Scaling—a Plenitude of Power Laws

Principles of Complex Systems | @pocsvox
CSYS/MATH 300, Fall, 2016 | #FallPoCS2016

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
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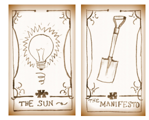
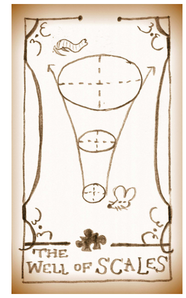
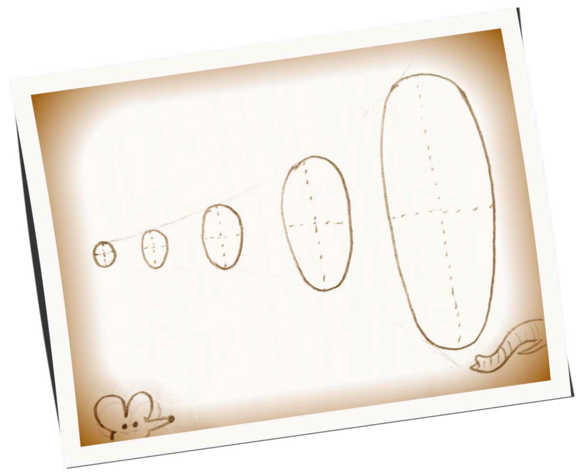
Outline





Archival object:



Scalingarama

General observation:

Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of scaling.

Outline—All about scaling:

- Basic definitions.
- Examples.

In CocoNuTs:

- Advances in measuring your power-law relationships.
- Scaling in blood and river networks.
- The Unsolved Allometry Theoricides.

Definitions

A power law relates two variables x and y as follows:

$$y = cx^{\alpha}$$

- α is the scaling exponent (or just exponent)
- α can be any number in principle but we will find various restrictions.
- c is the prefactor (which can be important!)

Definitions

- The prefactor c must balance dimensions.
- Imagine the height ℓ and volume v of a family of shapes are related as:

$$\ell = cv^{1/4}$$

- Using $[\cdot]$ to indicate dimension, then

$$[c] = [l]/[V^{1/4}] = L/L^{3/4} = L^{1/4}.$$

- More on this later with the Buckingham π theorem.

Looking at data

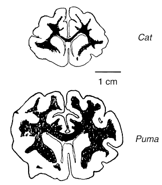
- Power-law relationships are linear in log-log space:

$$y = cx^{\alpha}$$

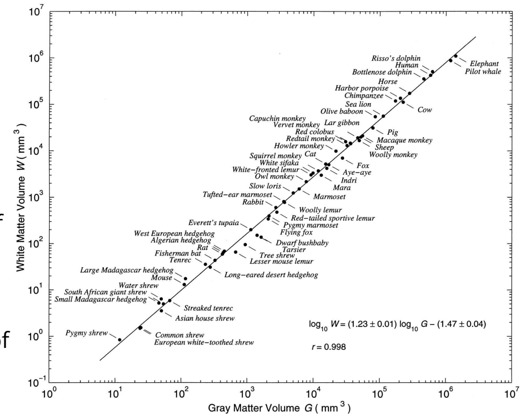
$$\Rightarrow \log_b y = \alpha \log_b x + \log_b c$$

with slope equal to α, the scaling exponent.
- Much searching for straight lines on log-log or double-logarithmic plots.
- Good practice: Always, always, always use base 10.
- Talk only about orders of magnitude (powers of 10).

A beautiful, heart-warming example:



- G = volume of gray matter: 'computing elements'
- W = volume of white matter: 'wiring'
- $W \sim cG^{1.23}$



- from Zhang & Sejnowski, PNAS (2000) [25]

Why is $\alpha \simeq 1.23$?

Quantities (following Zhang and Sejnowski):

- $G =$ Volume of gray matter (cortex/processors)
- $W =$ Volume of white matter (wiring)
- $T =$ Cortical thickness (wiring)
- $S =$ Cortical surface area
- $L =$ Average length of white matter fibers
- p = density of axons on white matter/cortex interface

A rough understanding:

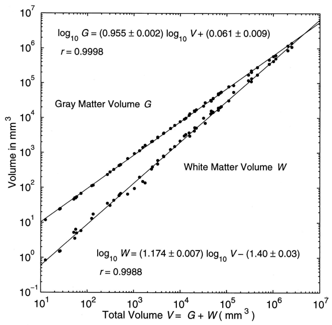
- $G \sim ST$ (convolutions are okay)
- $W \sim \frac{1}{2}pSL$
- $G \sim L^3 \leftarrow$ this is a little sketchy...
- Eliminate S and L to find $W \propto G^{4/3}/T$

Why is $\alpha \simeq 1.23$?

A rough understanding:

- We are here: $W \propto G^{4/3}/T$
- Observe weak scaling $T \propto G^{0.10 \pm 0.02}$.
- Implies $S \propto G^{0.9} \rightarrow$ convolutions fill space.
- $\Rightarrow W \propto G^{4/3}/T \propto G^{1.23 \pm 0.02}$

Tricksiness:



- With $V = G + W$, some power laws must be approximations.
- Measuring exponents is a hairy business...

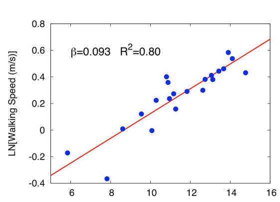
Good scaling:

General rules of thumb:

- *High quality:* scaling persists over three or more orders of magnitude for each variable.
- *Medium quality:* scaling persists over three or more orders of magnitude for only one variable and at least one for the other.
- *Very dubious:* scaling 'persists' over less than an order of magnitude for both variables.

Unconvincing scaling:

Average walking speed as a function of city population:



Two problems:

1. use of natural log, and
2. minute varation in dependent variable.

- from Bettencourt et al. (2007) [4]; otherwise totally great—see later.

Definitions

Power laws are the signature of scale invariance:

Scale invariant 'objects'
look the 'same'
when they are appropriately
rescaled.

- Objects = geometric shapes, time series, functions, relationships, distributions,...
- 'Same' might be 'statistically the same'
- To rescale means to change the units of measurement for the relevant variables

Scale invariance

Our friend $y = cx^\alpha$:

- If we rescale x as $x = rx'$ and y as $y = r^\alpha y'$,
- then

$$r^\alpha y' = c(rx')^\alpha$$

- $$\Rightarrow y' = c r^\alpha {x'}^\alpha r^{-\alpha}$$

- $$\Rightarrow y' = c{x'}^\alpha$$

Scale invariance

Compare with $y = ce^{-\lambda x}$:

- If we rescale x as $x = rx'$, then

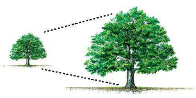
$$y = ce^{-\lambda r x'}$$

- Original form cannot be recovered.
- Scale matters for the exponential.

More on $y = ce^{-\lambda x}$:

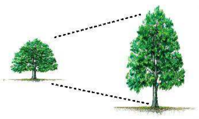
- Say $x_0 = 1/\lambda$ is the characteristic scale.
- For $x \gg x_0$, y is small, while for $x \ll x_0$, y is large.

Isometry:



- Dimensions scale linearly with each other.

Allometry:



- Dimensions scale nonlinearly.

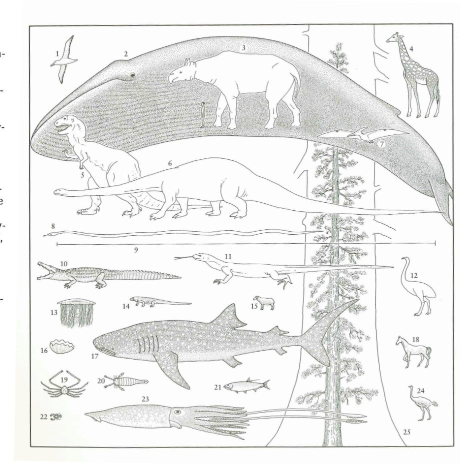
Allometry:

- Refers to differential growth rates of the parts of a living organism's body part or process.
- First proposed by Huxley and Teissier, Nature, 1936 "Terminology of relative growth" [10, 22]

The many scales of life:

The biggest living things (*left*). All the organisms are drawn to the same scale. *1*, The largest flying bird (albatross); *2*, the largest known animal (the blue whale), *3*, the largest extinct land mammal (*Baluchitherium*) with a human figure shown for scale; *4*, the tallest living land animal (giraffe); *5*, *Tyrannosaurus*; *6*, *Diplodocus*; *7*, one of the largest flying reptiles (*Pteranodon*); *8*, the largest extinct snake; *9*, the length of the largest tapeworm found in man; *10*, the largest living reptile (West African crocodile); *11*, the largest extinct lizard; *12*, the largest extinct bird (*Aepyornis*); *13*, the largest jellyfish (*Cyanea*); *14*, the largest living lizard (Komodo dragon); *15*, sheep; *16*, the largest bivalve mollusc (*Tridacna*); *17*; the largest fish (whale shark); *18*, horse; *19*, the largest crustacean (Japanese spider crab); *20*, the largest sea scorpion (Eurypterid); *21*, large tarpon; *22*, the largest lobster; *23*, the largest mollusc (deep-water squid, *Architeuthis*); *24*, ostrich; *25*, the lower 105 feet of the largest organism (giant sequoia), with a 100-foot larch superposed.

p. 2, McMahon and Bonner [17]



Definitions

Isometry versus Allometry:

- Iso-metry = 'same measure'
- Allo-metry = 'other measure'

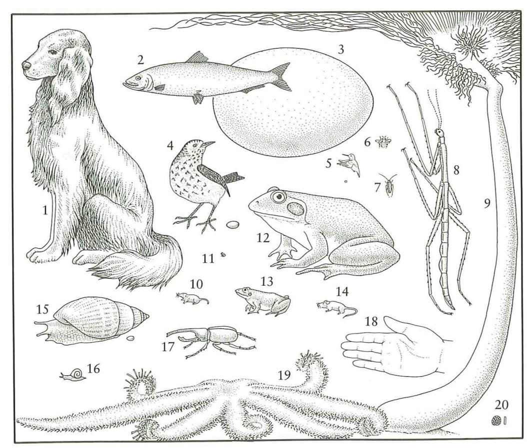
We use allometric scaling to refer to both:

1. Nonlinear scaling of a dependent variable on an independent one (e.g., $y \propto x^{1/3}$)
2. The relative scaling of correlated measures (e.g., white and gray matter).

The many scales of life:

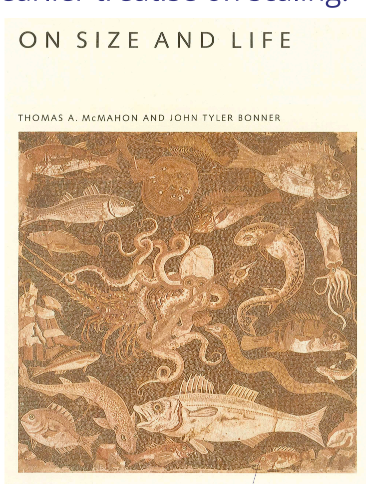
Medium-sized creatures (*above*). *1*, Dog; *2*, common herring; *3*, the largest egg (*Aepyornis*); *4*, song thrush with egg; *5*, the smallest bird (hummingbird) with egg; *6*, queen bee; *7*, common cockroach; *8*, the largest stick insect; *9*, the largest polyp (*Branchiocerianthus*); *10*, the smallest mammal (flying shrew); *11*, the smallest vertebrate (a tropical frog); *12*, the largest frog (goliath frog); *13*, common grass frog; *14*, house mouse; *15*, the largest land snail (*Achatina*) with egg; *16*, common snail; *17*, the largest beetle (goliath beetle); *18*, human hand; *19*, the largest starfish (*Luidia*); *20*, the largest free-moving protozoan (an extinct nummulite).

p. 3, McMahon and Bonner [17]
More on the Elephant Bird here.



An interesting, earlier treatise on scaling:

McMahon and Bonner, 1983 [17]

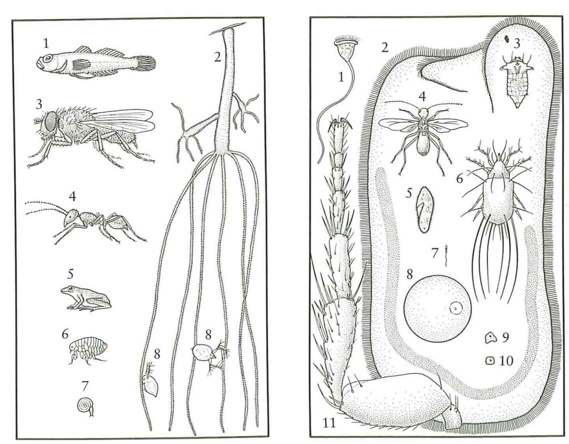


The many scales of life:

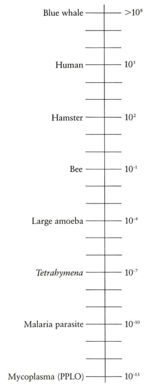
Small, "naked-eye" creatures (*lower left*). *1*, One of the smallest fishes (*Trimmatom nanus*); *2*, common brown hydra, expanded; *3*, housefly; *4*, medium-sized ant; *5*, the smallest vertebrate (a tropical frog, the same as the one numbered *11* in the figure *above*); *6*, flea (*Xenopsylla cheopis*); *7*, the smallest land snail; *8*, common water flea (*Daphnia*).

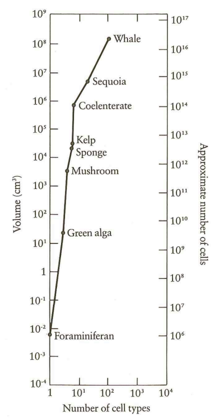
The smallest "naked-eye" creatures and some large microscopic animals and cells (*below right*). *1*, *Vorticella*, a ciliate; *2*, the largest ciliate protozoan (*Bursaria*); *3*, the smallest many-celled animal (a rotifer); *4*, smallest flying insect (*Elaphis*); *5*, another ciliate (*Paramecium*); *6*, cheese mite; *7*, human sperm; *8*, human ovum; *9*, dysentery amoeba; *10*, human liver cell; *11*, the foreleg of the flea (numbered *6* in the figure to the *left*).

3, McMahon and Bonner [17]



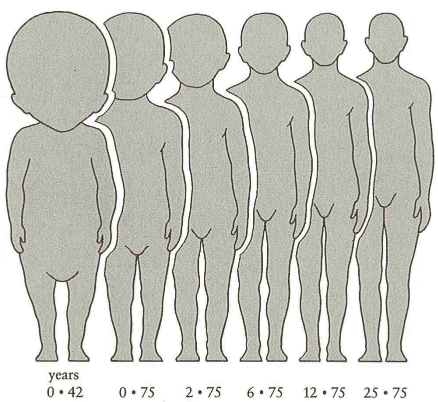
Size range (in grams) and cell differentiation:





10^{-13} to 10^{8} g, p. 3, McMahon and Bonner [17]

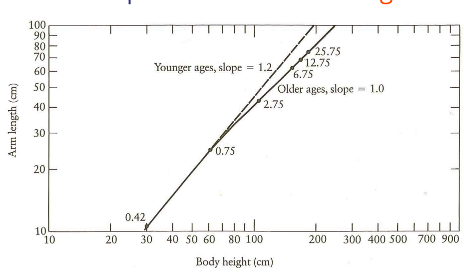
Non-uniform growth:



p. 32, McMahon and Bonner [17]

Non-uniform growth—arm length versus height:

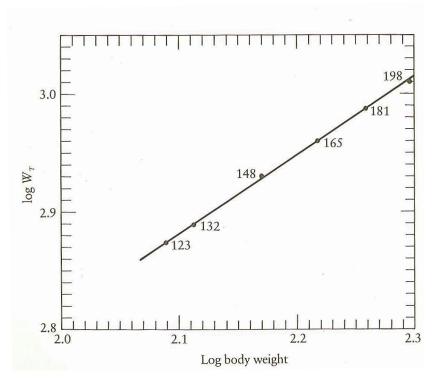
Good example of a break in scaling:



A crossover in scaling occurs around a height of 1 metre.

p. 32, McMahon and Bonner [17]

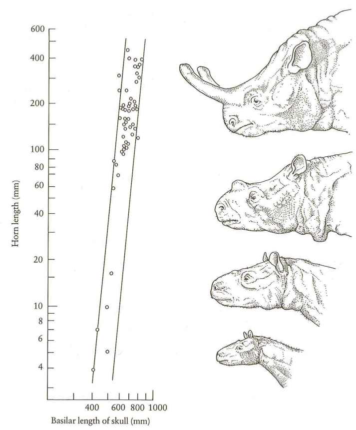
Weightlifting: $M_{\text{world record}} \propto M_{\text{lifter}}^{2/3}$



Idea: Power $\sim$ cross-sectional area of isometric lifters.

p. 53, McMahon and Bonner [17]

Titanothere horns: $L_{\text{horn}} \sim L_{\text{skull}^4}$



p. 36, McMahon and Bonner [17]; a bit dubious.

Animal power

Fundamental biological and ecological constraint:

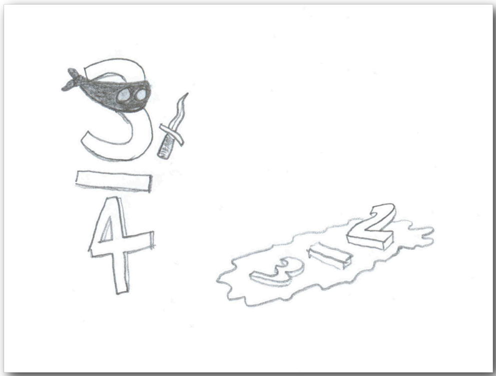
$$P = c\,M^{\alpha}$$

$P =$ basal metabolic rate

$M =$ organismal body mass

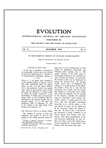
  

Stories—The Fraction Assassin:



Ecology—Species-area law:

Allegedly (data is messy): [12, 11]

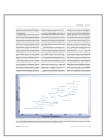


"An equilibrium theory of insular zoogeography"
MacArthur and Wilson,
Evolution, **17**, 373–387, 1963. [12]

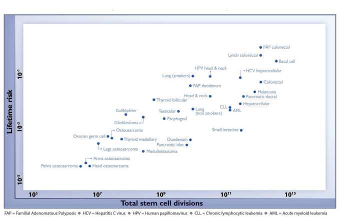
$$N_{\text{species}} \propto A^{\beta}$$

- According to physicists—on islands: $\beta \approx 1/4$.
- Also—on continuous land: $\beta \approx 1/8$.

Cancer:



"Variation in cancer risk among tissues can be explained by the number of stem cell divisions"
Tomasetti and Vogelstein,
Science Magazine, **347**, 78–81, 2015. [23]



Roughly: $p \sim r^{2/3}$ where p = life time probability and r = rate of stem cell replication.

"How fast do living organisms move: Maximum speeds from bacteria to elephants and whales"
Meyer-Vernet and Rospars,
American Journal of Physics, **83**, 719–722, 2015. [18]

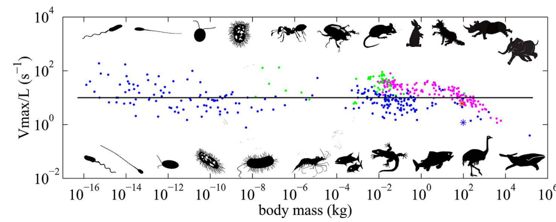
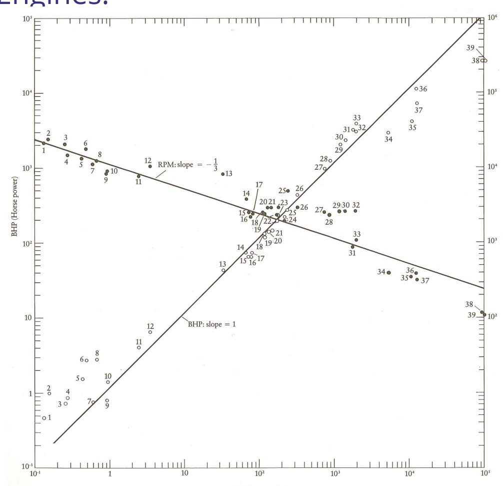


Fig. 1. Maximum relative speed versus body mass for 202 running species (157 mammals plotted in magenta and 45 non-mammals plotted in green), 127 swimming species and 91 micro-organisms (plotted in blue). The sources of the data are given in Ref. 16. The solid line is the maximum relative speed [Eq. (13)] estimated in Sec. III. The human world records are plotted as asterisks (upper for running and lower for swimming). Some examples of organisms of various masses are sketched in black (drawings by François Meyer).

Insert question from assignment 1

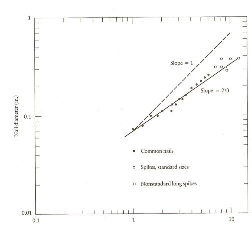
Engines:



BHP = brake horse power

The allometry of nails:

Observed: Diameter $\propto$ Length$^{2/3}$ or $d \propto \ell^{2/3}$.



Since $\ell d^2 \propto$ Volume v:

- Diameter $\propto$ Mass$^{2/7}$ or $d \propto v^{2/7}$.
- Length $\propto$ Mass$^{3/7}$ or $\ell \propto v^{3/7}$.
- Nails lengthen faster than they broaden (c.f. trees).

p. 58–59, McMahon and Bonner [17]

The allometry of nails:

A buckling instability?:

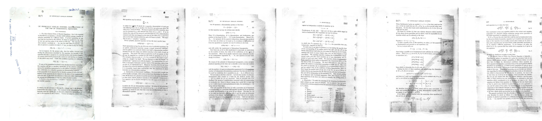
- Physics/Engineering result: Columns buckle under a load which depends on d^4/ℓ^2.
- To drive nails in, posit resistive force $\propto$ nail circumference = πd.
- Match forces independent of nail size: $d^4/\ell^2 \propto d$.
- Leads to $d \propto \ell^{2/3}$.
- Argument made by Galileo [7] in 1638 in "Discourses on Two New Sciences." Also, see here.
- Another smart person's contribution: Euler, 1757
- Also see McMahon, "Size and Shape in Biology," Science, 1973. [16]

Dimensional Analysis:

The Buckingham π theorem:[1]

"On Physically Similar Systems: Illustrations of the Use of Dimensional Equations"
E. Buckingham,
Phys. Rev., **4**, 345–376, 1914. [5]

As captured in the 1990s in the MIT physics library:

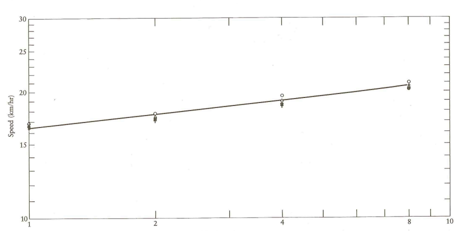


[1] Stigler's Law of Eponymy applies. See here.

Rowing: Speed $\propto$ (number of rowers)$^{1/9}$

Shell dimensions and performances.

No. of oarsmen	Modifying description	Length, l (m)	Beam, b (m)	l/b	Boat mass per oarsman (kg)	Time for 2000 m (min) I	II	III	IV
8	Heavyweight	18.28	0.610	30.0	14.7	5.87	5.92	5.82	5.73
8	Lightweight	18.28	0.598	30.6	14.7				
4	With coxswain	12.80	0.574	22.3	18.1				
4	Without coxswain	11.75	0.574	21.0	18.1	6.33	6.42	6.48	6.13
2	Double scull	9.76	0.381	25.6	13.6				
2	Pair-oared shell	9.76	0.356	27.4	13.6	6.87	6.92	6.95	6.77
1	Single scull	7.93	0.293	27.0	16.3	7.16	7.25	7.28	7.17



Dimensional Analysis:[2]

Fundamental equations cannot depend on units:

- System involves n related quantities with some unknown equation $f(q_1, q_2, \dots, q_n) = 0$.
- Geometric ex.: area of a square, side length ℓ: $A = \ell^2$ where $[A] = L^2$ and $[\ell] = L$.
- Rewrite as a relation of $p \le n$ independent dimensionless parameters where p is the number of independent dimensions (mass, length, time, luminous intensity ...):

$$F(\pi_1, \pi_2, \dots, \pi_p) = 0$$

- e.g., $A/\ell^2 - 1 = 0$ where $\pi_1 = A/\ell^2$.
- Another example: $F = ma \Rightarrow F/ma - 1 = 0$.
- Plan: solve problems using only backs of envelopes.

[2] Length is a dimension, furlongs and smoots are units

Physics:

Scaling in elementary laws of physics:

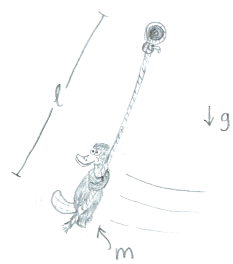
- Inverse-square law of gravity and Coulomb's law:

$$F \propto \frac{m_1 m_2}{r^2} \quad \text{and} \quad F \propto \frac{q_1 q_2}{r^2}.$$

- Force is diminished by expansion of space away from source.
- The square is $d - 1 = 3 - 1 = 2$, the dimension of a sphere's surface.

Example:

Simple pendulum:



- Idealized mass/platypus swinging forever.
- Four quantities:
  1. Length ℓ,
  2. mass m,
  3. gravitational acceleration g, and
  4. pendulum's period τ.
- Variable dimensions: $[\ell] = L$, $[m] = M$, $[g] = LT^{-2}$, and $[\tau] = T$.
- Turn over your envelopes and find some π's.

A little formalism:

- Game: find all possible independent combinations of the $\{q_1, q_2, \dots, q_n\}$, that form dimensionless quantities $\{\pi_1, \pi_2, \dots, \pi_p\}$, where we need to figure out $p \le n$.
- Consider $\pi_i = q_1^{x_1} q_2^{x_2} \cdots q_n^{x_n}$.
- We (desperately) want to find all sets of powers x_j that create dimensionless quantities.
- Dimensions: want $[\pi_i] = [q_1]^{x_1}[q_2]^{x_2}\cdots[q_n]^{x_n} = 1$.
- For the platypus pendulum we have $[q_1] = L$, $[q_2] = M$, $[q_3] = LT^{-2}$, and $[q_4] = T$, with dimensions $d_1 = L$, $d_2 = M$, and $d_3 = T$.
- So: $[\pi_i] = L^{x_1} M^{x_2} (LT^{-2})^{x_3} T^{x_4}$.
- We regroup: $[\pi_i] = L^{x_1+x_3} M^{x_2} T^{-2x_3+x_4}$.
- We now need: $x_1 + x_3 = 0$, $x_2 = 0$, and $-2x_3 + x_4$.
- Time for matrixology ...

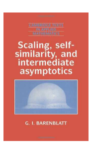
Well, of course there are matrices:

- Thrillingly, we have:

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- A nullspace equation: $\mathbf{A}\vec{x} = \vec{0}$.
- Number of dimensionless parameters = Dimension of null space = $n - r$ where n is the number of columns of $\mathbf{A}$ and r is the rank of $\mathbf{A}$.
- Here: $n = 4$ and $r = 3 \rightarrow F(\pi_1) = 0 \rightarrow \pi_1$ = const.
- In general: Create a matrix $\mathbf{A}$ where ijth entry is the power of dimension i in the jth variable, and solve by row reduction to find basis null vectors.
- We (you) find: $\pi_1 = \ell/g\tau^2 = \text{const.}$ Upshot: $\tau \propto \sqrt{\ell}$.

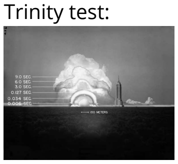
Insert question from assignment 1



"Scaling, self-similarity, and intermediate asymptotics" by G. I. Barenblatt (1996). [2]

G. I. Taylor, magazines, and classified secrets:

1945
New Mexico
Trinity test:



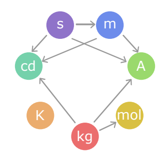
Self-similar blast wave:

- Radius: $[R] = L$,
  Time: $[t] = T$,
  Density of air: $[\rho] = M/L^3$,
  Energy: $[E] = ML^2/T^2$.
- Four variables, three dimensions.
- One dimensionless variable: $E = \text{constant} \times \rho R^5/t^2$.
- Scaling: Speed decays as $1/R^{3/2}$.

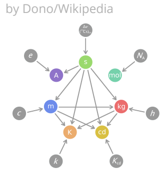
Related: Radiolab's Elements on the Cold War, the Bomb Pulse, and the dating of cell age (33:30).

We're still sorting out units:

Proposed 2018 revision of SI base units:



by Dono/Wikipedia

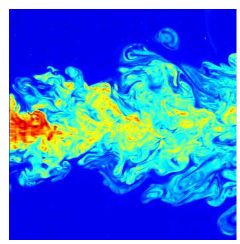


by Wikipetzi/Wikipedia

- Now: kilogram is an artifact in Sèvres, France.
- Future: Defined by fixing Planck's constant as $6.62606X \times 10^{-34}$ s^{-1}·m^2·kg.[3]
- Metre chosen to fix speed of light at 299792458 m·s^{-1}.
- Radiolab piece: $\le$ kg

[3] X = still arguing ...

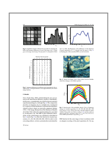
Turbulence:



Big whirls have little whirls
That heed on their velocity,
And little whirls have littler whirls
And so on to viscosity.

— Lewis Fry Richardson

- Image from here.
- Jonathan Swift (1733): "Big fleas have little fleas upon their backs to bite 'em, And little fleas have lesser fleas, and so, ad infinitum." The Siphonaptera.



"Turbulent luminance in impassioned van Gogh paintings"
Aragón et al.,
J. Math. Imaging Vis., **30**, 275–283, 2008. [1]

- Examined the probability pixels a distance R apart share the same luminance.
- "Van Gogh painted perfect turbulence" by Phillip Ball, July 2006.
- Apparently not observed in other famous painter's works or when van Gogh was settled.
- Oops: Small ranges and natural log used.

Advances in turbulence:

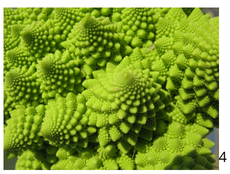
Kolmogorov, armed only with dimensional analysis and an envelope figures this out in 1941:

$$E(k) = C\epsilon^{2/3}k^{-5/3}$$

- $E(k)$ = energy spectrum function.
- ϵ = rate of energy dissipation.
- $k = 2\pi/\lambda$ = wavenumber.

- Energy is distributed across all modes, decaying with wave number.
- No internal characteristic scale to turbulence.
- Stands up well experimentally and there has been no other advance of similar magnitude.

"The Geometry of Nature": Fractals



- "Anomalous" scaling of lengths, areas, volumes relative to each other.
- The enduring question: how do self-similar geometries form?

- Robert E. Horton: Self-similarity of river (branching) networks (1945). [8]
- Harold Hurst—Roughness of time series (1951). [9]
- Lewis Fry Richardson—Coastlines (1961).
- Benoît B. Mandelbrot—Introduced the term "Fractals" and explored them everywhere, 1960s on. [13, 14, 15]

[d] Note to self: Make millions with the "Fractal Diet"

Scaling in Cities:

"Growth, innovation, scaling, and the pace of life in cities"
Bettencourt et al.,
Proc. Natl. Acad. Sci., **104**, 7301–7306, 2007. [4]

- Quantified levels of
  - Infrastructure
  - Wealth
  - Crime levels
  - Disease
  - Energy consumption

as a function of city size N (population).

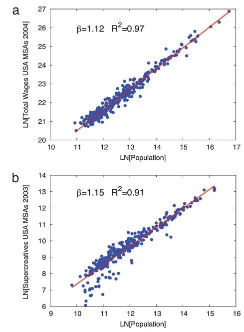


Fig. 1. Examples of scaling relationships. (*a*) Total wages per MSA in 2004 for the U.S. (blue points) vs. metropolitan population. (*b*) Supercreative employment per MSA in 2003, for the U.S. (blue points) vs. metropolitan population. Best-fit scaling relations are shown as solid lines.

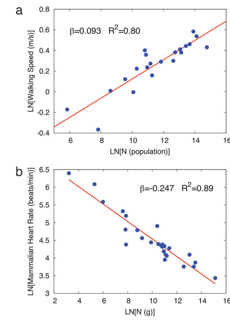


Fig. 2. The pace of urban life increases with city size in contrast to the pace of biological life, which decreases with organism size. (*a*) Scaling of walking speed vs. population for cities around the world. (*b*) Heart rate vs. the size (mass) of organisms.

Scaling in Cities:

Table 1. Scaling exponents for urban indicators vs. city size

Y	β	95% CI	Adj-R^2	Observations	Country–year
New patents	1.27	[1.25,1.29]	0.72	331	U.S. 2001
Inventors	1.25	[1.22,1.27]	0.76	331	U.S. 2001
Private R&D employment	1.34	[1.29,1.39]	0.92	266	U.S. 2002
"Supercreative" employment	1.15	[1.11,1.18]	0.89	287	U.S. 2003
R&D establishments	1.19	[1.14,1.22]	0.77	287	U.S. 1997
R&D employment	1.26	[1.18,1.43]	0.93	295	China 2002
Total wages	1.12	[1.09,1.13]	0.96	361	U.S. 2002
Total bank deposits	1.08	[1.03,1.11]	0.91	267	U.S. 1996
GDP	1.15	[1.06,1.23]	0.96	295	China 2002
GDP	1.26	[1.09,1.46]	0.64	196	EU 1999–2003
GDP	1.13	[1.03,1.23]	0.94	37	Germany 2003
Total electrical consumption	1.07	[1.03,1.11]	0.88	392	Germany 2002
New AIDS cases	1.23	[1.18,1.29]	0.76	93	U.S. 2002–2003
Serious crimes	1.16	[1.11, 1.18]	0.89	287	U.S. 2003
Total housing	1.00	[0.99,1.01]	0.99	316	U.S. 1990
Total employment	1.01	[0.99,1.02]	0.98	331	U.S. 2001
Household electrical consumption	1.00	[0.94,1.06]	0.88	377	Germany 2002
Household electrical consumption	1.05	[0.89,1.22]	0.91	295	China 2002
Household water consumption	1.01	[0.89,1.11]	0.96	295	China 2002
Gasoline stations	0.77	[0.74,0.81]	0.93	318	U.S. 2001
Gasoline sales	0.79	[0.73,0.80]	0.94	318	U.S. 2001
Length of electrical cables	0.87	[0.82,0.92]	0.75	380	Germany 2002
Road surface	0.83	[0.74,0.92]	0.87	29	Germany 2002

Data sources are shown in *SI Text*. CI, confidence interval; Adj-R^2, adjusted R^2; GDP, gross domestic product.

Scaling in Cities:

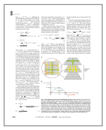
Intriguing findings:

- Global supply costs scale sublinearly with N ($\beta < 1$).
  - Returns to scale for infrastructure.
- Total individual costs scale linearly with N ($\beta = 1$)
  - Individuals consume similar amounts independent of city size.
- Social quantities scale superlinearly with N ($\beta > 1$)
  - Creativity (# patents), wealth, disease, crime, ...

Density doesn't seem to matter...

- Surprising given that across the world, we observe two orders of magnitude variation in area covered by agglomerations of fixed populations.

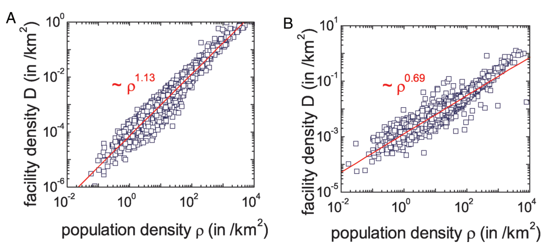
A possible theoretical explanation?



"The origins of scaling in cities"
Luís M. A. Bettencourt,
Science, **340**, 1438–1441, 2013. [3]

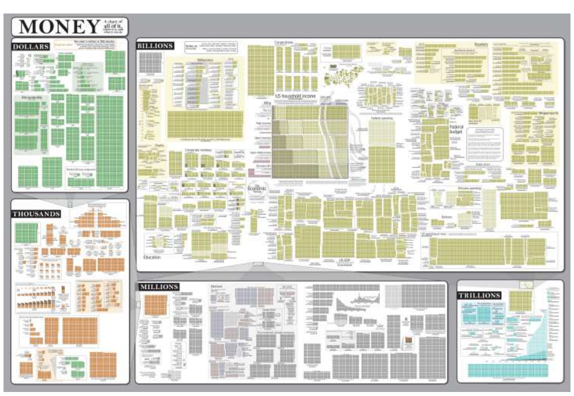
#sixthology

Density of public and private facilities:



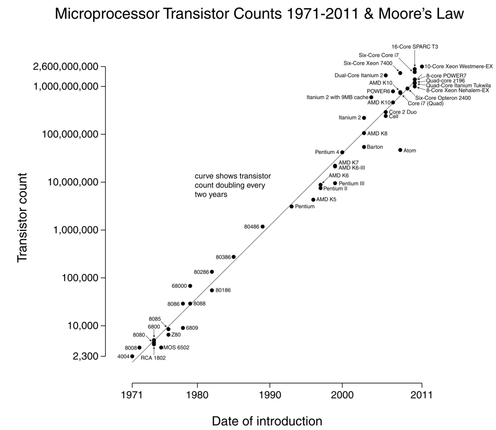
$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{\alpha}$$

- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.



Explore the original zoomable and interactive version here: http://xkcd.com/980/.

Moore's Law:



Scaling laws for technology production:

- "Statistical Basis for Predicting Technological Progress [20]" Nagy et al., PLoS ONE, 2013.
- y_t = stuff unit cost; x_t = total amount of stuff made.
- Wright's Law, cost decreases as a power of total stuff made: [24]

$$y_t \propto x_t^{-w}.$$

- Moore's Law, framed as cost decrease connected with doubling of transistor density every two years: [19]

$$y_t \propto e^{-mt}.$$

- Sahal's observation that Moore's law gives rise to Wright's law if stuff production grows exponentially: [21]

$$x_t \propto e^{gt}.$$

- Sahal + Moore gives Wright with $w = m/g$.

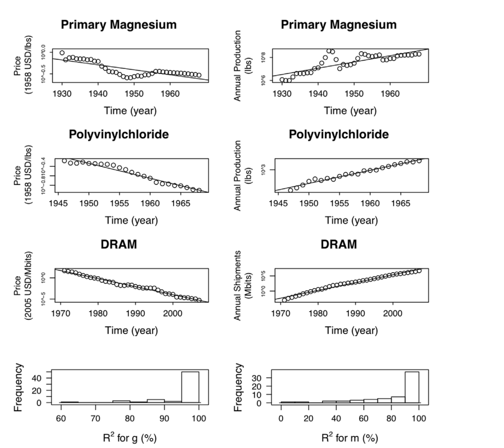


Figure 3. Three examples showing the logarithm of price as a function of time in the left column and the logarithm of production as a function of time in the right column, based on industry-wide data. We have chosen these examples to be representative: The top row contains an example with one of the worst fits, the second row an example with an intermediate goodness of fit, and the third row one of the best examples. The fourth row of the figure shows histograms of R^2 values for fitting g and m for the 62 datasets.
doi:10.1371/journal.pone.0052669.g003

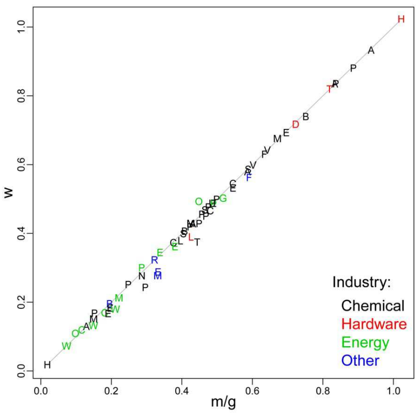


Figure 4. An illustration that the combination of exponentially increasing production and exponentially decreasing cost are equivalent to Wright's law. The value of the Wright parameter w is plotted against the prediction m/g based on the Sahal formula, where m is the exponent of cost reduction and g the exponent of the increase in cumulative production.
doi:10.1371/journal.pone.0052669.g004

TABLE 1
*Summary of results**

Network	Node	No. data points	Range of log N	Log–log R^2	Semi-log R^2	p_{power}/p_{log}	Relationship between C and N	Comb. degree	Exponent v for type-net scaling	Figure in text
Selected networks										
Electronic circuits	Component	373	2.12	0.747	0.602	0.05/4e−5	Power law	2.29	0.92	2
Legos™	Piece	391	2.65	0.903	0.732	0.09/1e−7	Power law	1.41	—	3
Businesses										
military vessels	Employee	13	1.88	0.971	0.832	0.05/3e−3	Power law	1.60	—	4
military offices	Employee	8	1.59	0.964	0.789	0.16/0.16	Increasing	1.13	—	4
universities	Employee	9	1.55	0.786	0.749	0.27/0.27	Increasing	1.37	—	4
insurance co.	Employee	52	2.30	0.748	0.685	0.11/0.10	Increasing	3.04	—	4
Universities										
across schools	Faculty	112	2.72	0.695	0.549	0.09/0.01	Power law	1.81	—	5
history of Duke	Faculty	46	0.94	0.921	0.892	0.09/0.05	Increasing	2.07	—	5
Ant colonies										
caste = type	Ant	46	6.00	0.481	0.454	0.11/0.04	Power law	8.16	—	6
size range = type	Ant	22	5.24	0.658	0.548	0.17/0.04	Power law	8.00	—	6
Organisms	Cell	134	12.40	0.249	0.165	0.08/0.02	Power law	17.73	—	7
Neocortex	Neuron	10	0.85	0.520	0.584	0.16/0.16	Increasing	4.56	—	9
Competitive networks										
Biotas	Organism	—	—	—	—	—	Power law	≈3	0.3 to 1.0	—
Cities	Business	82	2.44	0.985	0.832	0.08/8e-8	Power law	1.56	—	10

*(1) The kind of network, (2) what the nodes are within that kind of network, (3) the number of data points, (4) the logarithmic range of network sizes N (i.e. $\log(N_{max}/N_{min})$), (5) the log–log correlation, (6) the semi-log correlation, (7) the serial-dependence probabilities under, respectively, power-law and logarithmic models, (8) the empirically determined best-fit relationship between differentiation C and organization size N (if one of the two models can be refuted with $p<0.05$; otherwise we just write "increasing" to denote that neither model can be rejected), (9) the combinatorial degree (i.e. the inverse of the best-fit slope of a log–log plot of C versus N), (10) the scaling exponent for how quickly the edge-degree δ scales with type-network size C (in those places for which data exist), (11) figure in this text where the plots are presented. Values for biotas represent the broad trend from the literature.

Scaling of Specialization:

"Scaling of Differentiation in Networks: Nervous Systems, Organisms, Ant Colonies, Ecosystems, Businesses, Universities, Cities, Electronic Circuits, and Legos"
M. A. Changizi, M. A. McDannald and D. Widders [6]
J. Theor. Biol., 2002.

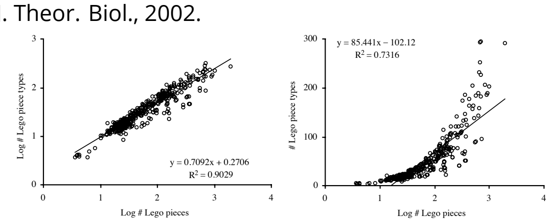


FIG. 3. Log–log (base 10) (left) and semi-log (right) plots of the number of Lego piece types vs. the total number of parts in Lego structures ($n = 391$). To help to distinguish the data points, logarithmic values were perturbed by adding a random number in the interval [−0.05, 0.05], and non-logarithmic values were perturbed by adding a random number in the interval [−1, 1].

- Nice 2012 wired.com write-up

Shell of the nut:

- Scaling is a fundamental feature of complex systems.
- Basic distinction between isometric and allometric scaling.
- Powerful envelope-based approach: Dimensional analysis.
- "Oh yeah, well that's just dimensional analysis" said the [insert your own adjective] physicist.
- Tricksiness: A wide variety of mechanisms give rise to scalings, both normal and unusual.

$C \sim N^{1/d}$, $d \geq 1$:

- C = network differentiation = # node types.
- N = network size = # nodes.
- d = combinatorial degree.
- Low d: strongly specialized parts.
- High d: strongly combinatorial in nature, parts are reused.
- Claim: Natural selection produces high d systems.
- Claim: Engineering/brains produces low d systems.

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Scaling

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