \hat{u}_2^T + \cdots + \lambda_n \hat{u}_n \hat{u}_n^T$$

outer products
= projection operators

A = sum of n rank 1 matrices.

For SVD:

$$A = \underset{m\times m}{\begin{bmatrix} | & | & & | \\ \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_m \\ | & | & & | \end{bmatrix}} \underset{m\times n}{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \sigma_r & \\ & & & & 0 \ddots 0 \end{bmatrix}} \underset{n\times n}{\begin{bmatrix} — & \hat{v}_1^T & — \\ — & \hat{v}_2^T & — \\ & \vdots & \\ — & \hat{v}_n^T & — \end{bmatrix}}$$

$$= \underset{m\times m}{\begin{bmatrix} | & | & & | \\ \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_m \\ | & | & & | \end{bmatrix}} \underset{m\times n}{\begin{bmatrix} — & \sigma_1 \hat{v}_1^T & — \\ — & \sigma_2 \hat{v}_2^T & — \\ — & \sigma_r \hat{v}_r^T & — \\ — & \vec{0}^T & — \\ & \vdots & \\ — & \vec{0}^T & — \end{bmatrix}} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} r \text{ rows non-zero} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} m-r \text{ rows of zilchos} \end{matrix}$$

$$= \sigma_1 \underset{m\times 1}{\hat{u}_1} \underset{1\times n}{\hat{v}_1^T} + \sigma_2 \underset{m\times 1}{\hat{u}_2} \underset{1\times n}{\hat{v}_2^T} + \cdots + \sigma_r \underset{m\times 1}{\hat{u}_r} \underset{1\times n}{\hat{v}_r^T}$$

$$\boxed{A = \sum_{i=1}^{r} \sigma_i \hat{u}_i \hat{v}_i^T}$$

See A as a superposition of r outer product rank 1 matrices of diminishing significance

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

- Each rank 1 matrix is a piece of Scottish Tartan
- SVD makes approximation of large matrices vigorous
- Speak of best rank 1, best rank 2, … approximations

SVD Example Calculation #1:

For $\underset{\substack{2\times2\\m\times n}}{A} = \begin{bmatrix}3 & -1\\ 3 & -1\end{bmatrix}$ find $A = \underset{\substack{m\times m\\2\times2}}{U}\underset{\substack{m\times n\\2\times2}}{\Sigma}\underset{\substack{n\times n\\2\times2}}{V^T}$

(1) Find $\hat{v}_i$'s and σ_i's using A^TA

- $A^TA = \begin{bmatrix}3 & 3\\ -1 & -1\end{bmatrix}\begin{bmatrix}3 & -1\\ 3 & -1\end{bmatrix} = \begin{bmatrix}18 & -6\\ -6 & 2\end{bmatrix}$ ← note symmetry
- Solve $|A^TA - \lambda I| = 0$

$$\Rightarrow 0 = \begin{vmatrix}18-\lambda & -6\\ -6 & 2-\lambda\end{vmatrix} = (18-\lambda)(2-\lambda) - 36$$
$$= \cancel{36} - 20\lambda + \lambda^2 - \cancel{36}$$
$$= \lambda(\lambda - 20)$$

$\Rightarrow \begin{matrix}\lambda_1 = 20 = \sigma_1^2\\ \lambda_2 = 0 = \sigma_2^2\end{matrix} \Rightarrow \begin{matrix}\sigma_1 = \sqrt{20}\\ \text{"}\sigma_2\text{"} = 0\end{matrix}$ ← row space ← null space $A\vec{x} = \vec{0}$

- $\lambda_1 = 20$: Solve $(A^TA - 20I)\vec{v}_1 = \vec{0}$

$$\Rightarrow \left[\begin{array}{cc|c}-2 & -6 & 0\\ -6 & -18 & 0\end{array}\right] \Rightarrow \hat{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix}3\\ -1\end{bmatrix}$$

rows must be multiples of each other for 2x2's

as promised, $\hat{v}_1 \perp \hat{v}_2$

- $\lambda_2 = 0$: Solve $(A^TA - \cancel{0}I)\vec{v}_2 = \vec{0}$

$$\Rightarrow \left[\begin{array}{cc|c}18 & -6 & 0\\ -6 & 2 & 0\end{array}\right] \Rightarrow \hat{v}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix}1\\ 3\end{bmatrix}$$

So far:

$$V = \begin{bmatrix}| & |\\ \hat{v}_1 & \hat{v}_2\\ | & |\end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix}3 & 1\\ -1 & 3\end{bmatrix}$$

$$\Sigma = \begin{bmatrix}\sqrt{20} & 0\\ 0 & 0\end{bmatrix}$$

$r = 1$ ↑ rank $\qquad \sigma_1 = \sqrt{20}$

- Need U as well

Either solve for eigenthings of $\underset{\substack{\shortparallel\\ \begin{bmatrix}10 & 10\\ 10 & 10\end{bmatrix}}}{AA^T}$ $\nearrow \lambda_1 = 20 \; \hat{u}_1$ $\searrow \lambda_2 = 0 \; \hat{u}_2$

- Better: Use $A\hat{v}_i = \sigma_i\hat{u}_i$

$$\Rightarrow \hat{u}_i = \frac{1}{\sigma_i}A\hat{v}_i$$

$$\hat{u}_1 = \frac{1}{\underset{\sigma_1}{\sqrt{20}}}A\underbrace{\frac{1}{\sqrt{10}}\begin{bmatrix}3\\ -1\end{bmatrix}}_{\hat{v}_1} = \frac{1}{\sqrt{20}}\frac{1}{\sqrt{10}}\begin{bmatrix}3 & -1\\ 3 & -1\end{bmatrix}\begin{bmatrix}3\\ -1\end{bmatrix}$$
$$= \frac{1}{\sqrt{20}}\frac{1}{\sqrt{10}}\begin{bmatrix}10\\ 10\end{bmatrix} = \frac{1}{\sqrt{2}}\frac{1}{10}\begin{bmatrix}10\\ 10\end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$

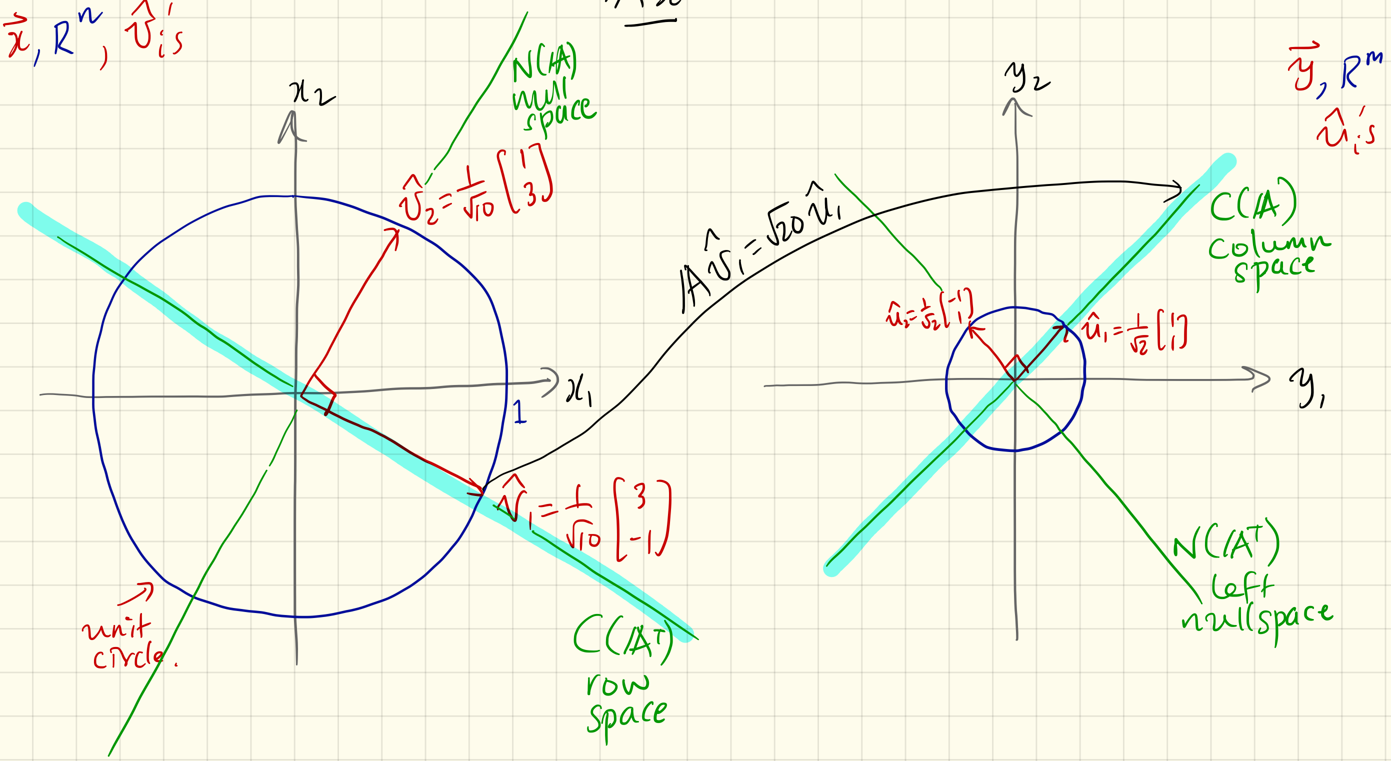
For $\hat{u}_2$, we just need a vector orthogonal to $\hat{u}_1$

By inspection: $\hat{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}$

better way to represent A

$$A = \begin{bmatrix}3 & -1\\ 3 & -1\end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}\sqrt{20} & 0\\ 0 & 0\end{bmatrix}}_{\Sigma}\underbrace{\frac{1}{\sqrt{10}}\begin{bmatrix}3 & -1\\ 1 & 3\end{bmatrix}}_{V^T}$$

$\mathbb{A}\vec{x}$



- See $\mathbb{A}$ sends $C(\mathbb{A}^T)$ to $C(\mathbb{A})$ with a stretch factor of $\sqrt{20}$.
- $\mathbb{A}$'s action between $C(\mathbb{A}^T)$ & $C(\mathbb{A})$ is invertible

SVD Example Calculation #2

E24cp1

Factorize $A = \frac{1}{5}\begin{bmatrix} 2 & 11 \\ 10 & 5 \end{bmatrix}$ as $U \Sigma V^T$

- Diagonalize A^TA

$$\underbrace{A^TA}_{\text{symmetric}} = \frac{1}{5}\begin{bmatrix} 2 & 10 \\ 11 & 5 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 2 & 11 \\ 10 & 5 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 104 & 72 \\ 72 & 146 \end{bmatrix} = \frac{2}{25}\begin{bmatrix} 52 & 36 \\ 36 & 73 \end{bmatrix}$$

SMT #731

if $B\vec{v} = \lambda \vec{v}$

then

$cB\vec{v} = c\lambda\vec{v}$

$\Rightarrow B'\vec{v} = (c\lambda)\vec{v}$

If $\vec{v}$ is an eigenvector of B with eigenvalue λ then $\vec{v}$ is an eigenvector of cB with eigenvalue $c\lambda$

Find λ's for $\begin{bmatrix} 52 & 36 \\ 36 & 73 \end{bmatrix}$

Solve $|A^TA - \lambda I| = 0$

$$0 = \begin{vmatrix} 52-\lambda & 36 \\ 36 & 73-\lambda \end{vmatrix} = (52-\lambda)(73-\lambda) - (36)^2$$

$$= 3796 - 125\lambda + \lambda^2 - 1296$$

$$= \lambda^2 - \underset{\uparrow \atop 25+100}{125}\lambda + 2500$$

$$= (\lambda - 25)(\lambda - 100) \Rightarrow \begin{matrix} \lambda_1 = 100 \\ \lambda_2 = 25 \end{matrix} \nearrow \text{for } \tfrac{25}{2}A^TA$$

$$\times \frac{2}{25} \Rightarrow \begin{matrix} \lambda_1 = 8 = \sigma_1^2 \\ \lambda_2 = 2 = \sigma_2^2 \end{matrix} \Rightarrow \begin{matrix} \sigma_1 = \sqrt{8} \\ \sigma_2 = \sqrt{2} \end{matrix}$$

$\swarrow$ for A^TA

- $\lambda_1 = 8$: $\frac{2}{25}\left[\begin{array}{cc|c} -48 & 36 & 0 \\ 36 & -27 & 0 \end{array}\right] \Rightarrow \hat{v}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Solve $(A^TA - \lambda I)\vec{v} = \vec{0}$

- $\lambda_2 = 2$: $\frac{2}{25}\left[\begin{array}{cc|c} 27 & 36 & 0 \\ 36 & 48 & 0 \end{array}\right] \Rightarrow \hat{v}_2 = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

(could choose $\hat{v}_2 = -\frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$)

now have

$$V = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \quad \& \quad \Sigma = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

- Now find $\hat{u}_1$ & $\hat{u}_2$

$$\hat{u}_i = \frac{1}{\sigma_i} A \hat{v}_i$$ ← best way

$$\hat{u}_1 = \frac{1}{\sqrt{8}} \frac{1}{5}\begin{bmatrix} 2 & 11 \\ 10 & 5 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{\sqrt{8}} \frac{1}{25}\begin{bmatrix} 50 \\ 50 \end{bmatrix} = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

unit vectors guaranteed

$$\hat{u}_2 = \frac{1}{\sqrt{2}} \frac{1}{5}\begin{bmatrix} 2 & 11 \\ 10 & 5 \end{bmatrix} \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{25}\begin{bmatrix} 25 \\ -25 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ ✓.

Could also diagonalize AA^T:

E24cp2

$$AA^T = \frac{1}{5}\begin{bmatrix} 2 & 11 \\ 10 & 5 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 2 & 10 \\ 11 & 5 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

Find $\lambda_1 = 8$, $\lambda_2 = 2$

$$\hat{u}_1 = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \hat{u}_2 = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

not sure about signs

⇒ Still have to compute $\hat{u}_i = \frac{1}{\sigma_i} A \hat{v}_i$ ✓.

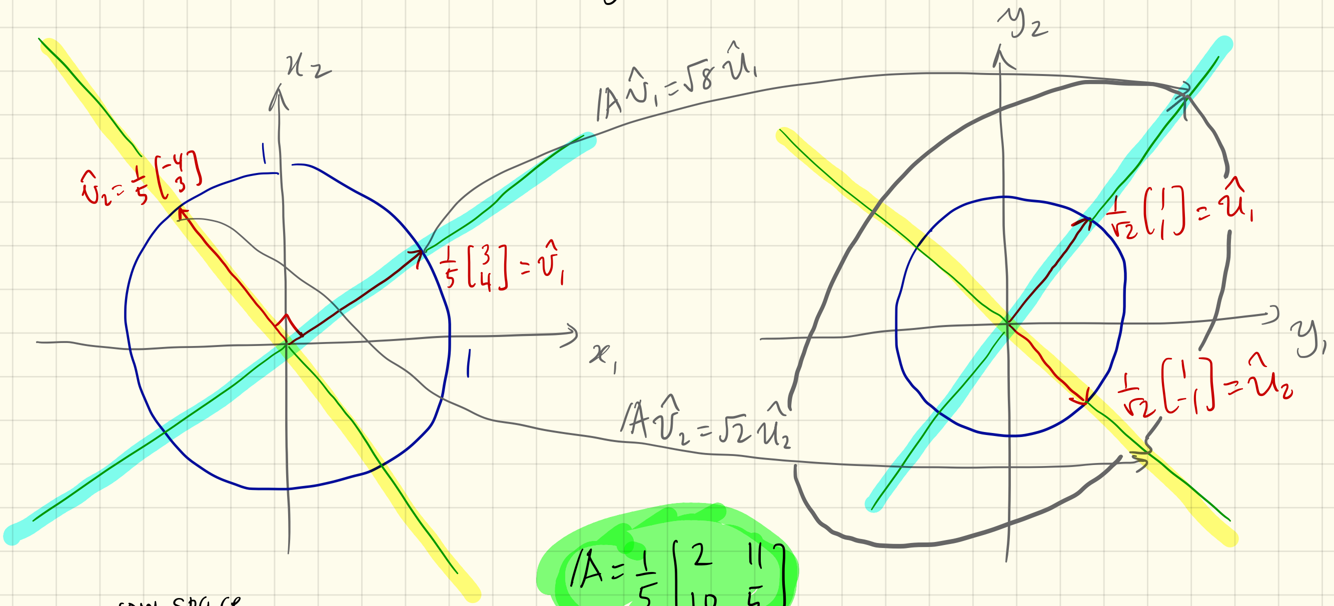
Overall:

$$A = \frac{1}{5}\begin{bmatrix} 2 & 11 \\ 10 & 5 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}}_{\Sigma} \underbrace{\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}}_{V^T}$$

$\mathbb{R}^2, \mathbb{R}^n, \vec{x}$'s

$$A\vec{x} = \vec{b}$$
or
$$\vec{y} = A\vec{x}$$

E24cp3



$$A = \frac{1}{5}\begin{bmatrix}2 & 11\\10 & 5\end{bmatrix}$$

row space

$$C(A^T) \equiv \mathbb{R}^2$$

$$N(A) = \{\vec{0}\}$$

Big deal: Circle ↦ Ellipse

↑
generalizes to higher dimensions

alphabetic, u, v
⟶
note $U \Sigma V^T$
↙
but wrong order of operations

E24cp4

how $\vec{y} = A\vec{x}$ works

$$A = U \Sigma V^T$$

③ Change from $\{\hat{u}\}$ to standard basis

② Does the work of A

Stretch/Shrink by σ_i factors in r dimensions

↗

$C(A^T)$ & $C(A)$

① Change $\vec{x}$'s representation from standard to $\{\hat{u}\}$ basis

Fundamental Theorem of Matrixology

From E13bp3 :

- $\dim C(A) = r$ ← rank   column space
- $\dim N(A^T) = m - r$ left null space
- $\dim C(A^T) = r$ row space
- $\dim N(A) = n - r$ nullspace
- $C(A)$ and $N(A^T)$ are orthogonal complements in R^m $C(A) \oplus N(A^T)$
- $C(A^T)$ and $N(A)$ are orthogonal complements in R^n $C(A^T) \oplus N(A)$
- The bases of $C(A)$ & $N(A^T)$ combine to give a basis of R^m
- The bases of $C(A^T)$ & $N(A)$ combine to give a basis of R^n

E25ap1

Now we also have:

- Row space has a "natural" orthonormal basis $\{\hat{v}_1, \dots, \hat{v}_r\}$, eigenvectors of A^TA
- Nullspace has a "natural" orthonormal basis $\{\hat{v}_{r+1}, \dots, \hat{v}_n\}$, eigenvectors of A^TA
- Column Space has a "natural" orthonormal basis $\{\hat{u}_1, \dots, \hat{u}_r\}$, eigenvectors of AA^T
- Left Nullspace has a "natural" orthonormal basis $\{\hat{u}_{r+1}, \dots, \hat{u}_m\}$, eigenvectors of AA^T
- The transformation between the "best" bases for row space and column space is diagonal with positive entries:

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & 0 \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$

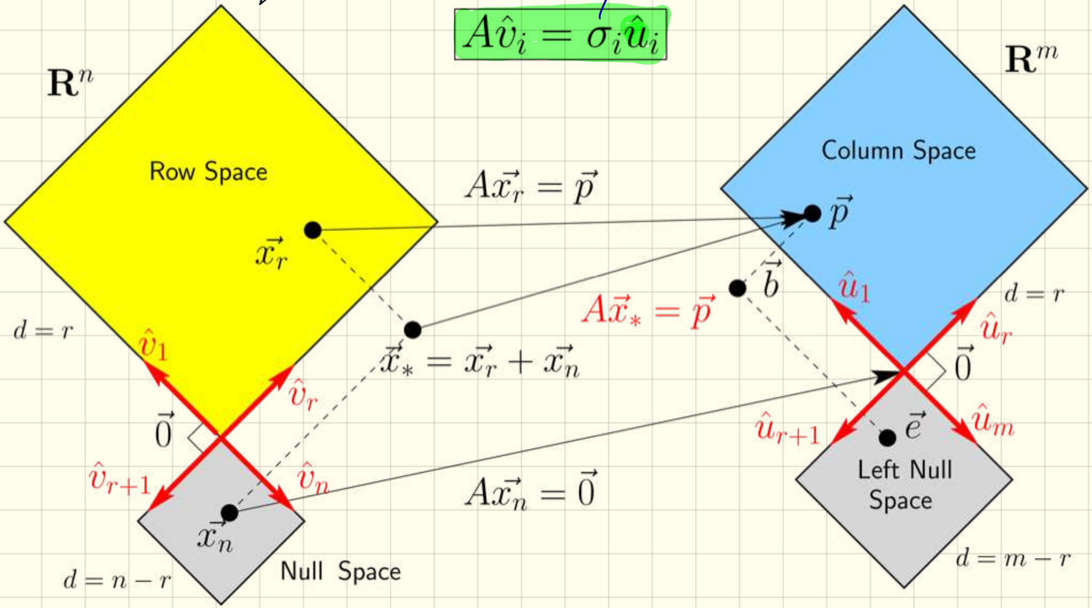
E25ap2

$\left[\begin{matrix} I \heartsuit \\ A\vec{x}=\vec{b} \end{matrix}\right]$

Show me the SVD!!

$\sigma_i > 0$

$$\boxed{A\hat{v}_i = \sigma_i \hat{u}_i}$$



Time for a nap:

E25ap3